Noncommutative Deformation of General Relativity

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- Main Idea (Matrix Deformation)
- Origin of Riemannian Geometry
- Hyperbolic Systems and Causal Structure
- Finsler Geometry and Matrix Geometry
- Invariant Action Functionals
Main Idea: Matrix Deformation

Einstein General Relativity $\Rightarrow$ Matrix General Relativity (MGR)

Gravitational Chromodynamics $GCD$

Riemannian Geometry $\Rightarrow$ Matrix Geometry

Riemannian metric $\Rightarrow$ matrix-valued tensor field (collection of Finsler metrics)

New features:

- dynamical causal structure of spacetime (collection of light cones)
- new gravicolor degrees of freedom
- new gauge symmetry and corresponding charges
Origing of Riemannian Geometry

light propagation

⇓

wave equation

⇓

Hamilton-Jacobi equation

⇓

characteristics of the wave equation

⇓

causal structure of spacetime

⇓

(pseudo)-Riemannian metric of spacetime

⇓

basic notions of general relativity
Origin of Matrix Geometry

Propagation of collection of fields
(with internal structure)

\[ \Downarrow \]

Hyperbolic system of linear second-order
partial differential equations

\[ \Downarrow \]

Matrix Geometry
Linear Hyperbolic System

Linear self-adjoint hyperbolic PDO with matrix-valued coefficients

\[ L\varphi = [a^{\mu\nu}(x)\partial_\mu \partial_\nu + b^\mu(x)\partial_\mu + c(x)] \varphi = 0 \]

Covariance under diffeomorphisms and gauge transformations

\[ \varphi(x) \longrightarrow U(x)\varphi(x) \]

Matrix-valued contravariant symmetric two-tensor

\[ a^{\mu\nu} = a^{\nu\mu}, \quad (a^{\mu\nu})^\dagger = a^{\mu\nu} \]

Principal symbol (matrix Hamiltonian)

\[ H(x, \xi) = a^{\mu\nu}(x)\xi_\mu \xi_\nu \]

Strict Hyperbolicity: for any \( x \) and \( \xi = (\lambda, \zeta_k) \) with \( \zeta \neq 0 \) the roots \( \lambda_j(x, \zeta) \) of the characteristic equation

\[ \det H(x, \xi) = 0 \]

are real
Causal Structure

Eigenvalues $h_i(x, \xi)$ of the matrix Hamiltonian $H(x, \xi)$ are real and distinct (or with constant multiplicities)

Hamilton-Jacobi equations $h_i \left( x, \frac{\partial S}{\partial x} \right) = 0$

Characteristics and Hamiltonian systems

$$\frac{dx^\mu}{ds} = \frac{\partial}{\partial \xi_\mu} h_i(x, \xi), \quad \frac{d\xi_\mu}{ds} = -\frac{\partial}{\partial x^\mu} h_i(x, \xi)$$

Null geodesics $h_i(x, \nu) = 0$

Causal Cones $C_i(x)$

Past and Future $\mathcal{I}_i^-(x)$ and $\mathcal{I}_i^+(x)$

Interior of the cone $\mathcal{I}_i(x) = \mathcal{I}_i^-(x) \cup \mathcal{I}_i^+(x)$

Exterior of the cone $\mathcal{E}_i(x)$
Causal Set
\[ \mathcal{I}(x) = \bigcup_{i=1}^{s} \mathcal{I}_i(x) \]

Absolute Past and Future
\[ \mathcal{I}^\pm(x) = \bigcup_{i=1}^{s} \mathcal{I}_i^\pm(x) \]

Acausal (Causally Disconnected) Set
\[ \mathcal{E}(x) = \bigcap_{i=1}^{s} \mathcal{E}_i(x) \]

Causal Structure of spacetime
\[ M = \mathcal{I}^-(x) \cup \mathcal{I}^+(x) \cup \partial \mathcal{I}(x) \cup \mathcal{E}(x) \]

Remark: Causal structure is dynamic (varies from point to point)
**Finsler Geometry**

**Homogeneity**

\[ h_i(x, \lambda \xi) = \lambda^2 h_i(x, \xi) \]

**Finsler form**

\[ h_i(x, \xi) = g_i^{\mu\nu}(x, \xi) \xi_\mu \xi_\nu \]

**Finsler metrics**

\[ g_i^{\mu\nu}(x, \xi) = \frac{1}{2} \frac{\partial^2 h_i(x, \xi)}{\partial \xi_\mu \partial \xi_\nu} \]

**Homogeneity**

\[ g_i^{\mu\nu}(x, \lambda \xi) = g_i^{\mu\nu}(x, \xi) \]

**Hyperbolicity**

\[ \text{sign} g_i^{\mu\nu}(x, \xi) = (- + \cdots +) \]

**Remark:**

same hyperbolic direction

**Tangent vector**

\[ \dot{x}^\mu = g_i^{\mu\nu}(x, \xi) \xi_\nu \]

\[ \xi_\mu = g_{i\mu\nu}(x, \dot{x}) \dot{x}^\nu \]

**Covariant metric**

\[ g_{i\mu\nu}(x, \dot{x}) g_i^{\nu\alpha}(x, \xi) = \delta_\alpha^\nu \]

**Interval**

\[ ds_i^2 = g_{i\mu\nu}(x, \dot{x}) dx^\mu dx^\nu \]
Matrix Geometry

Matrix connection $A_{\alpha\beta} = (A_{\alpha\beta}^A B)$

Yang-Mills field $B_{\mu}$

Compatibility condition

\[ \partial_{\mu} a^{\alpha\beta} + [B_{\mu}, a^{\alpha\beta}] + A_{\lambda\mu} a^{\lambda\beta} + A_{\beta\mu} a^{\alpha\lambda} = 0 \]

Symmetry condition in the commutative limit

\[ A_{\alpha\lambda\mu} = \frac{1}{2} b_{\lambda\sigma} \left\{ \left[ a^{\alpha\gamma} \partial_\gamma a^{\rho\sigma} + a^{\alpha\gamma} [B_\gamma, a^{\rho\sigma}] ight] \
- a^{\rho\gamma} \partial_\gamma a^{\sigma\alpha} - a^{\rho\gamma} [B_\gamma, a^{\sigma\alpha}] \right. \]

\[ \left. - a^{\sigma\gamma} \partial_\gamma a^{\alpha\rho} - a^{\sigma\gamma} [B_\gamma, a^{\alpha\rho}] \right\} b_{\rho\mu} \]

where $b_{\mu\nu} = (b_{\mu\nu}^A B)$ is defined by

\[ a^{\mu\nu} b_{\nu\lambda} = b_{\mu\nu} a^{\nu\lambda} = \delta_{\lambda}^{\mu} \]
Matrix Curvature

\[ R^\lambda_{\alpha\mu\nu} = \partial_\mu A^\lambda_{\alpha\nu} + [B_\mu, A^\lambda_{\alpha\nu}] - \partial_\nu A^\lambda_{\alpha\mu} - [B_\nu, A^\lambda_{\alpha\mu}] + A^\lambda_{\beta\mu} A^\beta_{\alpha\nu} - A^\lambda_{\beta\nu} A^\beta_{\alpha\mu} \]

Yang-Mills Curvature

\[ F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \]

Matrix Torsion

\[ T^\lambda_{\mu\nu} = A^\lambda_{\mu\nu} - A^\lambda_{\nu\mu} \]
Invariant Action Functionals

Matrix measure (not unique)

\[ \rho = (\psi^\dagger \psi)^{-1/4} \]

where

\[ \psi = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \varepsilon_{\nu_1 \ldots \nu_n} a_{\mu_1 \nu_1} \cdots a_{\mu_n \nu_n} \]

Action (not unique)

\[ S = \frac{1}{16\pi G N} \int dx \text{ tr } \rho (a^{\nu \mu} R^\alpha_{\mu \alpha \nu} - 2\Lambda) \]

\[ + (\text{torsion})^2 \]

\[ + \text{Yang \dash Mills} \]

\[ + \text{matter} \]
Invariant Functionals from Spectral Asymptotics

Invariant elliptic self-adjoint second-order PDO $F$ with positive definite non-scalar principal symbol $H(x, \xi)$

Heat trace asymptotic expansion as $t \to 0$

$$\text{Tr}_{L^2} \exp(-tF') \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} A_k$$

Spectral Invariants $A_k$

For operators with scalar leading symbol

$$A_0 = \int_M dx g^{1/2} N, \quad A_1 = \int_M dx g^{1/2} \text{tr} \left( \frac{1}{6} R - Q \right)$$

Invariant Action Functional of Matrix General Relativity

$$S = \frac{1}{16\pi G N} (6A_1 - 2\Lambda A_0)$$
Conclusions

Spontaneous breakdown of gauge symmetry

Broken phase: one tensor field (metric of the space-time)

Unbroken phase: no preferred metric in the usual sense

Confinement of gravicolor degrees of freedom

Only the invariants (graviwhite states) are visible at large distances