9.4 Frontal advance for unsteady 1D displacement

The unsteady state displacement of oil by water is due to the change in water saturation with time. This can be visualized by looking at the schematics in Figure 9.6. These schematics represent

Figure 9.6 Progression of water displacing oil for immiscible, 1D snapshots in time of the frontal boundary as water is displacing oil. In sequence, A depicts the initial state of the sample (or reservoir) where saturations are separated into irreducible water, residual oil and mobile oil components. After a given time of injection, the front advances to a position as shown in B. Ahead of the front water saturation is at irreducible, but behind the front water saturation is increased. Continuing in time, eventually the water will breakthrough the end of the core (reservoir) and both oil and water will be produced simultaneously, C. Continued injection will increase the displacing phase saturation in the core (reservoir), D.

Two methods to predict the displacement performance are 1) the analytical solution by Buckley – Leverett (1941), and 2) applying numerical simulation. Only the analytical solution will be described in this chapter.

9.4.1 Buckley – Leverett (1941)

The derivation begins from the 1D, multiphase continuity equations.
\[
\frac{\partial}{\partial x}(\rho_o u_{ox}) = -\frac{\partial}{\partial t}(\rho_o S_o \phi) \quad (9.19)
\]

\[
\frac{\partial}{\partial x}(\rho_w u_{wx}) = -\frac{\partial}{\partial t}(\rho_w S_w \phi) \quad (9.20)
\]

In terms of volumetric flow rate,

\[
\frac{\partial}{\partial x}(\rho_o q_o) = -A \frac{\partial}{\partial t}(\rho_o S_o \phi) \quad (9.21)
\]

\[
\frac{\partial}{\partial x}(\rho_w q_w) = -A \frac{\partial}{\partial t}(\rho_w S_w \phi) \quad (9.22)
\]

Assume the fluids are incompressible and the porosity is constant. Eqs. (9.21) and (9.22) simplify to,

\[
\frac{\partial q_o}{\partial x} = -A \phi \frac{\partial S_o}{\partial t} \quad (9.23)
\]

\[
\frac{\partial q_w}{\partial x} = -A \phi \frac{\partial S_w}{\partial t} \quad (9.24)
\]

Combining,

\[
\frac{\partial (q_w + q_o)}{\partial x} = -A \phi \frac{\partial (S_w + S_o)}{\partial t} = 0 \quad (9.25)
\]

The result is \( q_T = q_o + q_w = \) constant. That is, the total flow rate is constant at each cross-section.

From the definition of fractional flow,

\[
q_w = f_w q_T \\
q_o = (1 - f_w) q_T \quad (9.26)
\]

Substitute into Darcy’s equation for each phase,

\[
q_o = (1 - f_w) q_T = -\frac{k_o A}{\mu_o} \left( \frac{\partial p_o}{\partial x} + \rho_o g \sin \alpha \right) \quad (9.27)
\]

\[
q_w = f_w q_T = -\frac{k_w A}{\mu_w} \left( \frac{\partial p_w}{\partial x} + \rho_w g \sin \alpha \right) \quad (9.28)
\]

Rearranging Eqs. (9.27) and (9.28), we can substitute for the pressure gradient terms.

\[
\frac{dp_o}{dx} = \frac{dp_w}{dx} - \frac{dp_w}{dx}
\]

Solving the resulting equation provides the complete fractional flow equation.
\[
f_w = \frac{1}{1 + \frac{k_w \mu_o}{k_a \mu_w}} \left( \sigma + \frac{k_w A}{\mu_w q_r} \frac{\partial p_c}{\partial x} + \Delta \rho g \sin \alpha \right) \tag{9.28}
\]

In the analytical solution it is difficult to analyze the derivative term \(\frac{dp_c}{dx}\). If we expand this derivative to,

\[
\left( \frac{\partial p_c}{\partial x} \right) = \left( \frac{\partial p_c}{\partial S_w} \right) \cdot \left( \frac{\partial S_w}{\partial x} \right) \tag{9.29}
\]

and assume \(\frac{dp_c}{dx} \approx 0\), then either term on the right-hand side must be small and approach zero as well. In linear displacement, \(\frac{dp_c}{dS_w} \rightarrow 0\) at moderate to high water saturations. This is evident by inspection of a capillary pressure curve such as shown in Figure 9.7.

![Figure 9.7 Capillary pressure curve illustrating flat transition region at moderate to high water saturations.](image)

If the derivative term is negligible, and flow is in the horizontal direction such that no gravity term is present, then the fractional flow equation reduces to,

\[
f_w = \frac{1}{1 + \frac{k_w \mu_o}{k_a \mu_w}} \tag{9.30}
\]

If we define mobility ratio as,

\[
M = \frac{k_w \mu_o}{k_a \mu_w} \tag{9.31}
\]

then \(f_w = \frac{1}{1+1/M}\).
If we return to Eq. (9.24) and substitute for \( q_w \), we obtain,

\[
\frac{\partial f_w}{\partial x} = -\frac{A \phi}{q_T} \frac{\partial S_w}{\partial t} \tag{9.32}
\]

To develop a solution, Eq. (9.32) must be reduced to one dependent variable, either \( S_w \) or \( f_w \). Observe, \( S_w = S_w(x,t) \) or,

\[
dS_w = \left( \frac{\partial S_w}{\partial x} \right)_t \ dx + \left( \frac{\partial S_w}{\partial t} \right)_x \ dt \tag{9.33}
\]

By tracing a fixed saturation plane through a core, that is, \( \frac{dS_w(x,t)}{dt} \bigg|_{S_w} = 0 \), then

\[
\left( \frac{dx}{dt} \right)_{S_w} = -\left( \frac{\partial S_w/}{\partial t} \right)_x \left( \frac{\partial S_w/}{\partial x} \right)_t \tag{9.34}
\]

where the right-hand side is the velocity of the saturation front as it moves through the porous media.

Observe \( f_w = f_w(S_w) \) only, then,

\[
\left( \frac{\partial f_w}{\partial x} \right)_t = \left( \frac{\partial f_w}{\partial S_w} \right)_t \left( \frac{\partial S_w}{\partial x} \right)_t \tag{9.35}
\]

Substitution of Eqs. (9.34) and (9.35) into Eq. (9.32), results in the frontal advance equation.

\[
\frac{dx}{dt} \bigg|_{S_w} = q_T \left( \frac{\partial f_w}{\partial S_w} \right)_t \tag{9.36}
\]

Equation (9.36) represents the velocity of the saturation front. Basic assumptions in the derivation are incompressible fluid, \( f_w(S_w) \) only and immiscible fluids. Furthermore, only oil is displaced; i.e., the initial water saturation is immobile, and no initial free gas saturation exists; i.e., not a depleted reservoir.

The location of the front can be determined by integrating the frontal advance equation,

\[
\int_{x_0}^{x_f} dx \bigg|_{S_w} = \frac{1}{\phi A} \int_0^t \left( \frac{\partial f_w}{\partial S_w} \right)_t \cdot q_t \ dt \tag{9.37}
\]
If injection rate is constant and if \( df_w/dS_w = f(S_w) \) only, then
\[
x|_{S_w} = \frac{q_i \cdot t}{\phi A} \left( \frac{\partial f_w}{\partial S_w} \right)_{S_w}
\]  
(9.38)

We can evaluate the derivative from the fractional flow equation (Eq. 9.28), either graphically or analytical. Figure 9.8 illustrates the graphical solution.

**Figure 9.8 Fractional flow curve**

The fractional flow of water at the front, \( f_{wf} \), is determined from the tangent line originating at \( S_{wc} \). The corresponding water saturation at the front is \( S_{wf} \). The average water saturation behind the front at breakthrough, \( S_{wbt} \), is given by the intersection at \( f_w = 1 \). The location of the front is determined by Eq. (9.38), with the slope of the tangent to the fractional curve used for the derivative function.

### 9.4.2 Displacement Performance (constant injection rate)

The displacement performance can be separated into two intervals, before and after breakthrough. Until breakthrough, the volume of oil produced is equal to the volume of water injected. After breakthrough, water saturation gradients exists, thus the volume of water in the system can be defined as;

\[
V_w = \int_{x_1}^{x_2} S_w A \phi dx
\]  
(9.39)

and the volume of oil displaced,

\[
V_o = V_w - A \phi (x_2 - x_1) S_{wf}
\]  
(9.40)
Figure 9.9 illustrates the recovery of oil both before and after water breakthrough. Note the linear slope up to breakthrough and then the decrease in slope (reduced performance) after breakthrough.

Figure 9.9 Typical oil recovery performance plot for immiscible displacement

A solution for waterflood performance was developed by Welge in 1952. Define the volumetric average water saturation as,

$$
\overline{S}_w = \frac{\int_{x_1}^{x_2} S_w A \phi dx}{\int_{x_1}^{x_2} A \phi dx}
$$

(9.41)

For constant cross-sectional area (A) and porosity (\(\phi\)), Equation (9.41) reduces to,

$$
\overline{S}_w = \frac{\int_{x_1}^{x_2} S_w dx}{x_2 - x_1}
$$

(9.42)

The integrand can be expanded and the equation rearranged such that,

$$
\overline{S}_w = \frac{x_2 S_{w2} - x_1 S_{w1}}{x_2 - x_1} - \frac{1}{x_2 - x_1} \int_1^2 x \cdot dS_w
$$

(9.43)

Substitute the frontal advance equation (Eq. 9.35) for the integral and solve,

$$
\int_1^2 x \cdot dS_w = \frac{q_f \cdot t}{\phi A} \left( \int_1^2 \frac{\partial f_w}{\partial S_w} \bigg|_{S_w} \right) \cdot dS_w
$$

$$
= \frac{q_f \cdot t}{\phi A} (f_{w2} - f_{w1})
$$

(9.44)

Thus the general Welge equation is,
\[ \bar{S}_w = \frac{x_2 S_{w2} - x_1 S_{w1}}{x_2 - x_1} - \frac{q_f \cdot t}{A \phi} \left( f_{w2} - f_{wl} \right) \]  \hspace{1cm} (9.45)

A useful simplification is to consider \( x_1 = 0 \) at the inlet and \( x_2 = L \) at the outlet end of the core,

\[ \bar{S}_w = S_{w2} + \frac{q_f \cdot t}{A \phi L} (1 - f_{w2}) \]  \hspace{1cm} (9.46)

where \( f_{w1} \) is assumed to be one at the inlet.

Define the total volume injected, \( W_i = q_f t \), and the pore volume, \( V_p = A \phi L \). Combining gives the number of pore volumes injected, \( Q_i \),

\[ Q_i = \frac{W_i}{V_p} \]  \hspace{1cm} (9.47)

Thus we can write Eq. (9.46) in terms of \( Q_i \).

\[ \bar{S}_w = S_{w2} + Q_i (1 - f_{w2}) \]  \hspace{1cm} (9.48)

The cumulative oil displaced, \( N_p \), can be expressed in terms of the difference in the average water saturation and the exit end saturation, i.e.,

\[ N_p = V_p \left( \bar{S}_w - S_{w2} \right) \]  \hspace{1cm} (9.49)

Consider a special case immediately before breakthrough. In this case, \( S_{w2} = S_{wi} \) and \( f_{w2} = 0 \). Subsequently, Eq. (9.48) can be written as:

\[ \bar{S}_{wbt} - S_{wi} = Q_{ibt} \]  \hspace{1cm} (9.50)

and the cumulative oil displaced:

\[ N_p = V_p \left( \bar{S}_{wbt} - S_{wi} \right) \]  \hspace{1cm} (9.51)