Math 382

Practice Exam 1.

1. Consider a well-shuffled deck of cards consisting of 52 distinct cards, 4 of which are aces. Draw one card from the deck and put it aside. Then draw another card. Denote $A_1 =$ "First card is an ace"  $A_2 =$ "Second card is an ace"

i. Find $P(A_1) = \frac{4}{52}$

ii. Find $P(A_2 \mid A_1) = \frac{3}{51}$

iii. Find $P(A_1 \mid A_2)$: $\frac{3/51}{1/2} = \frac{3}{51}$. You could also use Bayes' formula:

$$p(A_1 \mid A_2) = \frac{P(A_1, A_2)}{P(A_2)} = \frac{P(A_2 \mid A_1) \times P(A_1)}{P(A_2 \mid A_1) \times P(A_1) + P(A_2 \mid A_1) \times P(A_1)}$$

iv. Are events $A_1$, $A_2$ independent? Explain.

no since e.g. $P(A_2 \mid A_1) \neq P(A_2) = \frac{3}{51} \neq \frac{4}{52}$

v. What is the probability that you draw your first ace on your third try?

$= P(\overline{A_1} \overline{A_2} A_3) = \frac{48}{52} \times \frac{47}{51} \times \frac{4}{50}$

(not geometric (no independence))

2. A computer manufacturer uses chips from three sources, equal amount from each source. Chips from sources A, B and C are defective with probabilities 0.001, 0.005 and 0.01, respectively.

a) What proportion of all chips used by manufacturer will be defective?

$P(D) = P(D \mid A) \times P(A) + P(D \mid B) \times P(B) + P(D \mid C) \times P(C) = 0.0053$ (Bottom part of Bayes' formula)

b) If a chip is found to be defective, find the probability that the source was C.

$P(C \mid D) = \frac{P(D \mid C) \times P(C)}{P(D)} = \frac{0.01 \times \frac{1}{3}}{0.0053} = 0.629$
3. Multiple choice: circle the appropriate answer.

a) If the A and B are mutually exclusive then A, B are independent. Is this true?
Always  Never  Sometimes, only if \( P(A) = P(B) = 0 \)

b) If the sample space consists of \( n \) outcomes, then each outcome has the probability \( 1/n \).
Always  Never  Sometimes

c) The following is a graph for CDF of \( Y \). Then \( P(Y = 3) \) equals:

\[
\begin{array}{cccc}
F(5) & 0.5 & 0 & 0.25 & 0.55 \\
\end{array}
\]

\[
P(Y = 3) = F(3) - F(2) = 0.55 - 0.3 = 0.25
\]

d) A box contains 6 red balls and 13 white balls. Four balls are drawn at random. The probability that we will get exactly 2 red balls and 2 white ones is

\[
\begin{array}{cccc}
0.000258 & 0.0373 & 0.3021 & 4/19
\end{array}
\]

\[
\binom{6}{2} \binom{13}{2} = \frac{6 \times 5 \times 13 \times 12}{2 \times 1 \times 2 \times 1} = \frac{78 \times 12}{2 \times 2} = 495
\]

\[
\binom{19}{4} = \frac{19 \times 18 \times 17 \times 16}{4 \times 3 \times 2 
\]

\[
is 44 \text{ to } 60 \text{ hours}
\]

e) \( P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2} = 0.75 \) (Chebyshev's), \( k = 2 \) interval is \( 52 \pm 2 \times 4 \)

4. The average number of accidents at a busy intersection is 41.6 per year.

a) Find the expected number of accidents in a week.

\[
\frac{41.6}{52} = 0.8
\]

b) Find the probability that there will be no accidents in a week

\[
P(Y = 0) = \frac{0.8^0}{0!} e^{-0.8} = 0.449
\]

c) Find the probability that there will be 2 or more accidents in a week

\[
P(Y \geq 2) = 1 - \left[ P(0) + P(1) \right] = 1 - F(1) = 1 - 0.809 = 0.191
\]
5. An insurance company is selling a “one-sum” policy on your car. You pay a premium of $500 in the beginning of the year. In the event your car gets seriously damaged, you will obtain a payment of $9000. The probability that a car will get seriously damaged is estimated as 4%.

a) Find the distribution of the random variable \( X = \) amount of money you receive from the company in a year

<table>
<thead>
<tr>
<th>( X )</th>
<th>( p(x) )</th>
<th>( E(x) = 360 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.96</td>
<td></td>
</tr>
<tr>
<td>9000</td>
<td>.04</td>
<td></td>
</tr>
</tbody>
</table>

\[
-\bar{E}(X) + 500 = -9000 \times (.04) + 500 = $140
\]

b) Find the expected profit of the insurance company from the policy.

c) If the insurance company sells 500 such policies, what is its expected profit?

\[
\text{Profit} = 140 \times 500 = $70,000
\]

d) Find the variance of the profit.

\[
\text{Var}(\text{Profit}) = (\mu^2 - \sigma^2) \text{Var}(X) = 3.1 \times 10^6
\]

6. Shaquille O’Neal makes 50% of his free throws. He attempts 10 in a game.

a) Find the probability he will make 8 or more.

\[
\text{Binomial} \quad n=10, \quad p=0.5 \quad \text{P}(X \geq 8) = 1 - \text{P}(7) = 1 - .945 = .055
\]

b) Find the standard deviation of the number of free throws he makes.

\[
\sigma = \sqrt{np(1-p)} = 1.58
\]

c) What assumption should one make to be able to evaluate a), b) ?

independent attempts

d) Find the probability that he needs 5 free throws to make his 2nd.

\[
\text{Neg. Bin.} \quad \text{P}(Y = 5) = \binom{4}{1} 0.5^2 (1-0.5)^3 = 0.125
\]
7. a) Assuming that components act independently, find the probability that the following system works: (each component has reliability of 0.7)

\[ (1 - 0.3^2)^2 = 0.8281 \]

b) Suppose the system has \( n \) components, all connected in parallel. How large should \( n \) be to make the reliability of the entire system higher than 99.9%? (each component has reliability of 0.7)

Show your reasoning.

\[ P(\text{At least one}) = 1 - P(\text{none work}) = 1 - 0.3^n \geq 0.999 \]

\[ 0.3^n \leq 0.001 \quad \therefore n \ln(0.3) \geq \ln(0.001) \]

\[ n \geq 5.7 \quad \text{or} \quad 6 \]

Extra credit.

Refer to problem 2. Suppose that the entire deck is distributed equally among four players. Find the probability that each player gets an ace.

\[
\begin{align*}
\text{ace} & \downarrow \quad \text{other cards} \\
4 \times \left( \frac{48}{12} \right) \times 3 \times \left( \frac{36}{12} \right) \times 2 \times \left( \frac{24}{12} \right) \times 1 \left( \frac{12}{12} \right) \\
\left( \frac{52}{13} \right) \left( \frac{39}{13} \right) \left( \frac{26}{13} \right) \left( \frac{13}{13} \right) & \approx \text{all possible choices} \\
\approx 0.1055
\end{align*}
\]