Zeta Function
and Riemann Hypothesis

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History of Prime Numbers

1. Euclid (300 BC)
2. Fermat (1640)
3. Euler (1737)
4. Dirichlet (1837)
5. Gauss (1849)
6. Chebyshev (1848)
7. Riemann (1859)
8. Dr X (2020)?
Millenium Prize Problem

- B. Riemann, “On the number of primes less that a given magnitude” (1859), 8 pages
- One of twenty-three most important mathematical problems composed by David Hilbert in 1900.
- One of the seven Millennium Prize Problems of the Clay Mathematics Institute (1 Million Dollar prize!) composed in 2000.
Evidence for the Riemann Hypothesis

- Strong numerical evidence. Riemann hypothesis holds for over $10^8$ zeros at heights up to $10^{20}$.

- The exceptions to the Riemann hypothesis, if they exist, must be rare.

- More than 40% of the nontrivial zeros are simple and satisfy the Riemann hypothesis.

- Profoundly deep relation to such diverse areas of mathematics as: analytic number theory, fractal geometry, noncommutative geometry, random matrix theory, chaotic dynamical systems (classical and quantum), quantum theory, solid state physics, quasi-crystals, etc
Prime Numbers

- A positive integer $d$ divides a positive integer $n$, denoted by $d|n$, if there is an integer $k$ such that $n = kd$.

- A positive integer $d$ is the greatest common divisor of positive integers $a$ and $b$, denoted by $d = (a, b)$ if it is a common divisor of $a$ and $b$ and is divisible by any other common divisor.

- Two positive integers $a$ and $b$ are relatively prime if $(a, b) = 1$.

- A positive integer $p > 1$ is prime if it is divisible only by 1 and $p$. 
Notation

- Summation over all positive divisors of $n$ is denoted by
  \[ \sum_{d|n} f(d) \]

- Summation over all positive integers relatively prime to $n$ is denoted by
  \[ \sum_{(k,n)} f(k) \]

- Summation over all primes is denoted by
  \[ \sum_{p} f(p) \]

- Summation over all powers of primes is denoted by
  \[ \sum_{p^k} f(n) \]
Fundamental Theorem of Arithmetic

- The primes form an increasing sequence \((p_j)_{j=1}^{\infty}\)

\[ p_1 < p_2 < p_3 < \cdots \]

- The study of this sequence as \(j \to \infty\) is central problem of number theory.

- **Unique Factorization Theorem.** Every positive integer \(n > 1\) can be identified with a sequence of non-negative integers \((n_j)_{j=1}^{\infty}\), \(n_j \geq 0\), containing only finitely many non-zero terms such that

\[
n = \prod_{j=1}^{\infty} p_j^{n_j}
\]
Examples

- If all \( n_j = 0 \), then \( n = 1 \)
- If only one \( n_j \) is nonzero and \( n_j = 1 \), then \( n = p_j \) is a prime
- If only one \( n_j \) is non-zero and \( n_j = k > 1 \), then \( n = p_j^k \) is a power of prime
- If all non-zero \( n_j \) are equal to 1, then \( n \) is the product of distinct primes
Arithmetic Functions

- **Arithmetic function** is a real valued function

  \[ f : \mathbb{Z}_+ \rightarrow \mathbb{R} \]

  defined on the set of positive integers that expresses some arithmetical property of integers.

- All arithmetic functions can be extended to functions of positive real numbers, \( f : \mathbb{R} \rightarrow \mathbb{R} \) as step functions by

  \[ f(x) = f([x]) \]

  if \( x \) is not an integer and equal to the average of the left and right values at the jumps.
Examples of Arithmetic Functions

- The unit function
  \[ e(1) = 1, \quad \text{and} \quad e(n) = 0 \quad \text{for} \quad n > 1 \]

- Function \( \omega(n) \) is defined as the number of distinct prime divisors of \( n \).

- Möbius function
  \[ \mu(1) = 1, \quad \text{and} \quad \mu(n) = (-1)^{\omega(n)} \]
  if all primes divisors of \( n > 1 \) are distinct and zero otherwise.

- The Von Mangoldt function
  \[ \Lambda(n) = \log p \]
  if \( n = p^k \) for some prime \( p \) with some \( k \geq 1 \) and zero otherwise.
Dirichlet Convolution

- The **Dirichlet convolution** $h = f \ast g$ of two arithmetic functions, $f, g$, is defined by

$$h(n) = \sum_{d | n} f(d)g\left(\frac{n}{d}\right)$$

- The **convolution inverse** of an arithmetic function $f$ is an arithmetic function $g$ such that its convolution with $f$ is equal to the unit function,

$$f \ast g = e,$$

that is,

$$\sum_{d | n} f(d)g\left(\frac{n}{d}\right) = e(n)$$

In particular, $g(1) = 1/f(1)$. So, if $f(1) = 0$, then it does not exist.
Möbius Inversion

**Fundamental Property of the Möbius Function**

The Möbius function is the convolution inverse of the constant function 1, that is, \( \mu \* 1 = e \) or

\[
\sum_{d\mid n} \mu(d) = e(n)
\]

- It allows us to invert relations between arithmetic functions

\[
F(n) = \sum_{d\mid n} f(d) \quad \text{iff} \quad f(n) = \sum_{d\mid n} \mu(d) F\left(\frac{n}{d}\right)
\]

- One can also show that

\[
g(x) = \sum_{n=1}^{\infty} f(nx) \quad \text{iff} \quad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx)
\]
Summatory Functions

- Given an arithmetic function $f$ one defines a function of a real variable $x \geq 0$,

$$M_f(x) = \sum_{n \leq x} f(n),$$

- One is interested in the asymptotic behavior as $x \to \infty$
Examples

- The **Mertens function**
  \[ M(x) = \sum_{n \leq x} \mu(n) \]

- The **prime counting function**
  \[ \pi(x) = \sum_{p \leq x} 1. \]

- The **weighted prime counting function**
  \[ \Pi(x) = \sum_{p^k \leq x} \frac{1}{k} = \sum_{n \leq x} \frac{\Lambda(n)}{\log n}. \]

- The **Chebyshev function**
  \[ \psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n). \]
Dirichlet Series

- The asymptotics of a summatory function $M_f$ of an arithmetic function $f$ as $x \to \infty$ are described by the *analytical properties* of the function of a complex variable $s = \sigma + it$,

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \sigma > c$$

called the **Dirichlet Series** of $f$.

- The Dirichlet series $D_f(s)$ and the summatory function $M_f(x)$ are related by the Mellin transform

$$D_f(s) = s \int_0^\infty dx \ x^{-s-1} M_f(x),$$

$$M_f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \ x^s D_f(s)$$
Dirichlet Series of a Convolution

- The Dirichlet series of a Dirichlet convolution \( h = f \ast g \) is equal to the product of Dirichlet series,

\[
D_h(s) = D_f(s)D_g(s)
\]

- The Dirichlet series of a convolution inverse \( g \) of a function \( f \) is equal to the reciprocal of the Dirichlet series of \( f \), that is,

\[
\frac{1}{D_f(s)} = D_g(s) \quad \text{iff} \quad f \ast g = e
\]
An arithmetic function \( f \) is called **multiplicative** if \( f(1) = 1 \) and for any relatively prime integers \( a \) and \( b \)

\[
f(ab) = f(a)f(b).
\]

The Dirichlet series of a multiplicative function \( f \) can be represented as the product over primes

\[
D_f(s) = \prod_p \left( 1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right)
\]
Zeta Function

Riemann zeta function

\[ \zeta(s) = D_1(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \quad \sigma > 1 \]

- By the convolution inverse one can immediately obtain

\[ \frac{1}{\zeta(s)} = D_{\mu}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1 \]

- The **Euler product** for the zeta function has the form

\[ \zeta(s) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^{ks}}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1 \]
Zeta Function

- This means that

\[ \log \zeta(s) = \sum_{k=1}^{\infty} \sum_{p} \frac{1}{k} \frac{1}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \frac{1}{n^s}, \quad \sigma > 1 \]

- And, therefore,

\[ -\frac{\zeta'(s)}{\zeta(s)} = D_\Lambda(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \sigma > 1 \]
Integral Representations of Zeta Function

- Zeta function is the Mellin transform of the integer part function $[x]$

$$\zeta(s) = s \int_1^\infty dx \ x^{-s-1}[x], \quad \sigma > 1$$

- It can be written in the form

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty dx \ x^{-s-1}\{x\}, \quad \sigma > 0$$

where $\{x\} = x - [x]$ is the fractional part of $x$, which reveals a simple pole at $s = 1$.

- Zeta function is directly related to the prime counting functions

$$\log \zeta(s) = s \int_1^\infty dx \ x^{-s-1}\Pi(x) = s \int_1^\infty dx \ \frac{\pi(x)}{x(x^s - 1)}, \quad \sigma > 1$$
Integral Representations of Zeta Function

- More importantly

**Mellin Transform of Mertens Function**

\[
\frac{1}{\zeta(s)} = s \int_{1}^{\infty} dx \ x^{-s-1} M(x), \quad \sigma > 1
\]

- It is easy to show that

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dx \ x^{s-1} \frac{1}{e^x - 1}, \quad \sigma > 1
\]

**Analytic Continuation of Zeta Function**

\[
\zeta(s) = \frac{1}{2\pi i} \Gamma(1 - s) \int_{C} dz \ \frac{(-z)^s}{z \ e^z - 1},
\]

where \( C \) is the contour that goes from \( +\infty + i\varepsilon \) to \( +\infty - i\varepsilon \) around the origin in the counterclockwise direction.
Heat Kernel Representation of Zeta Function

- The heat trace on the circle $S^1$ is given by

\[ \theta(t) = \text{Tr} \exp(-tD^2) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi t} = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi t} \]

Fundamental Duality (Poisson Summation Formula)

\[ \theta(t) = \frac{1}{\sqrt{t}} \theta \left( \frac{1}{t} \right) \]

- It is easy to show

\[ \zeta(s) = \frac{1}{2} \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} \frac{dt}{t} t^{s/2} [\theta(t) - 1], \quad \sigma > 1 \]
Functional Equation for Zeta Function

- By using the duality one can get

\[ \zeta(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \frac{1}{2} \int_{1}^{\infty} \frac{dt}{t} \left( t^{s/2} + t^{(1-s)/2} \right) [\theta(t) - 1] \right\} \]

which provides an analytic continuation for any \( s \) and reveals a pole at \( s = 1 \).

- This immediately leads to

**Functional Equation**

\[ \xi(s) = \xi(1 - s) \]

where

\[ \xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \]
Zeta function is a meromorphic function with a simple pole at $s = 1$.

It has some trivial zeros at the negative even integer points, that is,

$$\zeta(-2n) = 0, \quad n = 1, 2, \ldots$$

All non-trivial roots coincide with the roots of the entire function $\xi$. 
Non-Trivial Roots of the Zeta Function

- There are no roots for $\sigma > 1$.
- A complex number $1 - \rho$ is a root if and only if $\rho$ is a root.
- The roots are located symmetrically with respect to the critical line
  \[ \sigma = \frac{1}{2}. \]
- There are no non-trivial roots for $\sigma < 0$.
- All roots are located in the critical strip
  \[ 0 \leq \sigma \leq 1. \]
- If $\rho$ is a root then $\bar{\rho}$ is also a root.
- The roots are located symmetrically with respect to the real axis.
Non-Trivial Roots of the Zeta Function

- There are infinitely many roots on the critical line.
- The number of zeros (with \( s = \sigma + it \)) in the rectangle

\[
0 \leq \sigma \leq 1, \quad 0 < t < T
\]

as \( T \to \infty \) is

\[
N(T) = \frac{1}{2\pi} T \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(\log T)
\]

- There holds

\[
\sum \frac{1}{|\rho|^2} < \infty, \quad \sum \frac{1}{|\rho|} \to \infty
\]

- There holds

\[
\psi(x) = x - \sum \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right)
\]
Prime Number Theorem

Equivalent formulations of **Prime Number Theorem**:  
As \( x \to \infty \)

- \( \pi(x) \sim \text{Li}(x) \), or \( \pi(x) \sim \frac{x}{\log x} \)

where \( \text{Li}(x) = \int_{0}^{x} \frac{dt}{\log t} \) is the integral logarithm

- \( \psi(x) \sim x \), or \( \lim_{x \to \infty} \sum_{\rho} \frac{1}{\rho} x^{\rho-1} = 0 \)

- There are no roots of zeta function on the line \( \sigma = 1 \).
Riemann Hypothesis

The equivalent formulations of the Riemann hypothesis

- There are no non-trivial roots for $\sigma \neq 1/2$.
- For any $\varepsilon > 0$
  \[ \pi(x) = \text{Li}(x) + O(x^{1/2} \log x) \]
- For any $\varepsilon > 0$
  \[ \psi(x) = x + O(x^{1/2+\varepsilon}) \]
Riemann Hypothesis

- For any $\varepsilon > 0$
  \[ \sum_{\rho} \frac{1}{\rho} x^\rho = O \left( x^{1/2+\varepsilon} \right) \]

- The series
  \[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \]
  converges for $\sigma > 1/2$.

- For any $\varepsilon > 0$ the Mertens function $M(x)$ grows less rapidly than $x^{1/2+\varepsilon}$, that is, for any $\varepsilon > 0$
  \[ M(x) = O \left( x^{1/2+\varepsilon} \right) \]

- For large $n$ that is equal to the product of distinct primes the probability of even and odd number of distinct primes are equal.
Hilbert-Polya Conjecture

There is a unbounded self-adjoint operator $H$ such that the spectrum of the operator

$$\frac{1}{2} + iH$$

coinsides with the zeros of the Riemann zeta function.

Hints:

- the dynamics is chaotic, that is, unstable and bounded,
- the dynamics is time-irreversible,
- periodic orbits have periods independent of energy,
- dynamics is quasi-one-dimensional.
Randomness in Prime Numbers

- Suppose an unbiased coin is flipped a large number of times.
- Then the probability of getting exactly \( H \) heads in \( N \) trials is given by the binomial distribution

\[
B(N, H) = \binom{N}{H} \frac{1}{2^N}
\]

**de Moivre-Laplace Theorem**

As \( N \to \infty \) the binomial distribution approaches the normal distribution with mean \( N/2 \) and standard deviation \( \sqrt{N}/2 \),

\[
B(N, H) \sim F(N, H) = \sqrt{\frac{2}{\pi N}} \exp \left[ -\frac{2}{N} \left( H - \frac{N}{2} \right)^2 \right]
\]
Randomness in Prime Numbers

- The probability that the number of heads deviates from the expected value \( \frac{N}{2} \) by \( KN^{1/2+\varepsilon} \) as \( N \to \infty \) is equal

\[
P \left( \left| H - \frac{N}{2} \right| < KN^{1/2+\varepsilon} \right) \sim \int_{N/2-KN^{1/2+\varepsilon}}^{N/2+KN^{1/2+\varepsilon}} dH \ F(N, H)
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{2} N^\varepsilon} dx \ e^{-x^2} \to 1
\]

- The probability that the number of heads deviates from the number of tails by \( 2KN^{1/2+\varepsilon} \) as \( N \to \infty \)

\[
P \left( |H - T| < 2KN^{1/2+\varepsilon} \right) \to 1
\]
Randomness in Prime Numbers

- With probability 1 the number of heads minus the number of tails grows less rapidly than $N^{1/2 + \varepsilon}$.
- Let $n$ be a large integer such that it is product of distinct primes, so that, $\mu(n) \neq 0$.
- If the number of factors is odd then $\mu(n) = -1$ and if the number of factors is even then $\mu(n) = 1$.
- Then the evaluation of the Mertens function $M(x)$ is like flipping a coin for each integer $n \leq x$ which is a product of distinct primes and subtracting the number of tails from the number of heads.
- If the values of $\mu(n)$ are independent, that is, with equal probability a large square-free integer has either an odd number of factors or an even number of factors, then for any given $\varepsilon > 0$ as $x \to \infty$ with probability 1, $M(x) < O\left(x^{1/2 + \varepsilon}\right)$.
- This is equivalent to Riemann hypothesis.
There are conjectures that relate the statistical behavior of the nontrivial zeros of the Riemann zeta function to the distribution of the eigenvalues of large $N \times N$ random Hermitian matrices $M$ with the probability measure

$$P(M) = C_N \exp \left\{ -\frac{N}{2} \text{tr} M^2 \right\}$$

Such a system is called the **Gaussian Unitary Ensemble**.

The zeros of the zeta function

$$\rho_k = \frac{1}{2} + it_k$$

are distributed like the eigenvalues of the eigenvalues of random Hermitian matrices.
Free Riemann Gas

- The free Riemann gas is a quantum theory of identical non-interacting particles.
- The particles can be in infinitely many different states labeled by primes $p$.
- The particles can be either **bosonic** or **fermionic**.
- Bosonic particles can be in the same state.
- Fermionic particles cannot be in the same state (**Pauli Exclusion Principle**).
- Every state of a system can be described by a positive integer $n = \prod_{p} p^{k_{p}}$.

In such a state there $k_{1}$ particles in state $p_{1}$, $k_{2}$ particles in state $p_{2}$, etc.
Free Riemann Gas

• States of bosonic particles are described by any integer.

• States of fermionic particles are described only by integers which are product of distinct primes, that is, $k_p$ are equal to 0 or 1.

• The states of fermionic particles can be even or odd depending on the number of particles, that is, the parity of the fermionic state is equal to the Möbius function $\mu(n)$. 

Free Riemann Gas

- The energy of a particle in the state \( p \) is \( \log p \).
- The energy of the state \( |n\rangle \) is

\[
E(n) = \sum_p k_p \log p = \log \prod_p p^{k_p} = \log n
\]

- The partition function of the **bosonic** system is

\[
Z(\beta) = \text{Tr} \exp(-\beta H) = \sum_{n=1}^{\infty} \exp[-\beta E(n)] = \zeta(\beta),
\]

where \( \beta = 1/T \) is the inverse temperature.

- The partition function of the **fermionic** system is

\[
Z(\beta) = \text{Str} \exp(-\beta H) = \sum_{n=1}^{\infty} \mu(n) \exp[-\beta E(n)] = \frac{1}{\zeta(\beta)}
\]