RELATIONS BETWEEN UNIT VECTORS

Cylindrical ↔ Cartesian:
\[ \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \]
\[ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \]
\[ \hat{z} = \hat{z} \]

Spherical ↔ Cartesian:
\[ \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \]
\[ \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \]
\[ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \]

Spherical ↔ Cylindrical:
\[ \hat{r} = \sin \theta \hat{s} + \cos \theta \hat{z} \]
\[ \hat{\theta} = \cos \theta \hat{s} - \sin \theta \hat{z} \]
\[ \hat{\phi} = \hat{\phi} \]

INTEGRATION IN VARIOUS COORDINATE SYSTEMS

Line Integration Element:
\[ \text{dl} = \begin{cases} 
\hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz & \text{Cartesian} \\
\hat{s} \, ds + \hat{\phi} \, s \, d\phi + \hat{z} \, dz & \text{Cylindrical} \\
\hat{r} \, dr + \hat{\theta} \, r \, d\theta + \hat{\phi} \, r \sin \theta \, d\phi & \text{Spherical} 
\end{cases} \]

1-Dimensional (Line) Integrals

Rules: Use \( \text{dl} \) as is, for integrals of the form \( \int \mathbf{v} \cdot \text{dl} \), where \( \mathbf{v} \) is some vector field. Choose one nonconstant variable to be the integration variable. Write down the (two) restrictions which define the line forming the integration path, and use these to substitute out all other variables besides the integration variable. Take the implicit differential (d) of both restriction equations, and use these to substitute out any differentials of variables besides the integration variable, if necessary.

Example: to integrate along the line specified by the two restriction equations \( y = 3x + 2 \) and \( z = 5 \), one would choose either \( x \) or \( y \) as the integration variable (since \( z \) is constant, it isn’t suitable to be integrated over.) Assume in the following that we decide to integrate over \( x \). Then whenever \( y \) appears in the integrand, we would replace it by \( 3x + 2 \). Whenever \( z \) appears, we replace it by \( 5 \). Taking the differentials of the restrictions tells us that if \( dy \) appears in the integrand, we are to replace it by \( 3dx \), and if \( dz \) appears, we replace it by \( 0 \).
2-Dimensional (Area) Integrals

Rules: Find the normal \( \hat{n} \) to the integration surface (\( \hat{n} \) needs to be the outward normal, if the surface is closed, in Gauss’ Law and the divergence theorem). For flux integrals (those of the form \( \iint \mathbf{v} \cdot \hat{n} \, d\mathbf{a} \), where \( \mathbf{v} \) is some vector field), construct \( d\mathbf{a} = d^2(r) \) by multiplying together the two components of \( d\mathbf{l} \) which are perpendicular to \( \hat{n} \). Substitute out the nonintegration variable using the restriction equation which defines the surface. For non-flux integrals, the rules to construct \( d\mathbf{a} \) are exactly the same, only \( \hat{n} \) is only used to find the perpendicular components of \( d\mathbf{l} \) (and not used in any dot product).

Example: The familiar expression for area of a sphere comes from choosing the surface of constant radius (\( r = R \)) in spherical coordinates. The outward unit normal vector is \( \hat{n} = \hat{r} \), so the differential area element is \( d\mathbf{a} = r^2 \sin \theta \, d\theta \, d\phi \). Since \( r = R \) is constant, we take it outside the integral and find that the area of the sphere is \( \iint d\mathbf{a} = R^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi = 4\pi R^2 \). It is often convenient to use the substitution \( \int_0^\pi \sin \theta \, d\theta = \int_{-1}^1 du \), where \( u = \cos \theta \).

3-Dimensional (Volume) Integrals

Rule: Use all 3 components of \( d\mathbf{l} \) multiplied together to construct \( d\tau \).

\[
d\tau = d^3(r) = \begin{cases} 
  dx \, dy \, dz & \text{Cartesian} \\
  s \, ds \, d\phi \, dz & \text{Cylindrical} \\
  r^2 \sin \theta \, dr \, d\theta \, d\phi & \text{Spherical} 
\end{cases}
\]

Notes:

- The overall length, area, or volume of an integrated region always comes from the limits of the integrals, and never from any alteration of \( d\mathbf{l} \), \( da \), or \( d\tau \).
- If the limits of a two- or three-dimensional region are not formed by surfaces where one coordinate is constant, then the ‘inner’ integral(s) (those integrated first) will have limits which depend upon the variables in the ‘outer’ integral(s) (those integrated later).

Special Note on VECTOR integrands:

- If integrating a vector field to get a vector answer out (for example: total (vector) force on a surface \( \mathbf{F} = \iint (\mathbf{f}/\text{area}) \, d\mathbf{a} \), with no dot product on \( \mathbf{f} \) to make it a scalar), express the vectors only in Cartesian components (even when using other coordinate systems to perform the integration)! (Otherwise, the vector components being ‘integrated’ together really belong pointing in different directions; Cartesian unit vectors are special because they always point the same direction). This need not be a concern in flux integrals, however; taking the ‘dot product’ of a vector integrand with \( \hat{n} \) turns it into a scalar integrand, and scalars have no direction and thus may be integrated with reckless abandon.