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Chapter 1

Review of Differential Geometry

1.1 Topological Spaces and Manifolds

- A topology $\mathcal{T}$ on a set $M$ is a collection of subsets of $M$, called open sets, satisfying the conditions
  1. $\mathcal{T}$ contains the empty set $\emptyset$ and the whole set $M$,
  2. the intersection of a finite collection of open sets is open,
  3. the union of any collection of open sets is open.

- A topological space $(M, \mathcal{T})$ is a set $M$ together with a topology $\mathcal{T}$.

- Any open set containing a point $x \in M$ is called a neighborhood of $x$.

- A sequence $(x_n)$ of points in $M$ is said to converge to a point $x$, and we write $x_n \to x$,
  if every neighborhood of $x$ contains all but a finitely many elements of the sequence, that is, for every neighborhood $U$ of $x$ there is a tail of the sequence contained in $U$.

- A map $f : M \to N$ from a metric space $M$ to a metric space $N$ is said to be continuous at a point $x \in M$ if for every neighborhood $U$ of the point $f(x) \in N$ there is a neighborhood $V$ of the point $x \in M$ such that $f(V) \subset U$.

- A map $f : M \to N$ from a metric space $M$ to a metric space $N$ is said to be continuous if it is continuous at every point.
• A bijective map $f : M \rightarrow N$ such that both $f$ and $f^{-1}$ are continuous is called a **homeomorphism**.

• A **metric** on a set $M$ is a function $d : M \times M \rightarrow \mathbb{R}$ satisfying the conditions: for any $x, y, z \in M$,

1. $d(x, y) \geq 0$,

2. $d(x, y) = 0$, if and only if $x = y$,

3. $d(x, y) = d(y, x)$,

4. $d(x, z) \leq d(x, y) + d(y, z)$.

• A **metric space** $(M, d)$ is a set $M$ together with a **metric** $d$.

• **Examples.**

  • An **open ball** of radius $\varepsilon$ centered at the point $x_0$ is the set
    $$B_\varepsilon(x_0) = \{ x \in M \mid d(x, x_0) < \varepsilon \}.$$  

  • A set $U \subset M$ is called **open** if for every point $x \in U$ there is an open ball centered at $x$ and contained in $U$.

  • Two metrics on a set $M$ are said to be **equivalent** if they define the same open sets, that is, a subset is open with respect to one metric if and only if it is open with respect to the other metric.

  • Every metric space is a topological space with the natural topology defined as follows: the open sets are the unions of open balls.

  • A set $A \subset M$ is called **closed** if its complement is open.

  • The limit of a convergent sequence in a closed set belongs to that set.

  • The **interior** $A^\circ$ of a set $A \subset M$ is the largest open set contained in $A$.

  • The **closure** $\bar{A}$ of a set $A \subset M$ is the smallest closed set that contains $A$. 
1.1. **TOPOLOGICAL SPACES AND MANIFOLDS**

- The boundary of a set $A \subset M$ is the set
  \[ \partial A = \bar{A} - A^o. \]

- Every subset $W$ of a topological space $M$ is a topological space with the **induced topology** defined as follows. A set $U \subset W$ is open in $W$ if and only if there is an open set $V \subset M$ such that $U = V \cap M$.

- A topological space $M$ is said to be **Hausdorff** if every two distinct points in $M$ have disjoint neighborhoods.

- A topological space is **discrete** if every set containing one point is open.

- An **open cover** of a topological space $M$ is a collection of open subsets of $M$ whose union is the whole space $M$.

- A topological space is called **compact** if every open cover of $M$ has a finite subcover.

- In a compact topological space $M$ every sequence has a convergent subsequence (with a limit in $M$).

- For metric spaces the converse is true, that is, if every sequence has a convergent subsequence then the space is compact.

- A compact subspace $W$ of a topological space $M$ is closed in $M$.

- Every closed subspace of a compact topological space is compact.

- A continuous image of a compact space is compact.

- Every bijective continuous map $f : M \to N$ from a compact space $M$ to a space $N$ is a homeomorphism.

- The Cartesian product $M \times N$ of topological spaces $M$ and $N$ is a topological space with the **product topology** defined as follows: the open sets in $M \times N$ are the unions of the sets of the form $U \times V$, where $U$ is an open set in $M$ and $V$ is an open set in $N$.

- The product of compact spaces is compact.
A topological space $M$ is called an $n$-dimensional **topological manifold** if every point of $M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

A homeomorphism $\varphi : U \to \varphi(U)$ from an open subset $U$ of $M$ to an open subset $\varphi(U)$ of $\mathbb{R}^n$ gives a **local coordinate system**, or a **chart** on $M$.

A topological space $M$ is a manifold if it is covered by charts.

The collection of all charts $\{U_\alpha\}_{\alpha \in A}$ is called an **atlas**.

The set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is an open subset in $\mathbb{R}^n$. The maps
\[ f_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \]
are called the **transition maps**.

An atlas is smooth if all transition maps are smooth.

A topological manifold is called a **smooth manifold** if it is Hausdorff and has a smooth atlas.

An atlas determines a **smooth structure** on the manifold $M$, which is a collection of all charts that are compatible with the given atlas.

Two smooth atlases are said to be **compatible** if their union is a smooth atlas.

Compatible atlases define the same smooth structure.

Let $F : M \to M$ be a homeomorphism of a topological space $M$. Then every atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)$ on $M$ defines the atlas
\[ F(\mathcal{A}) = (F(U_\alpha), \varphi_\alpha \circ F^{-1}). \]

Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on $M$ are said to be **equivalent** if there is a homeomorphism $F : M \to M$ such that the atlases $\mathcal{A}_1$ and $F(\mathcal{A}_2)$ are compatible.

Equivalent atlases define equivalent smooth structures.

Two smooth structures are equivalent if they are related by a homeomorphism.
Example.

A topological manifold could have inequivalent smooth structures (so called exotic smooth structures). This can only happen for \( n \geq 4 \).

A topological space \( M \) is called an \( n \)-dimensional complex manifold if it is Hausdorff and every point of \( M \) has a neighborhood homeomorphic to an open subset of \( \mathbb{C}^n \) with holomorphic transition maps.

The topological dimension of a \( n \)-dimensional complex manifold is \( 2n \).

### 1.2 Tangent Vectors

A tangent vector at a point \( p_0 \in M \) of a manifold \( M \) is a map that assigns to each coordinate chart \((U_\alpha, x_\alpha)\) about \( p_0 \) an ordered \( n \)-tuple \((X^1_\alpha, \ldots, X^n_\alpha)\) such that

\[
X^j_p = \sum_{j=1}^{n} \left( \frac{\partial x^j_\beta}{\partial x^j_\alpha}(p_0) \right) X^j_\alpha.
\]

Let \( f : M \to \mathbb{R} \) be a real-valued function on \( M \).

Let \( p \in M \) be a point on \( M \) and \( X \) be a tangent vector at \( p \).

Let \((U_\alpha, x_\alpha)\) be a coordinate chart about \( p \).

The (directional) derivative of \( f \) with respect to \( X \) at \( p \) (or along \( X \), or in the direction of \( X \)) is defined by

\[
X_p(f) = D_X(f) = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial x^j_\alpha}(p) \right) X^j_\alpha.
\]

\( D_X(f) \) does not depend on the local coordinate system.

There is a one-to-one correspondence between tangent vectors to \( M \) at \( p \) and first-order differential operators acting on real-valued functions in a local coordinate chart \((U_\alpha, x_\alpha)\) by

\[
X_p = \sum_{j=1}^{n} X^j_\alpha \left. \frac{\partial}{\partial x^j_\alpha} \right|_p.
\]
• Let $M$ be a manifold and $p \in M$ be a point in $M$. The tangent space $T_p M$ to $M$ at $p$ is the real vector space of all tangent vectors to $M$ at $p$.

• Let $(U, x)$ be a local chart about $p$. Then the vectors
  \[ \frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p) \]
  form a basis in the tangent space called the coordinate basis, or the coordinate frame.

• A vector field $X$ on an open set $U \subset M$ in a manifold $M$ is the differentiable assignment of a tangent vector $X_p$ to each point $p \in U$
  \[ X = \sum_{j=1}^{n} X^j(x) \frac{\partial}{\partial x^j}. \]

• Example.

1.3 Diffeomorphisms and Flows

• Let $M$ be an $n$-dimensional manifold and $N$ be an $m$-dimensional manifold and let $F : M \to N$ be a map from $M$ to $N$. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be an atlas in $M$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ be an atlas in $N$. We define the maps
  \[ F_{\alpha \beta} = \psi_\beta \circ F \circ \varphi_{\alpha}^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta) \]
  from open sets in $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by
  \[ y_\beta^a = F_{\alpha \beta}^a(x_\alpha^1, \ldots, x_\alpha^n), \]
  where $a = 1, \ldots, m$.

• The map $F$ is said to be smooth if $F_{\alpha \beta}^a$ are smooth functions of local coordinates $x_\alpha^i$, $i = 1, \ldots, n$.

• If the map $F : M \to N$ is bijective and both $F$ and $F^{-1}$ are differentiable, then $F$ is called a diffeomorphism. This can only happen if $m = n$.  


1.4. DUAL SPACE

• If this is only true in a neighborhood of a point \( p \in M \), then \( F \) is called a local diffeomorphism.

• Example.

• A flow on a manifold \( M \) corresponding to a vector field \( X \) is a one-parameter family of diffeomorphisms

\[
\varphi_t : M \rightarrow M
\]

such that

\[
\varphi_0 = \text{Id}
\]

and for any \( t, s \)

\[
\varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \varphi_{-t} = \varphi_{t}^{-1}
\]

and

\[
\frac{d}{dt} \varphi_t (x) = X(\varphi_t (x)).
\]

• The corresponding differential operator

\[
X(f)(x) = \left. \frac{d}{dt} f(\varphi_t (x)) \right|_{t=0} = \sum_{j=1}^{n} X^j(x) \frac{\partial f(x)}{\partial x^j}
\]

is the derivative along the streamline (the integral curve) of the flow through the point \( p \).

1.4 Dual Space

• Let \( E \) be a real \( n \)-dimensional vector space.

• Let \( \{e_i\} \) be a basis in \( E \).

• The set of all linear functionals \( \alpha : E \rightarrow \mathbb{R} \) on \( E \) is called the dual space and denoted by \( E^* \).

• Then the linear functionals \( \{\sigma^j\} \) defined by

\[
\sigma^j(e_i) = \delta^j_i
\]

form a basis in \( E^* \) called the dual basis.
Then for any vector $v \in E$ and any linear functional $\alpha \in E^*$

$$v = \sum_{j=1}^{n} v^j e_j,$$

$$\alpha = \sum_{j=1}^{n} a_j \sigma^j,$$

where

$$v^j = \sigma^j(v),$$

$$a_j = \alpha(e_j).$$

### 1.5 Covectors

- Let $M$ be a manifold and $p \in M$ be a point in $M$. The space $T_p^*M$ dual to the tangent space $T_pM$ at $p$ is called the **cotangent space**.

- Let $M$ be a manifold and $f : M \to \mathbb{R}$ be a real valued smooth function on $M$. Let $p \in M$ be a point in $M$. The **differential** $df \in T_p^*M$ of $f$ at $p$ is the linear functional

$$df : T_p \to \mathbb{R}$$

defined by

$$(df)(v) = v_p(f).$$

- In local coordinates $x^j$ the differential is defined by

$$(df)(v) = \sum_{j=1}^{n} v^j(x) \frac{\partial f}{\partial x^j}$$

In particular,

$$(dx^i)\left( \frac{\partial}{\partial x^j} \right) = \delta^i_j.$$

- The differentials $\{dx^i\}$ form a basis for the cotangent space $T_p^*M$ called the **coordinate basis**.
Therefore, every linear functional has the form
\[ \alpha = \sum_{j=1}^{n} a_j dx^j. \]

That is why, the linear functionals are also called differential forms, or 1-forms, or covectors, or covariant vectors.

A covector field \( \alpha \) is a differentiable assignment of a covector \( \alpha_p \in T^*_p M \) to each point \( p \) of a manifold
\[ \alpha = \sum_{j=1}^{n} a_j(x) dx^j. \]

Under a change of local coordinates \( x^j_\alpha = x^j_\alpha(x_\beta) \) the differentials transform according to
\[ dx^j_\alpha = \sum_{i=1}^{n} \frac{\partial x^j_\alpha}{\partial x^i_\beta} dx^i_\beta. \]

Therefore, the components of a covector transform as
\[ a^\alpha_i = \sum_{j=1}^{n} \frac{\partial x^\alpha_j}{\partial x^i_\beta} a^\beta_j. \]

1.6 Differential and Pull-Back

- Let \( M \) and \( N \) be two manifolds and \( F : M \to N \) be a map from \( M \) into \( N \). Let \( p_0 \in M \) be a point in \( M \) and \( X \in T_{p_0}M \) be a tangent vector to \( M \) at \( p_0 \). Let \( p = p(t), t \in (-\varepsilon, \varepsilon) \), be a curve in \( M \) such that
  \[ p(0) = p_0, \quad \dot{p}(0) = X. \]

- Then the differential of \( F \) is the map
  \[ F_* : T_{p_0}M \to T_{F(p_0)}N \]
  defined by
  \[ F_*X = \left. \frac{d}{dt} F(p(t)) \right|_{t=0}. \]
• $F_*$ does not depend on the curve $p(t)$.

• Let $x^i$ be a local coordinate system in a local chart about $p \in M$ and $y^\alpha$ be
a local coordinate system in a local chart about $F(p) \in N$ and $\partial_i$ and $\partial_\alpha$ be
the coordinate bases for $T_pM$ and $T_{F(p)}N$.

• The matrix of the linear transformation $F_*$ in terms of the coordinate bases
$\partial/\partial y^\alpha$ and $\partial/\partial x^i$ is the Jacobian matrix

$$ (F_*)^\alpha_i = \frac{\partial y^\alpha}{\partial x^i}, $$

• Then the action of the differential $F_*$ is defined by

$$ F_*(\frac{\partial}{\partial x^j}) = \sum_{\alpha=1}^{m} \frac{\partial y^\alpha}{\partial x^j} \frac{\partial}{\partial y^\alpha}. $$

• Let $X = \sum_{i=1}^{n} X^i \partial_i$. Then

$$ (F_*X)^\alpha = \sum_{i=1}^{n} \frac{\partial y^\alpha}{\partial x^i} X^i. $$

• The pullback is the linear transformation of the cotangent spaces

$$ F^* : T_{F(p)}^*N \to T_p^*M $$

taking covectors at $F(p) \in N$ to covectors at $p \in M$, defined as follows. If
$\alpha \in T_{F(p)}^*N$, then $F^*\alpha \in T_p^*M$ so that

$$ F^*\alpha = \alpha \circ F_*: T_pM \to \mathbb{R} $$

where $\alpha : T_{F(p)}N \to \mathbb{R}$. That is, for any vector $X \in T_pM$

$$ (F^*\alpha)(X) = \alpha(F_*X). $$

• Diagram.

• In local coordinates,

$$ [F^*(dy^\alpha)]_j = \sum_{\alpha=1}^{m} \frac{\partial y^\alpha}{\partial x^i} dx^j. $$
1.7. SUBMANIFOLDS

- Let $\sigma = \sum_{a=1}^{m} \sigma_{a} dy^{a}$. Then

$$F^{\ast} \sigma = \sum_{j=1}^{n} \sum_{a=1}^{m} \sigma_{a} \frac{\partial y^{a}}{\partial x^{j}} dx^{j},$$

that is, in components,

$$[F^{\ast} \sigma]_{j} = \sum_{a=1}^{m} \sigma_{a} \frac{\partial y^{a}}{\partial x^{j}}.$$

- **Remark.**

- In general, for a map $F : M \rightarrow N$, the following linear transformations are well defined: the differential $F_{\ast} : T_{p}M \rightarrow T_{F(p)}N$ and the pullback $F^{\ast} : T_{F(p)}N \rightarrow T_{p}M$.

- The maps $T_{p}M \rightarrow T_{F(p)}N$ and $T_{F(p)}N \rightarrow T_{p}M$ are not well defined, in general.

- If dim $M =$ dim $N$ and $F : M \rightarrow N$ is a diffeomorphism, then all these maps are well defined.

- Explain.

### 1.7 Submanifolds

- **Implicit description.** Let $M$ be an $n$-dimensional manifold and $W \subset M$ be a subset of $M$. Then $W$ is an $r$-dimensional **embedded submanifold** of $M$ if $W$ is locally described as the common locus of $(n-r)$ differentiable independent functions

$$F^{\alpha}(x^{1}, \ldots, x^{n}) = 0, \quad \alpha = 1, \ldots, n-r,$$

such that the Jacobian matrix has the maximal rank $(n-r)$ at each point of the locus, that is,

$$\text{rank} \left( \frac{\partial F^{\alpha}}{\partial x^{i}} (x) \right) = n - r, \quad \forall x \in W.$$
• More generally, let $M$ be an $n$-dimensional manifold and $N$ be an $(n - r)$-dimensional manifold with $n > r$. Let $F : M \to N$ be a smooth map. Let $q \in N$ be a point in $N$ such that the inverse image $W = F^{-1}(q) \neq \emptyset$ is nonempty. Suppose that for each point $p \in W$ the differential $F_*$ of the map $F$ is surjective, that is, has the maximal rank
\[ \text{rank } F_*(p) = n - r. \]

Then $W$ is an $r$-dimensional submanifold of $M$.

• The number $(n - r)$ is called the **codimension** of the submanifold $W$.

• **Explicit description.** Let $y = (y^1, \ldots, y^r)$ be local coordinates in a neighborhood of a point $q \in W$. Then the submanifold $W$ can be described by a map $f : W \to M$ (called the **inclusion map**) such that $p = f(q)$ and
\[ x^i = f^i(y), \quad i = 1, \ldots, n. \]

• The $r$ vectors
\[ e_\mu = \frac{\partial}{\partial y^\mu} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^\mu} \frac{\partial}{\partial x^i}, \quad \mu = 1, \ldots, r \]
are tangent vector to the submanifold $W$ and form the basis of the tangent space $T_q W$.

• Note that the differential $f_*$ has the maximal rank
\[ \text{rank } f_*(q) = r. \]

### 1.8 Riemannian Metric

• Let $E$ be a $n$-dimensional real vector space.

• The **inner product** (or **scalar product**) on $E$ is a symmetric bilinear positive-definite functional on $E \times E$.

• For a positive-definite inner product the **norm** of a vector $v$ is defined by
\[ \|v\| = \sqrt{\langle v, v \rangle} \]
1.8. RIEMANNIAN METRIC

- Let \( \{e_j\} \) be a basis in \( E \).

- Then the matrix \( g_{ij} \) defined by

\[
g_{ij} = \langle e_i, e_j \rangle
\]

is a metric tensor, more precisely it gives the components of the metric tensor in that basis.

- The matrix \( g_{ij} \) is symmetric and nondegenerate, that is,

\[
g_{ij} = g_{ji}, \quad \det g_{ij} \neq 0.
\]

For a positive definite inner product, this matrix is positive-definite, that is, it has only positive real eigenvalues. One says, that the metric has the signature \((+ \cdots +)\). In special relativity one considers metrics which are not positive definite but have the signature \((- + \cdots +)\).

- Two vectors \(v, w \in E\) are orthogonal if

\[
\langle v, w \rangle = 0.
\]

- A vector \(u \in E\) is called unit vector if

\[
||u|| = 1.
\]

- The basis is called orthonormal if

\[
g_{ij} = \delta_{ij}.
\]

- The inner product is given then by

\[
\langle v, w \rangle = \sum_{i,j=1}^{n} g_{ij} v^i w^j.
\]

- Let \(v \in E\). Then we can define a linear functional \(\nu \in E^*\) by

\[
\nu(w) = \langle v, w \rangle.
\]

- Therefore, each vector \(v \in E\) defines a covector \(\nu \in E^*\) called the covariant version of the vector \(v\).
• Given a basis \{e_j\} in \(E\) and the dual basis \(\sigma^i\) in \(E^*\) we find

\[ \nu_i = \sum_{j=1}^{n} g_{ij} v^j, \quad v^i = \sum_{j=1}^{n} g^{ij} v_j, \]

where \(g^{ij}\) is the matrix inverse to the matrix \(g_{ij}\).

• These operations are called lowering the index of a vector and raising the index of a covector.

• A Riemannian metric on \(M\) is a differentiable assignment of a positive definite inner product in each tangent space \(T_p M\) to the manifold at each point \(p \in M\).

• A Riemannian manifold is a pair \((M, g)\) of a manifold with a Riemannian metric on it.

• Let \(p \in M\) be a point in \(M\) and \((U, x_\alpha)\) be a local coordinate system about \(p\). Let \(\partial_i\) be the coordinate basis in \(T_p M\) and \(g^{\alpha i} = \langle \partial^\alpha_i, \partial^\alpha_j \rangle\) be the components of the metric tensor in the coordinate system \(x_\alpha\). Let \((U_\beta, x_\beta)\) be another coordinate system containing \(p\). Then the components of the metric tensor transform as

\[ g^{\alpha i} = \sum_{k,l=1}^{n} \frac{\partial x^k_\beta}{x^i_\alpha} \frac{\partial x^l_\beta}{x^j_\alpha} g_{kl}. \]

### 1.9 Tensors

• Let \(E\) be a vector space and \(E^*\) be its dual space.

• Let \(e_i\) be a basis for \(E\) and \(\sigma^i\) be the dual basis for \(E^*\).

• A tensor of type \((p, q)\) is a multi-linear real-valued functional

\[ T : E^* \times \cdots \times E^* \times E \times \cdots \times E \rightarrow \mathbb{R} \]

with \(p\) factors of \(E^*\) and \(q\) factors of \(E\).

• The components of the tensor \(T\) with respect to the basis \(e_i, \sigma^j\) are defined by

\[ T_{j_1 \cdots j_q}^{i_1 \cdots i_p} = T(\sigma^{j_1}, \ldots, \sigma^{j_q}, e_{i_1}, \ldots, e_{i_p}). \]
Then for any covectors
\[ \alpha_{(a)} = \sum_{j=1}^{n} \alpha_{(a)j} \sigma^j, \]
where \( a = 1, \ldots, p \), and any vectors
\[ v_b = \sum_{j=1}^{n} v^j_b e_k, \]
where \( b = 1, \ldots, q \), we have
\[ T(\alpha_{(1)}, \ldots, \alpha_{(p)}, v_1, \ldots, v_q) = \sum_{k_1, \ldots, k_q=1}^{n} \sum_{j_1, \ldots, j_p=1}^{n} T^{j_1 \ldots j_p}_{k_1 \ldots k_q} \alpha_{(1)j_1} \cdots \alpha_{(p)j_p} v^{k_1} \cdots v^{k_q}. \]

The collection of all tensors of type \((p, q)\) forms a vector space denoted by
\[ T^p_q = E \otimes \cdots \otimes E \otimes E^* \otimes \cdots \otimes E^*. \]

The dimension of the vector space \( T_p \) is
\[ \dim T^p_q = n^{p+q}. \]

The tensor product of a tensor \( Q \) of type \((p, q)\) and a tensor \( T \) of type \((r, s)\) is a tensor \( Q \otimes T \) of type \((p+r, q+s)\) defined by:
\[ (Q \otimes T)(\alpha_{1}, \ldots, \alpha_{p}, \beta_{1}, \ldots, \beta_{r}, v_{1}, \ldots, v_{q}, w_{1}, \ldots, w_{s}) = Q(\alpha_{1}, \ldots, \alpha_{p}, v_{1}, \ldots, v_{q})T(\beta_{1}, \ldots, \beta_{r}, w_{1}, \ldots, w_{s}). \]

The components of the tensor product \( Q \otimes T \) are
\[ (Q \otimes T)^{i_1 \ldots i_p j_1 \ldots j_r}_{k_1 \ldots k_q l_1 \ldots l_s} = Q^{i_1 \ldots i_p}_{k_1 \ldots k_q} T^{j_1 \ldots j_r}_{l_1 \ldots l_s}. \]

Thus,
\[ \otimes : T^p_q \times T^r_s \to T^{p+r}_{q+s}. \]

The tensor product is associative.
• The basis in the space $T^p_q$ is
  \[ e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_q}, \]
  where $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq n$.

• A tensor $T$ of type $(p, q)$ has the form
  \[ T = \sum_{j_1, \ldots, j_q=1}^n \sum_{i_1, \ldots, i_p=1}^n T^{i_1 \ldots i_p}_{j_1 \ldots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_q}. \]

• Let $p, q \geq 1$ and $1 \leq r \leq p$, $1 \leq s \leq q$. The $(r, s)$-contraction of tensors of type $(p, q)$ is the map
  \[ \text{tr}^r_s : T^p_q \to T^{p-1}_{q-1} \]
  defined by
  \[ (\text{tr}^r_s T)^{i_1 \ldots i_{p-1}}_{j_1 \ldots j_{q-1}} = \sum_{k=1}^n T^{i_1 \ldots i_r \ldots i_{p-1}}_{j_1 \ldots j_s \ldots j_{q-1}} \cdot \]

• A tensor field on a manifold $M$ is a smooth assignment of a tensor at each point of $M$.

• Let $x_i^\alpha = x_i^\alpha(x_\beta)$ be a local diffeomorphism.

• Let $T$ be a tensor of type $(p, q)$. Then
  \[ T_{(\alpha)}^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x_\alpha) = \sum_{k_1, \ldots, k_p=1}^n \sum_{l_1, \ldots, l_q=1}^n \frac{\partial x_{i_1}^{l_1}}{\partial x_{\alpha}^{k_1}} \cdots \frac{\partial x_{i_p}^{l_p}}{\partial x_{\alpha}^{k_p}} \frac{\partial x_{j_1}^{l_1}}{\partial x_{\alpha}^{k_1}} \cdots \frac{\partial x_{j_q}^{l_q}}{\partial x_{\alpha}^{k_q}} T^{(\beta)}_{k_1 \ldots k_p}(x_\beta). \]

**Einstein Summation Convention**

• In any expression there are two types of indices: **free indices** and **repeated indices**.

  • Free indices appear only once in an expression; they are assumed to take all possible values from 1 to $n$.

  • The position of all free indices in all terms in an equation must be the same.
1.10. **PERMUTATION GROUP**

- Repeated indices appear twice in an expression. It is assumed that there is a summation over each repeated pair of indices from 1 to \( n \). The summation over a pair of repeated indices in an expression is called the **contraction**.

- Repeated indices are **dummy indices**: they can be replaced by any other letter (not already used in the expression) without changing the meaning of the expression.

- Indices cannot be repeated on the same level. That is, in a pair of repeated indices one index is in upper position and another is in the lower position.

- There cannot be indices occurring three or more times in any expression.

### 1.10 Permutation Group

- A **group** is a set \( G \) with an associative binary operation, \( \cdot : G \times G \to G \) with identity, called the **multiplication**, such that each element has an inverse.

- A **transformation** of a set \( X \) is a bijective map \( g : X \to X \).

- The set of all transformations of a set \( X \) forms a group \( \text{Aut}(X) \), with composition of maps as group multiplication.

- Any subgroup of \( \text{Aut}(X) \) is a **transformation group** of the set \( X \).

- The transformations of a finite set \( X \) are called **permutations**.

- The group \( S_p \) of permutations of the set \( \mathbb{Z}_n = \{1, \ldots, p\} \) is called the **symmetric group of order** \( p \).

- The order of the symmetric group \( S_p \) is

\[
|S_p| = p!.
\]

- Any subgroup of \( S_p \) is called a **permutation group**.

- An **elementary permutation** is a permutation that exchanges the order of only two elements.

- Every permutation can be realized as a product of elementary permutations.
• A permutation that can be realized by an even number of elementary permutations is called an **even permutation**.

• A permutation that can be realized by an odd number of elementary permutations is called an **odd permutation**.

• The **parity of a permutation** does not depend on the representation of a permutation by a product of the elementary ones.

• The **sign of a permutation** \( \varphi \in S_p \), denoted by \( \text{sign}(\varphi) \) (or simply \((-1)^\varphi\)), is defined by

\[
\text{sign}(\varphi) = (-1)^\varphi = \begin{cases} 
+1, & \text{if } \varphi \text{ is even}, \\
-1, & \text{if } \varphi \text{ is odd}
\end{cases}
\]

### 1.11 Permutation of Tensors

• Let \( S_p \) be the symmetric group of order \( p \). Then every permutation \( \varphi \in S_p \) defines a map

\[
\varphi : T_p \to T_p,
\]

which assigns to every tensor \( T \) of type \((0, p)\) a new tensor \( \varphi(T) \), called a **permutation of the tensor** \( T \), of type \((0, p)\) by: \( \forall v_1, \ldots, v_p \)

\[
\varphi(T)(v_1, \ldots, v_p) = T(v_{\varphi(1)}, \ldots, v_{\varphi(p)}).
\]

• Let \((i_1, \ldots, i_p)\) be a \( p \)-tuple of integers. Then a permutation \( \varphi : \mathbb{Z}_p \to \mathbb{Z}_p \) defines an action

\[
\varphi(i_1, \ldots, i_p) = (i_{\varphi(1)}, \ldots, i_{\varphi(p)}).
\]

• The components of the tensor \( \varphi(T) \) are obtained by the action of the permutation \( \varphi \) on the indices of the tensor \( T \)

\[
\varphi(T)_{i_1 \ldots i_p} = T_{i_{\varphi(1)} \ldots i_{\varphi(p)}}.
\]

• The **symmetrization** of the tensor \( T \) of the type \((0, p)\) is defined by

\[
\text{Sym}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \varphi(T).
\]
• The symmetrization is also denoted by parenthesis. The components of the symmetrized tensor $\text{Sym}(T)$ are given by

$$T_{(i_1 \ldots i_p)} = \frac{1}{p!} \sum_{\varphi \in S_p} T_{i_{\varphi(1)} \ldots i_{\varphi(p)}}.$$ 

• The anti-symmetrization of the tensor $T$ of the type $(0, p)$ is defined by

$$\text{Alt}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \text{sign}(\varphi) T_{i_{\varphi(1)} \ldots i_{\varphi(p)}}.$$ 

• The anti-symmetrization is also denoted by square brackets. The components of the anti-symmetrized tensor $\text{Alt}(T)$ are given by

$$T_{[i_1 \ldots i_p]} = \frac{1}{p!} \sum_{\varphi \in S_p} \text{sign}(\varphi) T_{i_{\varphi(1)} \ldots i_{\varphi(p)}}.$$ 

• Examples.

• A tensor $T$ of type $(0, p)$ is called symmetric if for any permutation $\varphi \in S_p$

$$\varphi(T) = T.$$ 

• A tensor $T$ of type $(0, p)$ is called anti-symmetric if for any permutation $\varphi \in S_p$

$$\varphi(T) = \text{sign}(\varphi) T.$$ 

• An anti-symmetric tensor of type $(0, p)$ is called a $p$-form.

• Let $(i_1, \ldots, i_n)$ be an $n$-tuple of integers $1 \leq i_1, \ldots, i_n \leq n$. The completely anti-symmetric (alternating) Levi-Civita symbols are defined by

$$\varepsilon^{i_1 \ldots i_n} = \varepsilon_{i_1 \ldots i_n} = \begin{cases} 
1 & \text{if } (i_1, \ldots, i_n) \text{ is an even permutation of } (1, \ldots, n) \\
-1 & \text{if } (i_1, \ldots, i_n) \text{ is an odd permutation of } (1, \ldots, n) \\
0 & \text{otherwise}
\end{cases}.$$ 

• The product of Levi-Civita symbols is equal to

$$\varepsilon^{i_1 \ldots i_n} \varepsilon_{j_1 \ldots j_n} = n! \delta_{i_1 j_1} \cdots \delta_{i_n j_n}.$$
• The contraction of this identity over \( k \) indices gives

\[
\varepsilon^{i_1 \ldots i_{n-k} m_1 \ldots m_k} e_{j_1 \ldots j_{n-k} m_1 \ldots m_k} = k!(n-k)! \delta^{i_1}_{j_1} \ldots \delta^{i_{n-k}}_{j_{n-k}}.
\]

In particular,

\[
\varepsilon^{m_1 \ldots m_n} e_{m_1 \ldots m_n} = n!.
\]

• The **determinant** of a \( n \times n \) matrix \( A = (A^i_j) \) is defined by

\[
\det A = \sum_{\varphi \in S_n} \text{sign} (\varphi) A^1_{\varphi(1)} \cdots A^n_{\varphi(n)}.
\]

• It can be written as

\[
\det A = \frac{1}{n!} \varepsilon^{i_1 \ldots i_n} e_{j_1 \ldots j_n} A^{i_1}_{j_1} \cdots A^{i_n}_{j_n}.
\]

### 1.12 Exterior \( p \)-forms

• An **exterior \( p \)-form** (or simply a **\( p \)-form**) is an anti-symmetric covariant tensor \( \alpha \in T_p \) of type \((0, p)\).

• The collection of all \( p \)-forms forms a vector space \( \Lambda_p \), which is a vector subspace of \( T_p \)

\[
\Lambda_p \subset T_p.
\]

• In particular,

\[
\Lambda_0 = \mathbb{R} \quad \text{and} \quad \Lambda_1 = T_1 = E^*.
\]

• In other words, a 0-form is a smooth function, and a 1-form is a covector field.

• Let \( \alpha \in \Lambda_p \) be a \( p \)-form.

• Let \( e_i \) be a basis in \( E \) and \( \sigma^i \) be the dual basis in \( E^* \).

• The components of the \( p \)-form \( \alpha \) are

\[
\alpha_{i_1 \ldots i_p} = \alpha(e_{i_1}, \ldots, e_{i_p}).
\]

• The components are completely anti-symmetric in all indices.
• In particular, under a permutation of any two indices the form changes sign

\[ \alpha_{\ldots i \ldots j \ldots} = -\alpha_{\ldots j \ldots i \ldots}, \]

which means that the components vanish if any two indices are equal

\[ \alpha_{\ldots i \ldots i \ldots} = 0 \quad \text{(no summation!)}. \]

• Thus, all non-vanishing components have different indices.

• Therefore, the values of all components \( \alpha_{i_1, \ldots, i_p} \) are completely determined by the values of the components with the indices \( i_1, \ldots, i_p \) reordered in strictly increasing order

\[ 1 \leq i_1 < \cdots < i_p \leq n. \]

• Therefore, the dimension of the space \( \Lambda_p \) is equal to the number of distinct increasing \( p \)-tuples of integers from 1 to \( n \).

\[ \dim \Lambda_p = \binom{n}{p} = \frac{n!}{p!(n-p)!}. \]

• In particular,

\[ \dim \Lambda_0 = \dim \Lambda_n = 1, \]

\[ \dim \Lambda_1 = \dim \Lambda_{n-1} = n, \]

etc.

• There are no \( p \)-forms with \( p > n \).

• Similarly to the norm of vectors and covectors we define the \textbf{inner product of exterior \( p \)-forms} \( \alpha \) and \( \beta \) in a Riemannian manifold by

\[ (\alpha, \beta) = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p}. \]

• This enables one to define also the \textbf{norm of an exterior \( p \)-form} \( \alpha \) by

\[ \|\alpha\| = \sqrt{(\alpha, \alpha)}. \]
1.13 Exterior Product

- If $\alpha$ is an $p$-form and $\beta$ is an $q$-form then the exterior (or wedge) product of $\alpha$ and $\beta$ is an $(p+q)$-form $\alpha \wedge \beta$ defined by

$$\alpha \wedge \beta = \frac{(p+q)!}{p!q!} \text{Alt} (\alpha \otimes \beta).$$

- In components

$$(\alpha \wedge \beta)_{i_1 \cdots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \cdots i_p} \beta_{i_{p+1} \cdots i_{p+q}]}.$$

- The exterior product has the following properties

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \quad \text{(associativity)}$$

$$\alpha \wedge \beta = (-1)^{\text{deg}(\alpha)\text{deg}(\beta)} \beta \wedge \alpha \quad \text{(anticommutativity)}$$

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \quad \text{(distributivity)}.$$ 

- The exterior square of any $p$-form $\alpha$ of odd degree $p$ (in particular, for any 1-form) vanishes

$$\alpha \wedge \alpha = 0.$$

- The exterior algebra $\Lambda$ (or Grassmann algebra) is the set of all forms of all degrees, that is,

$$\Lambda = \Lambda_0 \oplus \cdots \oplus \Lambda_n.$$

- The dimension of the exterior algebra is

$$\dim \Lambda = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

- A basis of the space $\Lambda_p$ is

$$\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}, \quad (1 \leq i_1 < \cdots < i_p \leq n).$$
1.13. EXTERIOR PRODUCT

- A $p$-form $\alpha$ can be represented in one of the following ways

\[
\alpha = \frac{1}{p!} \alpha_{i_1 \ldots i_p} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p} = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \ldots i_p} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}.
\]

- The exterior product of a $p$-form $\alpha$ and a $q$-form $\beta$ can be represented as

\[
\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{i_1 \ldots i_p} \beta_{i_{p+1} \ldots i_{p+q}} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_{p+q}}.
\]

- A collection of 1-forms $\alpha^1, \ldots, \alpha^n \in \Lambda_1$ is linearly dependent if and only if

\[
\alpha^1 \wedge \cdots \wedge \alpha^n = 0.
\]

- More generally, let $\sigma^1, \ldots, \sigma^n \in \Lambda_1$ be a collection of 1-forms and

\[
\alpha^j = \sum_{i=1}^n A^j_i \sigma^i, \quad 1 \leq j \leq n,
\]

Then

\[
\alpha^1 \wedge \cdots \wedge \alpha^n = (\det A^j_i) \sigma^1 \wedge \cdots \wedge \sigma^n.
\]

- This means that for a local diffeomorphism $x^i_\alpha = x^i_\alpha(x_\beta), \ i = 1, \ldots, n$, there holds

\[
dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha = J_{\alpha \beta}(x_\beta) dx^1_\beta \wedge \cdots \wedge dx^n_\beta.
\]

where

\[
J_{\alpha \beta}(x_\beta) = \det \left( \frac{\partial x^j_\alpha}{\partial x^i_\beta} \right)
\]

is the Jacobian.

- The interior product of a vector $v$ and a $p$-form $\alpha$ is a $(p - 1)$-form $i_v \alpha$ defined by, for any $v_1, \ldots, v_{p-1}$,

\[
i_v \alpha(v_1, \ldots, v_{p-1}) = \alpha(v, v_1, \ldots, v_{p-1}).
\]
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• In particular, if $p = 1$, then $i_v \alpha$ is a scalar
  $i_v \alpha = \alpha(v)$

  and if $p = 0$, then by definition
  $i_v \alpha = 0$.

• In components,
  $(i_v \alpha)_{i_1 \cdots i_{p-1}} = v^j \alpha_{j i_1 \cdots i_{p-1}}$.

• For any $\alpha \in \Lambda^p$, $\beta \in \Lambda^q$,
  $i_v (\alpha \wedge \beta) = (i_v \alpha) \wedge \beta + (-1)^p \alpha \wedge i_v \beta$.

1.14 Orientation of a Vector Space

• Let $E$ be a vector space. Let $\{e_i\} = \{e_1, \ldots, e_n\}$ and $\{e'_j\} = \{e'_1, \ldots, e'_n\}$ be two different bases in $E$ related by
  
  $e_i = \Lambda^i_j e'_j$,

  where $\Lambda = (\Lambda^i_j)$ is a transformation matrix.

• Note that the transformation matrix is non-degenerate
  $\det \Lambda \neq 0$.

• Since the transformation matrix $\Lambda$ is invertible, then the determinant $\det \Lambda$ is either positive or negative.

• If $\det \Lambda > 0$ then we say that the bases $\{e_i\}$ and $\{e'_j\}$ have the same orientation, and if $\det \Lambda < 0$ then we say that the bases $\{e_i\}$ and $\{e'_j\}$ have the opposite orientation.

• If the basis $\{e_i\}$ is continuously deformed into the basis $\{e'_j\}$, then both bases have the same orientation.

• Since $\det I = 1 > 0$ and the function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is continuous, then a one-parameter continuous transformation matrix $\Lambda(t)$ such that $\Lambda(0) = I$ preserves the orientation.
• This defines an equivalence relation on the set of all bases on \( E \) called the **orientation** of the vector space \( E \).

• This equivalence relation divides the set of all bases in two equivalence classes, called the **positively oriented** and **negatively oriented** bases.

• A vector space together with a choice of what equivalence class is positively oriented is called an **oriented vector space**.

### 1.15 Orientation of a Manifold

• Let \( M \) be a manifold and let \( T_pM \) be the tangent space at a point \( p \in M \).

• Let \((U, x)\) be a local coordinate patch about a point \( p \in M \).

• Then the vectors 
  \[
  \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n
  \]
  form a basis in \( T_pM \).

• Let \((U', x')\) be another local coordinate system about a point \( p \), that is, there is a local diffeomorphism \( x^i = x^i(x') \).

• Then the vectors
  \[
  \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j}
  \]
  form another basis in \( T_pM \).

• The orientation of the bases \( \{\partial_i\} \) and \( \{\partial'_j\} \) is the same (or **consistent**) if
  \[
  \det \left( \frac{\partial x^i}{\partial x'^j} \right) > 0 .
  \]

• If it is possible to choose an orientation of all tangent spaces \( T_pM \) at all points in a continuous fashion, then the orientation of all tangent spaces is consistent.

• A manifold \( M \) is called **orientable** if there is an atlas such that the orientation of all charts of this atlas can be chosen consistently, that is, the Jacobians of all transition functions have positive determinant.
• Each connected orientable manifold has exactly two possible orientations. One orientation can be declared positive, then the other orientation is negative.

• An orientable manifold with a chosen orientation is called oriented.

• Remarks.

• If a manifold can be covered by a single coordinate chart then it is orientable.

• Not all manifolds are orientable.

• Transport of the orientation.

• Let \( p, q \in M \) be two points in a manifold \( M \) and \( C(t) \) be a curve in \( M \) connecting \( p \) and \( q \), i.e.

\[
C(0) = p, \quad C(1) = q.
\]

• Let \( \{e_i(t)\} \) be a basis in \( T_{C(t)}M \) that continuously depends on \( t \in [0, 1] \).

• Then the orientation of the basis \( e_i(1) \) is uniquely determined by the orientation of the basis \( e_i(0) \).

• Thus, the orientation is transported along a curve in a unique way.

• Note that the transportation of the basis is not unique, in general. Only the transportation of the orientation is!

• Given a point \( p \) and another point \( q \), the orientation at the point \( q \) does, in general, depend on the curve \( C(t) \) connecting the points \( p \) and \( q \).

• If a manifold is orientable, then the transportation of the orientation from one point to another does not depend on the curve connecting the points.

• If there exists a closed curve \( C(t) \) in \( M \) such that the transport of the orientation along \( C \) leads to a reversal of orientation, then \( M \) is nonorientable.

• Example. Möbius Band.
Let $e_i$ be a basis in a vector space $E$, $\sigma^j$ be the dual basis of 1-forms in $E^*$, $g = (g_{ij})$ be a Riemannian metric.

The $n$-form
$$\text{vol} = \sqrt{|g|} \sigma^1 \wedge \cdots \wedge \sigma^n$$
where $|g| = \det g_{ij}$
is called the **Riemannian volume element** (or volume form).

The components of the volume form are
$$\text{vol} (e_{i_1}, \ldots, e_{i_n}) = \sqrt{|g|} \epsilon_{i_1 \ldots i_n} = E_{i_1 \ldots i_n}.$$

The volume form allows one to define the **duality** of $p$-forms and $(n-p)$-vectors.

For each $p$-form $A_{i_1 \ldots i_p}$ one assigns the **dual** $(n-p)$-vector by
$$\tilde{A}^{j_1 \ldots j_{n-p}} = \frac{1}{p!} E^{j_1 \ldots j_{n-p} i_1 \ldots i_p} A_{i_1 \ldots i_p}.$$

Similarly, for each $p$-vector $A^{i_1 \ldots i_p}$ one assigns the **dual** $(n-p)$-form by
$$\tilde{A}_{j_1 \ldots j_{n-p}} = \frac{1}{p!} E_{j_1 \ldots j_{n-p} i_1 \ldots i_p} A^{i_1 \ldots i_p}.$$

By lowering and raising the indices of the dual forms we can define the duality of forms and poly-vectors separately.

The **Hodge star operator**
$$* : \Lambda_p \rightarrow \Lambda_{n-p}$$
maps any $p$-form $\alpha$ to a $(n-p)$-form $*\alpha$ **dual** to $\alpha$ defined as follows.

For each $p$-form $\alpha$ the form $*\alpha$ is the unique $(n-p)$-form such that
$$\alpha \wedge *\alpha = (\alpha, \alpha) \text{vol}.$$
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• In particular,
  \[ *1 = \text{vol}, \quad *\text{vol} = 1. \]

• In components, this means that
  \[
  (*\alpha)_{p_1 \ldots p_n} = \frac{1}{p!} \epsilon_{i_1 \ldots i_p j_1 \ldots j_p} \sqrt{|g|} g^{i_1 j_1} \ldots g^{i_p j_p} \alpha_{j_1 \ldots j_p} \\
  = \frac{1}{p!} \frac{1}{\sqrt{|g|}} g_{i_p j_p} \epsilon^{i_1 \ldots i_p j_1 \ldots j_p} \alpha_{j_1 \ldots j_p}.
  \]

• For any \( p \)-form \( \alpha \) there holds
  \[ *^2 \alpha = (-1)^{p(n-p)} \alpha. \]

In particular, if \( n \) is odd, then for any \( p \)
  \[ *^2 = \text{Id}. \]

• Let \( \alpha \) be a 1-form and \( \mathbf{v} \) be the corresponding vector, that is, \( v^i = g^{ij} \alpha_j \).
  Then
  \[ *\alpha = i_\mathbf{v} \text{vol}. \]

• A collection \( \{\omega_1, \ldots, \omega_{n-1}\} \) of \( (n-1) \) 1-forms defines a 1-form \( \alpha \) by
  \[ \alpha = * \left[ \omega_1 \wedge \cdots \wedge \omega_{n-1} \right]. \]

• Then
  \[ *\alpha = (-1)^{n-1} \omega_1 \wedge \cdots \wedge \omega_{n-1} \]
  and
  \[ \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \alpha = (\alpha, \alpha) \text{vol}. \]

• In components,
  \[ \alpha_j = g_{jk} E_{i_1 \ldots i_{n-1}, k} \omega_{i_1} \cdots \omega_{i_{n-1}}. \]

• If the 1-forms \( \{\omega_1, \ldots, \omega_{n-1}\} \) are linearly dependent, then \( \alpha = 0 \).

• If the collection of 1-forms \( \{\omega_1, \ldots, \omega_{n-1}\} \) is linearly independent, then \( \{\omega_1, \ldots, \omega_{n-1}, \alpha\} \) are linearly independent and form a basis in \( E^* \).
1.17 Exterior Derivative and Coderivative

- Similarly, a collection \{v(1), \ldots, v(n-1)\} of \((n-1)\) vectors defines a covector \(\alpha\) by: for any vector \(v\)

\[ \alpha(v) = \text{vol}(v(1), \ldots, v(n-1), v) \]

or, in components,

\[ \alpha_j = \sqrt{|g|} \varepsilon_{i_1 \ldots i_{n-1} j} v^{i_1}(1) \cdots v^{i_{n-1}} \]

and the corresponding vector \(N_i\) is orthogonal to all vectors \{v(1), \ldots, v(n-1)\}, that is,

\[ (N, v(j)) = g^{ik} N^k v^j = 0, \quad (j = 1, \ldots, n - 1) \]

- The \(n\)-tuple \{v(1), \ldots, v(n-1), N\} forms a positively oriented basis.

1.17 Exterior Derivative and Coderivative

- The **exterior derivative** is a linear map

\[ d : \Lambda_p \rightarrow \Lambda_{p+1} \]

defined as follows.

- Let \(\alpha\) be a \(p\)-form

\[ \alpha = \frac{1}{p!} \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \]

- The **exterior derivative** of \(\alpha\) is a \((p+1)\)-form \(d\alpha\) defined by

\[ d\alpha = \frac{1}{p!} \partial_i \alpha_{i_2 \ldots i_{p+1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{p+1}} \]

- In components

\[ (d\alpha)_{i_1 i_2 \ldots i_{p+1}} = (p+1)\partial_{[i_1} \alpha_{i_2 \ldots i_{p+1}]} \]

\[ = \sum_{k=1}^{p+1} (-1)^{k-1} \partial_{[i_1 \ldots i_{k-1}} \alpha_{i_k \ldots i_{p+1}]} \]
For any $p$-form
\[ d^2 = 0. \]

For any $p$-form $\alpha \in \Lambda_p$ and any $q$-form $\beta \in \Lambda_q$ there holds
\[ d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta). \]

The coderivative is a linear map
\[ \delta : \Lambda_p \to \Lambda_{p-1} \]
defined by
\[ \delta = *^{-1} d* = (-1)^{(n-p+1)(p-1)} * d* \]

The coderivative of a $p$-form $\alpha$ is the $(p-1)$-form $\delta \alpha$ defined by
\[ (\delta \alpha)_{i_1 \ldots i_{p-1}} = \sum_{j} g^i_{j_1} \cdots g^i_{j_{p-1}} \frac{1}{\sqrt{|g|}} \partial_j \left( \sqrt{|g|} g^{j_{k_1}} \cdots g^{j_{k_{p-1}}} \alpha_{k_{k_1} \ldots k_{p-1}} \right). \]

For any $p$-form
\[ \delta^2 = 0. \]

For a 1-form $\alpha$, $\delta \alpha$ is a 0-form
\[ \delta \alpha = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \alpha_j \right). \]

The Hodge Laplacian $\Delta$ is a linear map
\[ \Delta : \Lambda_p \to \Lambda_p \]
defined by
\[ \Delta = d\delta + \delta d. \]

It is easy to see that Laplacian commutes with both $d$ and $\delta$. 
1.18 Pullback of Forms

- Let $M$ be a $n$-dimensional manifold and $W$ be a $r$-dimensional manifold.
- Let $F : M \to N$ be a smooth map of a manifold $M$ to a manifold $N$.
- Let $p \in M$ be a point in $M$ and $q = F(p) \in N$ be the image of $p$ in $N$.
- Let $x^i$, $(i = 1, \ldots, n)$, be a local coordinate system about $p$ and $y^\mu$, $(\mu = 1, \ldots, r)$, be a local coordinate system about $q$ so that $y^\mu = y^\mu(x)$.

- The pullback of $p$-forms is the map
  
  $$F^* : \Lambda_p N \to \Lambda_p M$$
  
  defined as follows.

- Let $\alpha \in \Lambda_p N$ be a $p$-form on $N$. The pullback of $\alpha$ is a $p$-form $F^* \alpha$ on $M$ defined by: for any vectors $v_1, \ldots, v_p$
  
  $$(F^* \alpha)(v_1, \ldots, v_p) = \alpha(F_* v_1, \ldots, F_* v_p).$$

- In local coordinates
  
  $$F^* \alpha = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} (y(x)) dy^{\mu_1} \wedge \cdots \wedge dy^{\mu_p}$$

  $$= \frac{1}{p!} \frac{\partial y^{\mu_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\mu_p}}{\partial x^{i_p}} \alpha_{\mu_1 \ldots \mu_p} (y(x)) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

  - In components

  $$\left(F^* \alpha\right)_{i_1 \ldots i_p} = \frac{\partial y^{\mu_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\mu_p}}{\partial x^{i_p}} \alpha_{\mu_1 \ldots \mu_p} (y(x))$$

  - For any two forms $\alpha$ and $\beta$

  $$F^* (\alpha \wedge \beta) = (F^* \alpha) \wedge (F^* \beta)$$

  - $F^*$ commutes with exterior derivative, that is, for any $p$-form $\alpha$

  $$F^* (d\alpha) = d(F^* \alpha).$$
1.19 Integration of Differential Forms

- Let $M$ be an $n$-dimensional manifold with local coordinates $x^i$, $i = 1, \ldots, n$.

- Let $0 \leq p \leq n$ and $U$ be an oriented region in $\mathbb{R}^p$ with orientation $o$ and coordinates $u^\mu$, $\mu = 1, \ldots, p$.

- Let $F : U \to M$ be a smooth map given locally by
  \[ x^i = F^i(u). \]

- Then the image $F(U) \subset M$ of the set $U$ is called a **$p$-subset of $M$** and the collection $(U, o, F)$ is called an **oriented parametrized $p$-subset of $M$**.

- Usually, the differential $F_* \text{ has rank } p$ (that is, $F(U)$ is a submanifold) almost everywhere.

- Let $\alpha \in \Lambda_p$ be a $p$-form on $M$
  \[ \alpha = \frac{1}{p!} \alpha_{i_1 \ldots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \]

- The **integral of $\alpha$ over an oriented parametrized $p$-subset $F(U)$** is defined by
  \[ \int_{F(U)} \alpha = \int_U F^* \alpha. \]

- In more detail,
  \[ \int_{F(U)} \alpha = o(u) \int_U (F^* \alpha) \left( \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^p} \right) \, du^1 \cdots du^p \]
  \[ = o(u) \int_U \alpha \left( F_* \frac{\partial}{\partial u^1}, \ldots, F_* \frac{\partial}{\partial u^p} \right) \, du^1 \cdots du^p \]
  \[ = \frac{1}{p!} o(u) \int_U \alpha_{i_1 \ldots i_p} \left( x(u) \right) \frac{\partial x^{i_1}}{\partial u^1} \cdots \frac{\partial x^{i_p}}{\partial u^p} \, du^1 \cdots du^p. \]

- The integral of the form $\alpha$ over $U$ reverses sign if the orientation of $U$ is reversed.

- The integral is independent of the parametrization of a $p$-subset.
• Let $M$ be an $n$-dimensional manifold and $W$ be an $r$-dimensional manifold.
• Let $\varphi : M \to W$ be a smooth map.
• Let $U \subset \mathbb{R}^p$ be an oriented region in $\mathbb{R}^n$ and $F : U \to M$ be an oriented parametrized $p$-subset of $M$.
• Then $\psi = \varphi \circ F : U \to W$ is an oriented parametrized $p$-subset of $W$.
• Let $\alpha \in \Lambda_p W$ be a $p$-form on $W$.
• Then
  \[ \int_{\psi(U)} \alpha = \int_U \psi^* \alpha = \int_U (F^* \circ \varphi^*) \alpha = \int_U F^* (\varphi^* \alpha) = \int_{F(U)} \varphi^* \alpha \]
• Let $S = F(U)$ be an oriented subset of $M$. Then $\psi(U) = \varphi(F(U)) = \varphi(S)$ is an oriented subset of $W$.
• The general pullback formula takes the form
  \[ \int_{\varphi(S)} \alpha = \int_S \varphi^* \alpha. \]

1.20 Manifolds with Boundary and Stokes’ Theorem

• Recall that an open ball in $\mathbb{R}^n$ is the set
  \[ B_\varepsilon(x_0) = \{ x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon \} \]
• Let us consider also sets
  \[ H_\varepsilon(x_0) = \{ x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon, x^n - x'^n_0 \geq 0 \}. \]
  Such sets are called half-open balls.
• A $n$-dimensional manifold with boundary consists of the interior $M^o$ and the boundary $\partial M$. 
• The interior $M^o$ is a genuine $n$-dimensional manifold such that all its points have neighborhoods diffeomorphic to open balls in $\mathbb{R}^n$.

• The boundary $\partial M$ is a subset of $M$ such that all its points have neighborhoods diffeomorphic to half-open balls.

• Usually, the boundary $\partial M$ is itself an $(n-1)$-dimensional submanifold of $M$ (without boundary).

• Boundary may be disconnected. It can also be not smooth.

• Local coordinates $(x^1, \ldots, x^{n-1}, x^n)$ in $M$ in the neighborhoods of points on the boundary can always be chosen in such a way that $(x^1, \ldots, x^{n-1})$ are the coordinates along the boundary and $0 \leq x^n < \delta$ with some $\delta$.

• A compact manifold is a manifold which is closed and bounded (say, as a submanifold of some $\mathbb{R}^N$).

• A **closed manifold** is a manifold which is compact and does not have a boundary.

• Let $M$ be an $n$-dimensional orientable manifold with boundary $\partial M$, which is an $(n-1)$-dimensional manifold without boundary.

• Let $M$ be oriented.

• Then an orientation on $M$ naturally induces an orientation on $\partial M$.

• Let $p \in \partial M$ and $\{e_2, \ldots, e_n\}$ be a basis in $T_p \partial M$.

• Let $N \in T_p M$ be a tangent vector at $p$ that is transverse to $\partial M$ and points out of $M$.

• Then $\{N, e_2, \ldots, e_n\}$ forms a basis in $T_p M$.

• Then, by definition, the basis $\{e_2, \ldots, e_n\}$ has the same orientation as the basis $\{N, e_2, \ldots, e_n\}$. That is, $\{e_2, \ldots, e_n\}$ is positively oriented in $\partial M$ if $\{N, e_2, \ldots, e_n\}$ is positively oriented in $M$.

• **Stokes’ Theorem.** Let $M$ be an $n$-dimensional manifold and $V$ be a $p$-dimensional compact oriented submanifold with boundary $\partial V$ in $M$. Let $\omega \in \Lambda_{p-1} M$ be a smooth $(p-1)$-form in $M$. Then

\[
\int_V d\omega = \int_{\partial V} \omega.
\]
1.21 Lie Derivative

- Let $X$ be a vector field on a manifold $M$.

- The **Lie derivative** of a tensor of type $(p, q)$ along the vector $X$ is a linear map

  $$L_X : T^p_q M \to T^p_q M$$

  defines as follows.

- Let $\varphi_t : M \to M$ be the flow generated by $X$.

- Since the flow $\varphi_t : M \to M$ is a diffeomorphism it naturally acts on general tensors of type $(p, q)$, that is,

  $$\varphi_t^* : (T^p_q)_{\varphi_t(x)} M \to (T^p_q)_{x} M.$$

- Namely, $\varphi_t^* T$ is a tensor field of type $(p, q)$ defined by

  $$\left(\varphi_t^* T\right)_{i_1...i_p}^{k_1...k_p} = \frac{\partial \varphi_{i_1}^{j_1}(x)}{\partial x^i} \cdots \frac{\partial \varphi_{i_q}^{j_q}(x)}{\partial x^i} \frac{\partial x^{k_1}}{\partial \varphi_{j_1}^{i_1}(x)} \cdots \frac{\partial x^{k_p}}{\partial \varphi_{j_p}^{i_p}(x)} T_{i_1...i_q}^{m_1...m_p}(\varphi_t(x))$$

- Let $T$ be a tensor field of type $(p, q)$ on $M$. The **Lie derivative** of $T$ with respect to $X$ is a tensor field $L_X T$ of type $(p, q)$ defined by

  $$\left(L_X T\right)_x = \frac{d}{dt} \left(\varphi_t^* T\right)_x \bigg|_{t=0}.$$

- The Lie derivative of a tensor field $T$ of type $(p, q)$ with respect to a vector field $X$ is given in local coordinates by

  $$\left(L_X T\right)_{i_1...i_q}^{k_1...k_p} = X^j \partial_j T_{i_1...i_q}^{k_1...k_p} + T_{j_2...j_q}^{k_1...k_p} \partial_{i_1} X^j + \cdots + T_{j_1...j_p}^{k_1...k_p} \partial_{i_q} X^j - T_{i_1...i_q}^{j_1...j_p} \partial_j X^{k_1} - \cdots - T_{i_1...i_q}^{k_1...k_p-1} \partial_j X^{k_p}$$

- The **Lie derivative** of the function $f$ along the vector field $X$ is the rate of change of $f$ along the flow generated by $X$,

  $$L_X f = \mathbf{X}(f).$$
The Lie bracket of two vector fields $X$ and $Y$ is a vector field $[X, Y]$ such that for any smooth function $f$ on $M$

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Notice that $[X, Y] = -[Y, X].$

In local coordinates the Lie bracket is given by

$$[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i.$$

The Lie derivative of the vector field $Y$ along the vector field $X$ is given by

$$L_X Y = [X, Y].$$

Let $M$ be a manifold and $W$ be a submanifold of $M$. Let $X$ and $Y$ be vector fields on $M$ tangent to $W$. Then the Lie bracket $[X, Y]$ is also tangent to $W$.

The Lie derivative of a $p$-form $\alpha$ with respect to a vector field $X$ is given by

$$(L_X \alpha)_{i_1...i_p} = X^j \partial_j \alpha_{i_1...i_p} + \alpha_{j[i_1...i_p} \partial_i X^{j]} + \cdots + \alpha_{i_1...i_p]} \partial_i X^j.$$

In particular, for a 1-form $\alpha$ we have

$$(L_X \alpha)_i = X^j \partial_j \alpha_i + \alpha_j \partial_i X^j.$$

In particular, for a tensor $g_{ij}$ of type $(0, 2)$ we obtain

$$(L_X g)_{ij} = X^k \partial_k g_{ij} + g_{ik} \partial_j X^k + g_{kj} \partial_i X^k.$$

For any two tensors $T$ and $R$ and a vector field $X$ the Leibnitz rule holds

$$L_X(T \otimes R) = (L_X T) \otimes R + T \otimes (L_X R).$$

Let $\alpha$ be a $p$-form, $\beta$ be a $q$-form and $X$ be a vector field on $M$. Then the Leibnitz rule holds

$$L_X(\alpha \wedge \beta) = (L_X \alpha) \wedge \beta + \alpha \wedge (L_X \beta).$$
1.22. AFFINE CONNECTION AND COVARIANT DERIVATIVE

- The Lie derivative commutes with the exterior derivative, that is,
  \[ L_X d = dL_X. \]

- Cartan Formula.
  \[ L_X = i_X d + di_X \]

- Let \( g_{ij} \) be a Riemannian metric on an \( n \)-dimensional manifold \( M \) and
  \[ \text{vol} = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n \]
  be the Riemannian volume form. Let \( X \) be a vector field on \( M \). Then
  \[ L_X \text{vol} = d(i_X \text{vol}) = (\text{div} X) \text{vol}, \]
  where \( \text{div} X \) is a scalar function defined by
  \[ \text{div} X = \ast L_X \text{vol} = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} X^i \right). \]

- The scalar \( \text{div} X \) is called the divergence of the vector field \( X \).

1.22 Affine Connection and Covariant Derivative

- Let \( M \) be an \( n \)-dimensional manifold. An affine connection is an operator
  \[ \nabla : \mathcal{C}^\infty(TM) \times \mathcal{C}^\infty(TM) \rightarrow \mathcal{C}^\infty(TM) \]
  that assigns to two vector fields \( X \) and \( Y \) a new vector field \( \nabla_X Y \), that is linear in both variables, that is, for any \( a, b \in \mathbb{R} \) and any vector fields \( X, Y \) and \( Z \),
  \[
  \nabla_X (aY + bZ) = a \nabla_X Y + b \nabla_X Z \\
  \nabla_{aX + bY} Z = a \nabla_X Z + b \nabla_Y Z
  \]
  and satisfies the Leibnitz rule, that is, for any smooth function \( f \in \mathcal{C}^\infty(M) \) and any two vector fields \( X \) and \( Y \),
  \[
  \nabla_X (fY) = (X(f))Y + f \nabla_X Y \\
  = (df)(X)Y + f \nabla_X Y
  \]
• Let $x^\mu$, $\mu = 1, \ldots, n$, be local coordinates and $\partial_\mu$ be the basis of vector fields. It will be called a **coordinate frame** for the tangent bundle.

• A basis of vector fields $e_i = e^\mu_i \partial_\mu$, $i = 1, \ldots, n$, for the tangent bundle $TM$ is called a **frame**.

• Let $\sigma^j = \sigma^j_\mu dx^\mu$ be the dual frame of 1-forms.

• Then
  \[ \sigma^j(e_j) = \sigma^j_\mu e^\mu_j = \delta^i_j \]
  and
  \[ \sigma^j_\mu e^\nu_i = \delta^\nu_\mu. \]

• We will denote the action of frame vector fields on functions by
  \[ e_i(f) = e^\mu_i \partial_\mu = f_i. \]

• For any frame the commutator of the frame vector fields defines the **commutation coefficients**
  \[ [e_i, e_j] = C^k_{ij} e_k. \]

• In components,
  \[ e^\nu_j - e^\nu_i = e^\mu_i \partial_\mu e^\nu_j - e^\mu_j \partial_\mu e^\nu_i = C^k_{ij} e^\nu_k. \]
  That is,
  \[ C^k_{ij} = \sigma^k(\sigma^i_j e_k) = \sigma^k \left( e^\mu_i \partial_\mu e^\nu_j - e^\mu_j \partial_\mu e^\nu_i \right). \]

• The symbols $\omega^j_{ik}$ defined by
  \[ \omega^j_{ik} = \sigma^j(\nabla e_i e_k) \]
  are called the **coefficients of the affine connection**.

• We denote
  \[ \nabla_i = \nabla_{e_i}. \]

• Then
  \[ \nabla_i e_j = \omega^k_{ji} e_k, \]
Then, if \(X = X^i e_i\) is a vector field, then
\[\nabla_X = X^i \nabla_i.\]

If \(Y = Y^j e_j\) is another vector field then
\[\nabla_X Y = X^i \nabla_i Y^k e_k\]
where
\[\nabla_i Y^k = Y^k_{|i} + \omega^k_{ji} Y^j.\]

The tensor field \(\nabla Y\) of type \((1, 1)\) with components \(\nabla_i Y^k\) is called the covariant derivative of the vector field \(Y\).

In the coordinate frame \(e_i = \partial_i\) the covariant derivative takes the form
\[\nabla_i Y^k = \partial_i Y^k + \omega^k_{ji} Y^j.\]

### 1.23 Curvature, Torsion and Levi-Civita Connection

Let \(X\) and \(Y\) be vector fields on a manifold \(M\). Then the vector field
\[T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\]
defines a tensor field \(T\) of type \((1, 2)\), called the torsion, so that for any 1-form \(\sigma\)
\[T(\sigma, X, Y) = \sigma(T(X, Y)).\]

The affine connection is called torsion-free (or symmetric) if the torsion vanishes, that is, for any \(X, Y\),
\[\nabla_X Y - \nabla_Y X = [X, Y]\]

The components of the torsion tensor are defined by
\[T^i_{jk} = \sigma^i(T(e_j, e_k)).\]

In the coordinate frame the components of the torsion tensor are given by
\[T^i_{jk} = \omega^i_{kj} - \omega^j_{ik}.\]
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• Let $X$, $Y$ and $Z$ be vector fields on a manifold $M$. Then the vector field
  \[ R(X, Y)Z = \left[ [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \right] Z \]
defines a tensor field $R$ of type $(1, 3)$, called the \textbf{Riemann curvature}, so that for any 1-form $\sigma$
  \[ R(\sigma, Z, X, Y) = \sigma(R(X, Y)Z). \]

• The affine connection is called \textbf{flat} if the curvature vanishes, that is, for any $X, Y, Z$,
  \[ [\nabla_X, \nabla_Y]Z = \nabla_{[X,Y]}Z. \]

• The components of the curvature tensor are defined by
  \[ R^i_{jkl} = \sigma^j(R(e_k, e_l)e_j). \]

• The components of the curvature tensor have the form
  \[ R^i_{jkl} = \omega^j_{jk}\omega^i_{kl} + \omega^i_{mk}\omega^m_{jl} - \omega^i_{ml}\omega^m_{jk} - C^m_{kl}\omega^i_{jm}. \]

• In the coordinate frame the components of the curvature tensor are given by
  \[ R^i_{jkl} = \partial_k\omega^j_{jl} - \partial_l\omega^j_{jk} + \omega^i_{mk}\omega^m_{jl} - \omega^i_{ml}\omega^m_{jk}. \]

• For a Riemannian manifold $(M, g)$ the metric tensor $g$ has the components
  \[ g_{ij} = g(e_i, e_j) = g_{\mu\nu}e^i_{\mu}e^j_{\nu}. \]

• This metric is used to lower and raise the frame indices.

• Let $(M, g)$ be a Riemannian manifold and $\nabla$ be an affine connection on $M$. Then the connection $\nabla$ is called \textbf{compatible with the metric} $g$ if for any vector fields $X, Y$ and $Z$ it satisfies the condition
  \[ Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \]

• An affine connection that is torsion-free and compatible with the metric is called the \textbf{Levi-Civita connection}. 
• Each Riemannian manifold has a unique Levi-Civita connection.

• We define
  \[ \omega_{ijk} = g_{im} \omega^m_{jk} \quad C_{ijk} = g_{im} C^m_{jk}. \]

• The coefficients of the Levi-Civita connection are given by
  \[ \omega_{ijk} = \frac{1}{2} \left( g_{ij|k} + g_{ik|j} - g_{jk|i} + C_{kij} + C_{jik} - C_{ijk} \right). \]

• The coefficients of the Levi-Civita connection in a coordinate frame are called Christoffel symbols and denoted by \( \Gamma^i_{jk} \)
  \[ \Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk} \right). \]

Christoffel symbols have the following symmetry property
  \[ \Gamma^i_{jk} = \Gamma^i_{kj}. \]

• The coefficients of the Levi-Civita connection in an orthonormal frame have the form
  \[ \omega_{ijk} = \frac{1}{2} \left( C_{kij} + C_{jik} - C_{ijk} \right). \]

They have the following symmetry properties
  \[ \omega_{ijk} = -\omega_{jik}. \]

• In a coordinate basis for a torsion-free connection we have the following identities (called the Ricci identities):
  \[ [\nabla_i, \nabla_j]Y^k = R^k_{lij} Y^l. \]

### 1.24 Parallel Transport

• Let \( x_0 \) and \( x_1 \) be two points on a manifold \( M \) and \( C \) be a smooth curve connecting these points described locally by \( x^i = x^i(t) \), where \( t \in [0, 1] \) and \( x(0) = x_0 \) and \( x(1) = x_1 \). The tangent vector to \( C \) is defined by
  \[ X = \dot{x}(t), \]

where the dot denotes the derivative with respect to \( t \).
Let $Y$ be a vector field on $M$. We say that $Y$ is parallel transported along $C$ if

$$\nabla_X Y = 0.$$  

The vector field $Y$ is parallel transported along $C$ if its components satisfy the linear ordinary differential equation

$$\frac{d}{dt} Y^i(x(t)) + \omega^i_{jk}(x(t)) \dot{x}^j(t) Y^j(x(t)) = 0.$$  

A curve $C$ such that the tangent vector to $C$ is transported parallel along $C$, that is,

$$\nabla_{\dot{x}} \dot{x} = 0,$$

is called the geodesics.

The coordinates of the geodesics $x = x(t)$ satisfy the non-linear second-order ordinary differential equation

$$\ddot{x}^i + \omega^i_{jk}(x(t)) \dot{x}^j \dot{x}^k = 0.$$  

### 1.25 Covariant Derivative of Tensors

- The affine connection on a tensor bundle $T^p_q M$ is an operator

$$\nabla : C^\infty(TM) \times C^\infty(T^p_q M) \to C^\infty(T^p_q M)$$

that assigns to a vector field $X$ and a tensor field $T$ of type $(p, q)$ a new tensor field $\nabla_X T$ of type $(p, q)$.

- The covariant derivative of a tensor field $T$ of type $(p, q)$ is a linear operator

$$\nabla : C^\infty(T^p_q M) \to C^\infty(T^{p+1}_{q+1} M)$$

that assigns to a tensor field $T$ of type $(p, q)$ a new tensor field $\nabla T$ of type $(p, q + 1)$.

- First of all, the covariant derivative of a 1-form $\alpha$ on a manifold $M$ is a tensor $\nabla \alpha$ of type $(0, 2)$ such that for any two vector fields $X$ and $Y$

$$(\nabla \alpha)(X, Y) = (\nabla_X \alpha)(Y) = X[\alpha(Y)] - \alpha(\nabla_X Y).$$
Then the covariant derivative of a tensor \( T \) of type \((p, q)\) is a tensor \( \nabla T \) of type \((p, q + 1)\) such that for any vector fields \( X, Y_1, \ldots, Y_q \) and 1-forms \( \omega_1, \ldots, \omega_p \)

\[
(\nabla T)(X, Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p) = (\nabla_X T)(X, Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)
\]

\[
= X[T(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)]
\]

\[
- \sum_{i=1}^{q} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_q, \omega_1, \ldots, \omega_p)
\]

\[
- \sum_{j=1}^{p} T(Y_1, \ldots, Y_q, \omega_1, \ldots, \nabla_X \omega_j, \ldots, \omega_p)
\]

- Let \( e_i \) be a basis of vector fields and \( \sigma^j \) be the dual basis of 1-forms.
- The covariant derivative of a 1-form has the form

\[
(\nabla_i \alpha)_k = e_i(\alpha_k) - \omega^j_{ki} \alpha_l
\]

In the coordinate frame this simplifies to

\[
(\nabla_i \alpha)_k = \partial_i \alpha_k - \omega^j_{ki} \alpha_l
\]

- In a coordinate basis for a torsion-free connection we have the following identities (called the Ricci identities):

\[
[\nabla_i, \nabla_j] \alpha_k = -R^l_{ki} \alpha_l
\]

- The covariant derivative of a tensor of type \((p, q)\) in a coordinate basis has the form

\[
\nabla_i T_{k_1 \ldots k_q}^{j_1 \ldots j_p} = \partial_i T_{k_1 \ldots k_q}^{j_1 \ldots j_p} + \sum_{m=1}^{p} \omega_{im}^j T_{k_1 \ldots k_q}^{j_m \ldots j_{m+1} \ldots j_p} - \sum_{n=1}^{q} \omega_{nk}^j T_{k_1 \ldots k_{n-1} k_n k_q}^{j_1 \ldots j_p}
\]

- In a coordinate basis for a torsion-free connection we have the following identities (called the Ricci identities):

\[
[\nabla_i, \nabla_j] T_{k_1 \ldots k_q}^{j_1 \ldots j_p} = \sum_{m=1}^{p} R_{mij}^{kl} T_{k_1 \ldots k_q}^{j_n \ldots j_{m+1} \ldots j_p} - \sum_{n=1}^{q} R_{knj}^{l} T_{k_1 \ldots k_{n-1} k_n k_q}^{j_1 \ldots j_p}
\]
• The parallel transport of tensor fields is defined similarly to vector fields. Let \( C \) be a smooth curve on a manifold \( M \) described locally by \( x^i = x^i(t) \), where \( t \in [0, 1] \), with the tangent vector \( \mathbf{X} = \dot{x}(t) \).

• Let \( T \) be a tensor field on \( M \). We say that \( T \) is parallel transported along \( C \) if
\[
\nabla_\mathbf{X}T = 0.
\]

1.26 Properties of the Curvature Tensor

• Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold. We will restrict ourselves to the Levi-Civita connection below. We define some new curvature tensors.

• The Ricci tensor
\[
R_{ij} = R^k_{\ ij\ k}.
\]

• The scalar curvature
\[
R = g^{ij}R_{ij} = g^{ij}R^k_{\ ik\ j}.
\]

• The Einstein tensor
\[
G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R.
\]

• The trace-free Ricci tensor
\[
E_{ij} = R_{ij} - \frac{1}{n}g_{ij}R.
\]

• The Weyl tensor (for \( n > 2 \))
\[
C^{ij}_{\ kl} = R^{ij}_{\ kl} - \frac{4}{n-2}R^{ij}_{\ [k\delta^l]l} + \frac{2}{(n-1)(n-2)}R^i_{\ [k\delta^l]l}^j + \frac{2}{n(n-1)}R^i_{\ [k\delta^l]l}^j - \frac{2}{n(n-1)}R^i_{\ [k\delta^l]l}^j
\]

• The Riemann curvature tensor of the Levi-Civita connection has the following symmetry properties
1. \( R_{ijkl} = -R_{ijlk} \)
2. \( R_{ijkl} = -R_{jikl} \)
3. \( R_{ijkl} = R_{klij} \)
4. \( R^{i}_{[jkl]} = R^{i}_{jkl} + R^{i}_{kli} + R^{i}_{ljk} = 0 \)
5. \( R_{ij} = R_{ji} \)

- The Weyl tensor has the same symmetry properties as the Riemann tensor and all its contractions vanish, that is,
  \[ C^{i}{}_{jik} = 0 . \]

- The number of algebraically independent components of the Riemann tensor of the Levi-Civita connection is equal to
  \[ n^2(n^2 - 1) \frac{1}{12} . \]

- In dimension \( n = 2 \) the Riemann tensor has only one independent component determined by the scalar curvature. The trace-free Ricci tensor \( E_{ij} \) vanishes, that is,
  \[ R^{i}{}_{jkl} = R^{j}{}_{[k}{}_{l]} \]
  \[ R_{ij} = \frac{1}{2} R g_{ij} . \]

- In dimension \( n = 3 \) the Riemann tensor has six independent components determined by the Ricci tensor \( R_{ij} \). The Weyl tensor \( C_{ijkl} \) vanishes, that is,
  \[ R^{i}{}_{jkl} = 4 R^{i}{}_{[k}{}_{l]} + R^{j}{}_{[k}{}_{l]} . \]

- The Riemann tensor satisfies the following identities

\[ \nabla_{[m} R^{i}{}_{jkl]} = \nabla_{m} R^{i}{}_{jkl} + \nabla_{k} R^{i}{}_{m}{}_{ln} + \nabla_{l} R^{i}{}_{m}{}_{nk} = 0 . \]

These identities are called the **Bianchi identities**.

- The divergences of the Riemann tensor and the Ricci tensor have the form
  \[ \nabla_{i} R^{i}{}_{jkl} = \nabla_{i} R^{j}{}_{l} - \nabla_{j} R^{i}{}_{k} , \]
  \[ \nabla_{i} R^{j}{}_{i} = \frac{1}{2} \nabla_{j} R . \]

- The divergence of the Einstein tensor vanishes
  \[ \nabla_{i} G^{i}{}_{j} = 0 . \]
1.27 Cartan’s Structural Equations

- Let $\partial_\mu$ be a coordinate basis for the tangent bundle $TM$ and $e_i = \delta^\mu_i \partial_\mu$ be an orthonormal frame of vector fields and $\sigma^i = \sigma^i_\mu dx^\mu$ be the dual orthonormal basis so that

$$g^{ij} = g_{ij} = \delta_{ij}.$$ 

- The coefficients of the Levi-Civita connection in an orthonormal frame are given in terms of the commutation coefficients by

$$\omega_{ijk} = \frac{1}{2} \left( C^{kij} + C^{jik} - C^{ijk} \right).$$

- There holds

$$d\sigma^i = \omega^i_{jk} \sigma^j \wedge \sigma^k = -\frac{1}{2} C^{i}_{jk} \sigma^j \wedge \sigma^k.$$

- Now we define the connection 1-forms

$$\mathcal{A}^i_j = \omega^i_{jk} \sigma^k$$

and the curvature 2-forms

$$\mathcal{F}^i_j = \frac{1}{2} R^i_{jk} \sigma^k \wedge \sigma^j.$$

- Then we have Cartan’s first structural equation

$$d\sigma^i + \mathcal{A}^i_j \wedge \sigma^j = 0.$$ 

- The curvature 2-forms are obtained from the connection 1-forms by Cartan’s second structural equation

$$\mathcal{F}^i_j = d\mathcal{A}^i_j + \mathcal{A}^i_k \wedge \mathcal{A}^k_j.$$ 

- Cartan’s third structural equation

$$d\mathcal{F}^i_j + \mathcal{A}^i_k \wedge \mathcal{F}^k_j - \mathcal{F}^i_k \wedge \mathcal{A}^k_j = 0.$$ 

is equivalent to Bianci identities.
• **Exercise.** Let the dimension $n = 2k$ of the manifold $M$ be even. We define the following $2l$-forms

$$\Omega_{(l)} = \text{tr} \ F \wedge \cdots \wedge F$$

and the $n$-form

$$\Omega = \varepsilon^{i_1\ldots i_{2k}} F_{i_1 i_2} \wedge \cdots \wedge F_{i_{2k-1} i_{2k}}$$

1. Prove that these forms are independent of the orthonormal basis and are closed, that is,

$$d\Omega_{(l)} = d\Omega = 0$$

2. Find the expressions in local coordinates for these forms.

• These forms define so called **characteristic classes**, which are closed invariant forms whose integrals over the manifold do not depend on the metric and, therefore, are topological invariants of the manifold.
Chapter 2

Poincaré, Integrability, Degree

2.1 Poincaré Lemma

**Definition 2.1.1** Let $M$ be a manifold. A $p$-form $\alpha$ on $M$ is called **closed** if $d\alpha = 0$. A $p$-form $\alpha$ on $M$ is called **exact** if there is a $(p - 1)$-form $\beta$ such that $\alpha = d\beta$. The form $\beta$ is called a **potential** of $\alpha$. 
Theorem 2.1.1

1. Every exact form is closed (Poincaré Lemma).
   That is, if \( \alpha = d\beta \), then \( d\alpha = 0 \).

2. The exterior product of closed forms is closed.
   That is, if \( d\alpha = 0 \) and \( d\beta = 0 \), then \( d(\alpha \wedge \beta) = 0 \).

3. The exterior product of a closed form and an exact form is exact.
   That is, if \( d\alpha = 0 \) and \( \beta = d\gamma \), then there is \( \sigma \) such that \( \alpha \wedge \beta = d\sigma \).

4. Let \( M \) be an \( n \)-dimensional orientable compact manifold without boundary and \( \alpha \) be an exact \( n \)-form on \( M \), that is, \( \alpha = d\beta \) for some \((n-1)\)-form \( \beta \). Then
   \[
   \int_M \alpha = 0.
   \]

5. Let \( M \) be an \( n \)-dimensional oriented compact manifold with boundary \( \partial M \) and \( \alpha \) be a closed \((n-1)\)-form on \( M \), that is, \( d\alpha = 0 \). Then
   \[
   \int_{\partial M} \alpha = 0.
   \]

**Proof**: Use Stokes theorem.

- **Example.** On \( \mathbb{R}^2 - \{0\} \)

\[
\beta = \frac{xdy - ydx}{x^2 + y^2}
\]

Why is \( \beta \neq d\theta \)?

Then

\[
d\beta = 0
\]

\[
\oint_C d\beta = 2\pi
\]

if 0 is inside \( C \) and 0 if 0 is outside \( C \).
Definition 2.1.2  Let $M$ be a manifold. Suppose that for every closed oriented smooth curve $C$ there is a smooth oriented $2$-dimensional surface $S$ and a map $F : S \to M$ such that $\partial F(S) = C$, that is, the curve $C$ is the boundary of the surface $S$. Then the manifold $M$ is said to have first Betti number equal to zero, $b_1 = 0$.

Example. For $T^2$

$$b_1 \neq 0.$$  

Theorem 2.1.2  Let $M$ be a manifold with first Betti number equal to zero. Then every closed $1$-form on $M$ is exact. That is, if $\alpha$ is a $1$-form such that $d\alpha = 0$, then there is a function $f$ such that $\alpha = df$.

Proof:

1. Let $\alpha$ be a closed one-form on $M$.
2. Let $x$ and $y$ be two points in $M$ and $C_{xy}$ be an oriented curve with the initial point $y$ and the final point $x$.
3. Let $f$ be defined by
   $$f(x) = \int_{C_{xy}} \alpha.$$  
4. Then $f$ is independent on the curve $C_{xy}$ and so is well defined.
5. Finally, we show that $df = \alpha$.

\[\square\]

Theorem 2.1.3  Let $\alpha \in \lambda_1 M$. Suppose that for any closed curve $C$

$$\int_C \alpha = 0.$$  

Then $\alpha$ is exact.

Theorem 2.1.4  Let $\alpha$ be a closed $p$-form in $\mathbb{R}^n$. Then there is $(p-1)$-form $\beta$ in $\mathbb{R}^n$ such that $\alpha = d\beta$.

That is, every closed form in $\mathbb{R}^n$ is exact.

Proof:
1. Let $\alpha$ be a closed $p$-form in $\mathbb{R}^n$.
2. We define a $(p - 1)$-form $\beta$ by
   \[
   \beta_{i_1...i_{p-1}}(x) = \int_0^1 d\tau \tau^{p-1} x^j \alpha_{j i_1...i_{p-1}}(\tau x)
   \]
3. We can show that $d\beta = \alpha$.

**Corollary 2.1.1** Let $M$ be a manifold and $\alpha$ be a closed $p$-form on $M$. Then for every point $x$ in $M$ there is a neighborhood $U$ of $x$ and a $(p - 1)$-form $\beta$ on $M$ such that $\alpha = d\beta$ in $U$.

**Proof:**

1. Use the fact that a sufficiently small neighborhood of a point in $M$ is diffeomorphic to an open ball in $\mathbb{R}^n$.
2. Pullback the form $\alpha$ from $M$ to $\mathbb{R}^n$ by the pullback $F^*$ of the diffeomorphism $F : V \to U$, where $U \subset M$ and $V \subset \mathbb{R}^n$.
3. Use the previous theorem.

### 2.1.1 Complex Analysis

- Let $M = \mathbb{C}^2$ and
  \[ z = x + iy, \]
  Then
  \[ dz = dx + idy, \quad \bar{dz} = dx - idy \]
  \[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \]
  \[ \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \]
  and
  \[ dz \wedge \bar{dz} = -2i dx \wedge dy. \]
2.1. POINCARÉ LEMMA

• Let \( f = u + iv \) be a function. Then

\[
fdz = udx - vdy + i(udy + vdx)
\]

and

\[
d(fdz) = (-u_y - v_x)dx \wedge dy + i(u_x - v_y)dx \wedge dy = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz
\]

• So,

\[
\int_C fdz = \int_C [udx - vdy + i(udy + vdx)] = \int_S d(fdz)
\]

• Thus, \( fdz \) is closed if and only if \( f \) is holomorphic (satisfies cauchy-Riemann equations)

\[
u_x = v_y, \quad u_y = -v_x
\]

or

\[
\frac{\partial f}{\partial \bar{z}} = 0
\]

• The system of pde

\[
\partial_i A_j - \partial_j A_i = F_{ij} \quad (dA = F)
\]

can be solved if and only if

\[
\partial_{[k} F_{ij]} = 0 \quad (dF = 0).
\]
2.2 Degree of a Map

2.2.1 Gauss-Bonnet Theorem

- Let us consider a compact oriented two-dimensional manifold $M$ (a surface embedded in $\mathbb{R}^3$).
- Let $x^i, i = 1, 2, 3$, be the Cartesian coordinates in $\mathbb{R}^3$ and $u^\mu, \mu = 1, 2$, be the local coordinates on $M$.
- Let $F : M \to \mathbb{R}^3$ be the embedding map defined locally by
  $$x^i = F^i(u).$$
- The differential of the map $F$ is given by the matrix
  $$(F_*)^i_\mu = \frac{\partial x^i}{\partial u^\mu}.$$
- We assume that $F_*$ is onto, that is, rank $F_* = 2$.
- The tangent space $T_xM$ is spanned by the vectors $e_\mu, \mu = 1, 2$, with components
  $$e^i_\mu = \frac{\partial x^i}{\partial u^\mu}.$$
- Let $\delta_{ij}$ be the Euclidean metric in $\mathbb{R}^3$.
- Then the induced metric on $M$ is defined by
  $$g_{\mu\nu} = (e_\mu, e_\nu) = \delta_{ij} e^i_\mu e^j_\nu.$$  
  This matrix is also called the first fundamental form. It describes the intrinsic geometry of the surface.
- Let $N$ be the vector in $\mathbb{R}^3$
  $$N = e_1 \times e_2$$
  with components
  $$N_i = \varepsilon_{ijk} e^j_1 e^k_2 = \varepsilon_{ijk} \frac{\partial x^j}{\partial u^1} \frac{\partial x^k}{\partial u^2}.$$
• Then $N$ is a vector field that is everywhere normal to $M$.

• Notice that the norm of the normal vector is
  \[ ||N||^2 = ||e_1||^2 ||e_2||^2 - (e_1, e_2)^2. \]

• The **second fundamental form**, or the **extrinsic curvature** is defined by the matrix
  \[ b_{\mu\nu} = \frac{1}{||N||} \left( \frac{\partial e_\mu}{\partial u^\nu}, N \right) = \frac{1}{||N||} \frac{\partial^2 x^i}{\partial u^\mu \partial u^\nu} N_i. \]

• The second fundamental form describes the **extrinsic geometry** of the surface $M$.

• The **mean curvature** of $M$ is defined by
  \[ H = g^{\mu\nu} b_{\mu\nu}. \]

• The **Gauss curvature** of $M$ is defined by
  \[ K = \frac{\det b_{\mu\nu}}{\det g_{\alpha\beta}}. \]

• Gauss has shown that the $K$ is an intrinsic invariant. In fact,
  \[ K = R^{12}_{12} = \frac{1}{2} R, \]

  where $R^{12}_{12}$ is the only non-vanishing components of the Riemann curvature of the metric $g$ and $R$ is the scalar curvature. This will be discussed later.

• The **Gauss map** is the map $\varphi : M \to S^2$ from $M$ to $S^2$ defined by
  \[ \varphi(x) = \frac{N(x)}{||N(x)||}, \]

  that is, it associates to every point $x$ in $M$ the unit normal vector at that point.
Theorem 2.2.1  Let $M$ be a closed (compact without boundary) oriented 2-dimensional surface embedded in $\mathbb{R}^3$. Let $g_{\mu\nu}$ be the induced Riemannian metric, $d\text{vol} = \sqrt{|g|} dx$ be the Riemannian volume element on $M$, $K$ be the Gaussian curvature of $M$ and $\varphi : M \to S^2$ be the Gauss normal map. Then

$$\frac{1}{4\pi} \int_M d\text{vol} \ K = \deg(\varphi)$$

is an integer that does not depend on the metric and does not change under smooth deformations of the surface.

Proof: Later.

• Remark. For a 2-surface of genus $g$ (with $g$ holes)

$$\deg(\varphi) = 1 - g .$$

• The Euler characteristic of the surface $M$ is a topological invariant equal to

$$\chi = 2(1 - g) .$$

2.2.2 Laplacian

• Let $g_{ij}$ be a Riemannian metric on a manifold $V$. Let $h$ be a function and $X$ be its gradient vector field defined by

$$X^i = g^{ij} \partial_j h .$$

• The Laplacian operator $\Delta$ on the scalar function $h$ is defined by

$$\Delta h = \text{div} X .$$

• In local coordinates

$$\Delta h = g^{-1/2} \partial_i (g^{1/2} g^{ij} \partial_j h) ,$$

where $g = \det g_{ij}$.

• The operator $(-\Delta) : C^\infty(V) \to C^\infty(V)$ acting on smooth functions on a compact manifold $V$ without boundary can be extended to a self-adjoint operator on the Hilbert space $L^2(V)$.
• It has a discrete non-negative real spectrum \(\{\lambda_k\}_{k=0}^\infty\) bounded from below by zero

\[0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots\]

with finite multiplicities.

• The eigenfunctions \(\{h_k\}_{k=0}^\infty\) form an orthonormal basis in \(L^2(V)\), that is,

\[(h_k, h_l) = \int_V d\text{vol } h_k h_l = \delta_{kl}.

• For each \(f \in L^2(V)\) there is a Fourier series

\[f = \sum_{k=0}^\infty a_k h_k,
\]

where

\[a_k = (h_k, f) = \int_V f h_k.

• The lowest eigenvalue is 0. It is simple (has multiplicity 1). The corresponding eigenfunction is the constant \(h_0 = [\text{vol}(V)]^{-1/2}\).

\begin{lemma}
Let \(f\) be a function on a closed manifold \(M\) such that \(\int_M f = 0\). Let \(\{\lambda_k, h_k\}_{k=1}^\infty\) be the spectral resolution of the operator \((-\Delta)\).

Then the equation \(\Delta h = f\) has a unique solution given by the Fourier series

\[h = -\sum_{k=2}^\infty \frac{1}{\lambda_k} a_k h_k,
\]

where

\[a_k = (h_k, f) = \int_M f h_k.
\]

\textbf{Proof}: 1.

\end{lemma}
Lemma 2.2.2  Let $V$ be a closed (compact without boundary) oriented $n$-dimensional manifold and $\omega$ be an $n$-form on $V$. Then $\omega$ is exact if and only if

$$\int_{V} \omega = 0.$$ 

Proof:

1. (I). If $\omega$ is exact, then $\omega = d\sigma$ for some $(n-1)$-form $\sigma$.

2. Then by Stokes theorem

$$\int_{V} \omega = 0.$$ 

3. (II). Suppose that

$$\int_{V} \omega = 0.$$ 

4. Let us fix a Riemannian metric on $V$ and let $\text{vol}$ be a Riemannian volume element.

5. Then

$$\omega = f \text{vol}$$

for some function $f$ which satisfies

$$\int_{V} f = 0.$$ 

6. Let $h$ be the solution of the equation

$$\Delta h = f.$$ 

7. Let $X$ be the gradient of the function $h$.

8. We define an $(n-1)$-form $\sigma$ by

$$\sigma = i_{X} \text{vol}.$$ 

9. Then

$$d\sigma = \omega.$$ 

2.2.3 Brouwer Degree

- Let $M$ and $V$ be two closed (compact without boundary) oriented $n$-dimensional manifolds.
- Let $\varphi : M \to V$ be a smooth map.
- Let $\omega \in \Lambda_n V$ be an $n$-form on $V$.
- Suppose that
  \[ \int_V \omega \neq 0. \]

  Then we can normalize it so that
  \[ \int_V \omega = 1. \]

- Then we can consider the quantity
  \[ \frac{\int_M \varphi^* \omega}{\int_V \omega}, \]

  which can also be written as
  \[ \frac{\int_{\varphi(M)} \omega}{\int_V \omega}. \]

- This quantity counts how many times the image of $M$ wraps around $V$.

**Corollary 2.2.1** Let $M$ and $V$ be $n$-dimensional manifolds and $\varphi : M \to V$ be a smooth map. Let $\alpha$ and $\beta$ be $n$-forms on $V$ such that $\int_V \alpha \neq 0$ and $\int_V \beta \neq 0$. Then

\[ \frac{\int_M \varphi^* \alpha}{\int_V \alpha} = \frac{\int_M \varphi^* \beta}{\int_V \beta}. \]

**Proof:**

1. Let
   \[ \omega = \left( \frac{\alpha}{\int_V \alpha} - \frac{\beta}{\int_V \beta} \right). \]
2. We have
\[ \int_V \omega = 0. \]
3. Therefore, the form \( \omega \) is exact.
4. Hence, the form \( \varphi^* \omega \) is exact.
5. Thus,
\[ \int_M \varphi^* \omega = 0. \]

The quantity
\[ \deg(\varphi) = \frac{\int_M \varphi^* \omega}{\int_V \varphi} \]
does not depend on the choice of the form \( \omega \) but only on the map \( \varphi \). It is called the Brouwer degree of the map \( \varphi \).

**Example.**

In the case of one-dimensional manifolds the degree of the map \( \varphi : M \to S^1 \) is called the **winding number**.

**Picture.**

**Theorem 2.2.2** Let \( V \) and \( M \) be \( n \)-dimensional compact oriented manifolds without boundary. Let \( \varphi : M \to V \) be a smooth map. Let \( y \in V \) be a regular value of \( \varphi \) so that the differential \( \varphi_\ast : T_x M \to T_y V \) at any point \( x \in \varphi^{-1}(y) \) is bijective (isomorphism). Then
\[ \deg(\varphi) = \sum_{x \in \varphi^{-1}(y)} \text{sign } (\varphi_\ast(x)), \]
where
\[ \text{sign } (\varphi_\ast(x)) = \text{sign } (\det(\varphi_\ast)). \]

**Proof:**

1. (I). Claim: the preimage \( \varphi^{-1}(y) \) of a regular value is a finite set, that is,
\[ \varphi^{-1}(y) = \{ x_i \in M \mid \varphi(x_i) = y, \ i = 1, 2, \ldots, N \}. \]
2. Suppose $\varphi^{-1}(y)$ is infinite.

3. Then by compactness argument $\varphi^{-1}(y)$ has a limit point $x_0 \in M$, which is a regular point.

   Indeed, every sequence has a convergent subsequence. Thus, there is a sequence $(x_k)$, such that $x_k \in \varphi^{-1}(y)$, converging to some $x_0$. $x_k \to x_0$, so that $\varphi(x_0) = y$. Thus $x_0$ is a regular point of $M$.

4. Since $\varphi_* : T_{x_0}M \to T_yV$ is bijective at $x_0$, the map $\varphi : M \to V$ is a \textit{diffeomorphism} in a neighborhood of $x_0$.

   That is, $\det \varphi_*|_{x_0} \neq 0$.

5. This contradicts the fact that there is sequence of points $x_k \to x_0$ such that $\varphi(x_k) = y$.

   Since, otherwise there exist infinitely many points $x_k \in M$ such that $\varphi(x_k) = \varphi(x_0) = y$ contradicting the fact that $\varphi$ is bijective.

6. (II). Claim: The point $y \in V$ has a neighborhood whose inverse image is a disjoint union of neighborhoods of the preimages of $y$, each of which is diffeomorphic to $V_y$.

7. Let $W_i$ be disjoint neighborhoods of $x_i \in M$ such that $\varphi : W_i \to V_i = \varphi(W_i)$ are diffeomorphisms.

8. Let $S = \varphi(M - \bigcup_{i=1}^{N} W_i)$.

9. Since $M - \bigcup_{i=1}^{N} W_i$ is compact, then $S$ is compact (and closed in $V$).

10. Let $O = V - S$.

11. Then $O$ is open and is a neighborhood of $y$.

12. Now, we define

   $$V_y = O \cap \bigcap_{i=1}^{N} V_i.$$  

13. Then

   $$V_y \subset O, \quad V_y \subset V_i.$$  

14. Let

   $$U_i = \varphi^{-1}(O) \cap W_i.$$  

15. Then

   $$U_i \subset W_i \subset M.$$
16. Then
\[ \varphi^{-1}(V_y) = \bigcup_{i=1}^{N} U_i, \]
and \( \varphi : U_i \to V_y \) are diffeomorphisms.

17. (III). Let \( \omega \) be an \( n \)-form on \( V \) with support in \( V_y \) such that \( \int_{V_y} \omega = 1 \).

18. Let \( y^i, i = 1, \ldots, n, \) be local coordinates in \( V_y \).

19. Since each \( \varphi : U_i \to V_y \) are diffeomorphisms, one can use \( y^i \) as local coordinates on \( U_i \).

20. In such coordinates the diffeomorphism \( \varphi : U_i \to V_y \) is the identity map.

21. We notice that the orientation of \( U_i \) is described by \( \text{sign} \varphi(x_i) \).

22. Thus
\[
\text{deg}(\varphi) = \int_M \varphi^* \omega = \sum_{i=1}^{N} \int_{U_i} \varphi^* \omega = \sum_{i=1}^{N} \int_{\varphi(U_i)} \omega = \sum_{i=1}^{N} \text{sign} \varphi(x_i) \int_{V_y} \omega = \sum_{i=1}^{N} \text{sign} (\varphi(x_i)) .
\]

\[ \blacksquare \]

**Corollary 2.2.2**

1. The Brouwer degree \( \text{deg}(\varphi) \) of any map \( \varphi : M \to V \) is an integer.

2. The sum
\[ \sum_{x \in \varphi^{-1}(y)} \text{sign} (\varphi(x)) \]

is independent on the regular value \( y \in V \).

3. The Brouwer degree \( \text{deg}(\varphi) \) remains constant under continuous deformations of the map \( \varphi \).

**Exercise.** Let \( \omega \) be the volume form on \( S^n \) in \( \mathbb{R}^{n+1} \) given by
\[ \omega = \sum_{k=1}^{n+1} (-1)^k x^k dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1} . \]

Show that the antipodal map \( \varphi : S^n \to S^n \) has degree \( \text{deg}(\varphi) = (-1)^{n+1} \).
Exercise. Let $M$ be a closed oriented $n$-dimensional manifold and $\varphi : M \to S^n$ be a smooth map. We identify $\varphi(x)$ with a unit vector field $v$ on $S^n$ in $\mathbb{R}^{n+1}$. Let $\text{vol}$ be the volume $(n+1)$-form in $\mathbb{R}^{n+1}$. Let $\text{vol}(S^n)$ be the volume of the unit sphere $S^n$. Let $u^\mu, \mu = 1, \ldots, n$, be local coordinates on $M$. Show that

$$\text{deg}(\varphi) = \frac{1}{\text{vol}(S^n)} \int_M \text{vol} \left( v, \frac{\partial v}{\partial u^1}, \ldots, \frac{\partial v}{\partial u^n} \right) du^1 \wedge \cdots \wedge du^n$$

2.2.4 Index of a Vector Field

- Let $M$ be a closed (compact without boundary) $n$-dimensional submanifold of $\mathbb{R}^{n+1}$ that is a boundary of a compact region $U \subset \mathbb{R}^{n+1}$, that is,

$$M = \partial U$$

- Let $N$ be an outward-pointing normal to $M$.

- Then $N$ induces an orientation on $M$.

- Let $v$ be a unit vector field on $M$.

- Let $S^n$ be the unit $n$-sphere embedded in $\mathbb{R}^n$ centered at the origin.

- We identify the unit vectors in $\mathbb{R}^{n+1}$ with points in $S^n$.

- Let $\varphi : M \to S^n$ be a map defined for every $x \in M$ by

$$\varphi(x) = v(x)$$

- The Kronecker index of the vector field $v$ on $M$ is defined as the degree of the map $\varphi$:

$$\text{Index}(v) = \text{deg}(\varphi)$$

- In general, if $v$ is a nonvanishing vector field on $M$, then its index is defined as the Kronecker index of the unit vector field $v/\|v\|$.

- Example. $M = S^1$ in $\mathbb{R}^2$. \\

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• **Exercise.** Let $M = \partial U$, where $U \subset \mathbb{R}^{n+1}$ is a compact region as above. Show that if a vector field $\mathbf{v}$ on $M$ can be extended to a nonvanishing vector field on all of the interior region $U$, then $\text{Index}(\mathbf{v}) = 0$.

In other words, a vector field on $M$ with a non-zero index has a singularity (that is, it vanishes) at least at one point inside $M$.

• **Exercise.** Suppose that a unit vector field $\mathbf{v}$ in $\mathbb{R}^{n+1}$ is such that it is smooth in $U$ except for a finite number of points $x_a$, $a = 1, \ldots, N$. Let $B_a$ be sufficiently small balls around the points $x_a$ so that they are in $U$. Then $\mathbf{v}$ is non-vanishing in $U \setminus \bigcup_{a=1}^{N} B_a$. Then $\partial B_a$ are small spheres with outward-pointing normals. Let $\text{Index}(\mathbf{v}|_{\partial B_a})$ be the indices of $\mathbf{v}$ on $\partial B_a$. Show that

$$\text{Index}(\mathbf{v}) = \sum_{a=1}^{N} \text{Index}(\mathbf{v}|_{\partial B_a}).$$

That is, the index of a vector field on a closed manifold $M$ is equal to the sum of indices inside $M$.

**Theorem 2.2.3** Let $\mathbf{v}_t$, $t \in [0, 1]$, be a smooth family of non-vanishing vector field on a closed manifold $M$. Then

$$\text{Index}(\mathbf{v}_0) = \text{Index}(\mathbf{v}_1).$$

**Proof:**

1. Follows from the fact that the index is an integer.

• **Exercise.** Show that if a vector field $\mathbf{v}$ on a sphere $S^n$ in $\mathbb{R}^{n+1}$ never points to the origin, then

$$\text{Index}(\mathbf{v}) = 1.$$

• **Corollary.** The index of every non-vanishing vector field tangent to $S^n$ is equal to

$$\text{Index}(\mathbf{v}) = 1.$$

**Theorem 2.2.4** Brouwer Fixed Point Theorem. Let $B$ be a closed unit ball in $\mathbb{R}^{n+1}$ centered at the origin. Let $\varphi : B \to B$ be a smooth map. Then $\varphi$ has a fixed point.

**Proof:**
2.2. DEGREE OF A MAP

1. Let \( v \) be a vector field in \( \mathbb{R}^{n+1} \) defined by
   \[
   v(x) = \varphi(x) - x.
   \]

2. Then \( v \) never points to the origin. So, \( \text{Index}(v) = 1 \), and, therefore, \( \varphi \) has a fixed point in \( B \).

3. Alternatively, suppose that \( \varphi \) does not have a fixed point.

4. Then \( v \) is non-vanishing.

5. Let \( \psi : B \to \partial B = S^n \) be a map defined as follows. The point \( \psi(x) \) is the point on the sphere \( S^n \) where the line from the \( \varphi(x) \) to the point \( x \) intersects \( S^n \).

6. Then \( \psi(x) = x \) for any \( x \in S^n \).

7. Let \( \alpha \) be an \( n \)-form on \( S^n \) normalized by \( \int_{S^n} \alpha = 1 \).

8. Since \( \psi \) is an identity map on \( S^n \), the form \( \psi^*\alpha \) is an \( n \)-form on \( B \) whose restriction to \( S^n \) is equal to \( \alpha \).

9. Since also \( S^n = \partial B \) and \( d\alpha = 0 \), we have
   \[
   1 = \int_{S^n} \alpha = \int_{\partial B} \psi^*\alpha = \int_B d(\psi^*\alpha) = 0,
   \]
   which is a contradiction.

\[\blacksquare\]

• Exercise. Let \( M \) be a closed \( n \)-dimensional submanifold of \( \mathbb{R}^{n+1} \). Let \( v \) be a unit vector field on \( M \). Let \( \text{vol} \) be the volume \((n + 1)\)-form in \( \mathbb{R}^{n+1} \). Let \( \text{vol}(S^n) \) be the volume of the unit sphere \( S^n \). Let \( u^\mu, \mu = 1, \ldots, n \), be local coordinates on \( M \). Show that
   \[
   \text{Index}(v) = \frac{1}{\text{vol}(S^n)} \int_M \text{vol} \left( v, \frac{\partial v}{\partial u^1}, \ldots, \frac{\partial v}{\partial u^n} \right) du^1 \wedge \cdots \wedge du^n
   \]

• Exercise. If \( v \) is a non-vanishing vector field, not necessarily unit, then
   \[
   \text{Index}(v) = \frac{1}{\text{vol}(S^n)} \int_M \frac{1}{\|v\|^{n+1}} \text{vol} \left( v, \frac{\partial v}{\partial u^1}, \ldots, \frac{\partial v}{\partial u^n} \right) du^1 \wedge \cdots \wedge du^n
   \]
Corollary 2.2.3  Let $M$ be a closed $n$-dimensional submanifold of $\mathbb{R}^{n+1}$ such that $M$ is the boundary of a compact region $U \subset \mathbb{R}^{n+1}$. Let $f_i, i = 1, \ldots, (n + 1)$, be smooth functions on $U$. Let

$$\|f\|^2 = \sum_{i=1}^{n+1} |f_i|^2 .$$

Suppose the functions $f_i$ do not have common zeros on $M$, that is, $\|f\| \neq 0$ on $M$. Let $u^\mu, \mu = 1, \ldots, n$, be local coordinates on $M$. Let

$$\det(f, df) = \epsilon^{i_1i_2, \ldots, i_{n+1}} f_{i_1} \frac{\partial f_{i_2}}{\partial u^1} \cdots \frac{\partial f_{i_{n+1}}}{\partial u^n}$$

$$= \det \begin{pmatrix} f_1 & \frac{\partial f_1}{\partial u^1} & \cdots & \frac{\partial f_1}{\partial u^n} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & \frac{\partial f_n}{\partial u^1} & \cdots & \frac{\partial f_n}{\partial u^n} \end{pmatrix}$$

Suppose that

$$\int_M \frac{1}{\|f\|^{n+1}} \det(f, df) du^1 \wedge \cdots \wedge du^n \neq 0 .$$

Then the functions $f_i$ have a common zero in $U$, that is, the system of $(n + 1)$ equations

$$f_1 = \cdots = f_{n+1} = 0$$

has a solution in $U$.

\textbf{Proof}: Follows from above.

2.2.5  Linking Number

- Let $C_\mu : S^1 \to \mathbb{R}^3, \mu = 1, 2$, be two nonintersecting smooth closed curves (loops) in $\mathbb{R}^3$ described by

$$x = x_1(\theta_1), \quad x = x_2(\theta_2) .$$

- Let $T^2 = S^1 \times S^1$ be the torus with the local coordinates $\theta_1, \theta_2$. 

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2.2. DEGREE OF A MAP

- Let $\varphi : T \to S^2$ be a smooth map defined by
  \[ \varphi(\theta) = \frac{x_1(\theta_1) - x_2(\theta_2)}{||x_1(\theta_1) - x_2(\theta_2)||}. \]

- The **Gauss linking number** of the loops $C_1$ and $C_2$ is defined by
  \[ \text{Link}(C_1, C_2) = \deg(\varphi). \]

- **Exercise.** Let $x_{12} = x_2 - x_1$.
  Show that
  \[
  \text{Link}(C_1, C_2) = \frac{1}{4\pi} \int_{C_1} \left( \int_{C_2} \frac{1}{||x_{12}||^3} x_{12} \times dx_{12} \right) \cdot dx_1
  = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{\{|x_2(\theta_2) - x_1(\theta_1)| \times \dot{x}_2(\theta_2)| \cdot \dot{x}_1(\theta_1)|}{||x_2(\theta_2) - x_1(\theta_1)||^3}
  \]

- Let $V$ be an orientable surface in $\mathbb{R}^3$ such that $\partial V = C_1$.

- Let $N$ be the normal vector to $V$ consistent with the orientation of $C_1$.

- Let the curve $C_2$ intersect $V$ transversally.

- The **intersection number** $V \circ C_2$ of the curve $C_2$ and the surface $V$ is the signed number of intersections of $C_2$ and $V$, with an intersection being positive if the tangent vector to $C_2$ at the point of the intersection has the same direction as $N$, that is,
  \[ V \circ C_2 = \sum_{x_i \in V} \text{sign}(N, \dot{x}_i) \left| x_i \right|, \]
  where the sum goes over all intersection points $x_i$.

- **Exercise.** Show that
  \[ \text{Link}(C_1, C_2) = V \circ C_2. \]
Chapter 3

Groups

3.1 Groups

- A group is a set $G$ with a binary operation, $\cdot$, called the group multiplication, that is,
  1. associative,
  2. has an identity element,
  3. every element has an inverse.

- A group is Abelian if the group operation is commutative.

- For an Abelian group the group operation is called addition and is denoted by $+$. The identity element is called zero and denoted by $0$. The inverse element of an element $g \in G$ is denoted by $(-g)$.

- The left translation and right translation by an element $g \in G$ are the maps $L_g, R_g : G \to G$ defined by, for any $h \in G$
  
  \[ L_g h = gh, \quad R_g h = hg. \]

- A map $F : G \to E$ from a group $G$ to a group $E$ is called a homomorphism if for any $g, h \in G$
  
  \[ F(gh) = F(g)F(h). \]

  In particular,
  
  \[ F(g^{-1}) = (F(g))^{-1}. \]
• A homomorphism \( F : G \to E \) is called an **isomorphism** if it is bijective.

• An isomorphism \( F : G \to G \) of a group \( G \) to itself is called an **automorphism**.

• Two elements \( h, h' \in G \) are said to be **conjugated** if there is an element \( g \in G \) such that
  \[
  h' = ghg^{-1}.
  \]

• A map \( A_g : G \to G \) defined by, for any \( h \in G \)
  \[
  A_g h = ghg^{-1},
  \]
  is called an **inner automorphism** (or **conjugation**) by the element \( g \).

• Obviously,
  \[
  A_g = L_gR_g^{-1}.
  \]

• A subset \( H \) of a group \( G \) is called a **subgroup** of \( G \) if it contains the identity element and is closed under the group operation.

• Two subgroups \( H \) and \( H' \) of a group \( G \) are said to be **conjugate** if there is an element \( g \in G \) such that
  \[
  H' = gHg^{-1}.
  \]

• A subgroup \( H \) of a group \( G \) is called **normal subgroup** if it is invariant under inner automorphisms, that is, for any \( h \in H \) and \( g \in G \), \( ghg^{-1} \in H \).

• A subgroup \( H \) of a group \( G \) is normal if it does not have any conjugates other than itself.

• The **image** of a homomorphism \( F : G \to E \) is the set
  \[
  \text{Im} \ F = \{ h \in E \mid h = F(g) \text{ for some } g \in G \}.
  \]

• The image of a homomorphism \( F : G \to E \) is a subgroup of the group \( E \).

• The **kernel** of the homomorphism \( F : G \to E \) is the set of elements of \( G \) mapped to the identity element of \( E \),
  \[
  \text{Ker} \ F = F^{-1}(e) = \{ g \in G \mid F(g) = e \}.
  \]
3.2. **GROUP REPRESENTATIONS**

- The kernel of a homomorphism $F : G \to E$ is a normal subgroup of the group $E$.

- A homomorphism $F : G \to E$ is injective if and only if its kernel is trivial, that is, it consists of only the identity, $\text{Ker } F = \{ e \}$.

- A bijection of a set $X$ is called a **transformation** of the set $X$.

- The set of all transformations of a set $M$ forms a group called the **full transformation group** of $M$.

- Any subgroup of the full transformation group of a set $M$ is called a **transformation group** of the set $M$.

- A **topological group** $G$ is a group that is a topological space and the composition and the inversion are continuous maps.

- A subgroup of a topological group is a topological group.

- The **direct product** of the groups $G$ and $G'$ is the group $G \times G'$ with the multiplication law, for any $g, h \in G$ and $g', h' \in G'$,

  $$(g, g')(h, h') = (gh, g'h').$$

- The product of topological groups is a topological group with product topology.

### 3.2 Group Representations

- A **representation** of a group $G$ in a vector space $V$ (called the **representation space** of $G$) is a homomorphism of the group $G$ into the group of linear transformations $GL(V)$ of the vector space $V$, that is, it is a map

  $$T : G \to GL(V),$$

  that associates to every element $g \in G$ of the group $G$ a linear transformation $T_g : V \to V$ of the vector space $V$ such that for any $g, h \in G$

  $$T_g T_h = T_{g \circ h}.$$
in particular,

\[ T_e = I \]

and

\[ T_{g^{-1}} = (T_g)^{-1}. \]

- A subspace \( W \subset V \) is called an **invariant subspace** of the representation \( T \) if for any \( g \in G \), \( T_g W \subset W \).

- The restriction of the linear transformation \( T_g \) to an invariant subspace \( W \) of the representation \( T \),

\[ T^W_g : W \rightarrow W, \]

forms a representation

\[ T^W : G \rightarrow GL(W) \]

of \( G \) in \( W \).

- A representation \( T \) of \( G \) in a vector space \( V \) is called **irreducible** if \( V \) does not have any nontrivial invariant subspaces.

- The **direct sum** of two representations \( T : G \rightarrow GL(V) \) and \( T' : G \rightarrow GL(V') \) of a group \( G \) in the vector spaces \( V \) and \( V' \) is a representation

\[ T \oplus T' : G \rightarrow GL(V \oplus V') \]

defined by, for any \( g \in G \), \( v \in V \) and \( v' \in V' \),

\[ (T \oplus T')_g(v, v') = (T_g v, T'_g v'). \]

- Two representations

\[ T : G \rightarrow GL(V), \quad T' : G \rightarrow GL(V') \]

are said to be **equivalent** if there is an isomorphism

\[ \rho : V \rightarrow V', \]

such that for any \( g \in G \)

\[ T'_g = \rho T_g \rho^{-1}. \]

- A representation \( T : G \rightarrow GL(V) \) is **unitary** (or **orthogonal**) the linear transformations \( T_g \) are unitary (or orthogonal).
3.3. **QUOTIENT SPACES**

- If a representation $T : G \to GL(V)$ is unitary and $W$ is an invariant subspace of the representation $T$, then its orthogonal complement $W^\perp$ is also an invariant subspace and the representation $T$ is a direct sum of the restrictions to the subspaces $W$ and $W^\perp$.

- Every finite-dimensional representation of a compact group is equivalent to the direct sum of irreducible representations.

### 3.3 Quotient Spaces

- A **equivalence relation** on a set $X$ is a rule to decide if two elements of the set $X$ are equivalent (in writing $a \sim a$) satisfying the conditions: for any $a, b, c \in X$,
  
  - $a \sim a$,
  - if $a \sim b$ then $b \sim a$,
  - if $a \sim b$ and $b \sim c$ then $a \sim c$.

- The **equivalence class** of an element $a \in X$ is the set of all elements equivalent to $a$,
  
  $$[a] = \{x \in X \mid x \sim a\}$$

- An equivalence relation on a set $X$ defines a **partition** of $X$ into a disjoint union of equivalence classes.

- The set of all equivalence classes is called the **quotient** of $X$ by the equivalence relation $\sim$; denoted by $\tilde{X}$.

- The map
  
  $$\pi : X \to \tilde{X}$$

  defined by
  
  $$\pi(x) = [x]$$

  is called the **identification map**.

- A map $f : X \to Y$ defines a natural equivalence relation on $X$ defined by, for any $a, b \in X$
  
  $$a \sim b \quad \text{if and only if} \quad f(a) = f(b) .$$
• The equivalence classes of this equivalence relation are the inverse images
  \[ [a] = f^{-1}(a) . \]

• If the map \( f : X \to Y \) is surjective then there is a one-to-one correspondence
  between \( \tilde{X} \) and \( Y \)
  \[ g : \tilde{X} \to Y . \]

• If \( X \) is a topological space with an equivalence relation ~, then the quotient
  space \( \tilde{X} \) has a natural quotient topology: a map \( g : \tilde{X} \to Z \) to a topological
  space \( Z \) is continuous if and only if the map \( \tilde{g} = g \circ \pi : X \to Z \) is continuous.

• In other words, a set \( U \subset \tilde{X} \) is open if and only if its inverse image \( \pi^{-1}(U) \)
  is open in \( X \).

### 3.4 Group Actions

• A left action (or a nonlinear representation) of a group \( G \) on a set \( M \) is a
  homomorphism
  \[ L : G \to \text{Aut}(M), \]
  where \( \text{Aut}(M) \) is the full group of transformations of \( M \), that associates to
  each element \( g \) of \( G \) a transformation
  \[ L_g : M \to M \]
  such that
  \[ L_{gh} = L_g \circ L_h, \]
  in particular,
  \[ L_{g^{-1}} = (L_g)^{-1}, \quad L_e = \text{Id}. \]

• If the maps \( L_g \) are injective then we say that the group \( G \) acts on \( M \) effectively.

• If the set \( M \) is a vector space and the transformation \( L_g \) is linear then the left
  action of the group \( G \) is a representation of \( G \) in \( M \).
3.4. **GROUP ACTIONS**

- A **right action** of a group $G$ on a set $M$ is a homomorphism
  \[ R : G \rightarrow \text{Aut}(M), \]
  that associates to each element $g$ of $G$ a transformation
  \[ R_g : M \rightarrow M \]
such that
  \[ R_{gh} = R_h \circ R_g. \]

- Each right action defines a left action by, for any $g \in G$,
  \[ L_g = R_{g^{-1}}. \]

- We use the following shorthand notation, for any $g \in G$ and $x \in M$,
  \[ L_g(x) = gx, \quad R_g(x) = xg \]

- A group action of a group $G$ on a set $M$ naturally defines an equivalence relation by: two points for any $x, y \in M$ are equivalent, $x \sim y$, if there is an element $g \in G$, such that $y = gx$.

- The equivalence class $[x]$ of a point $x \in M$ is called the **orbit** of $x$.

- The set of orbits, denoted by, $M/G$, forms a partition of $M$.

- The group $G$ is said to act **transitively** on $M$ if there is only one orbit, that is, if every two points in $M$ are related by the group action.

- The **stabilizer** $H_x$ of a point $x \in M$ is the set of all elements of the group $G$ that leave the point $x$ fixed,
  \[ H_x = \{ g \in G \mid gx = x \} \]

- The stabilizer $H_x$ of any point $x \in M$ is a subgroup of $G$.

- It is easy to see that
  \[ H_{gx} = g^{-1}H_xg \]

- The action of a group $G$ on $M$ is **free** if the stabilizer subgroup $H_x$ is trivial for all $x \in M$, that is, for any $x \in M$ and any $g \neq e \in G$, $gx \neq x$. 
- Each subgroup $H \subset G$ of a group $G$ defines the left and right actions on $G$.
- The orbits of the left action of the subgroup $H$ on $G$ are called **left cosets** and the orbits of the right action are called **right cosets**.
- The **quotient space** (or the **coset space**) of $G$ by $H$ is the space of right cosets, $G/H$.
- The coset space $G/H$ is a group (called the **quotient** of $G$ by $H$) if and only if $H$ is a normal subgroup of $G$.
- If $H$ is a normal subgroup then the left and right cosets coincide.
- If a group $G$ acts transitively on a set $M$, then $M$ is called a **homogeneous space**.
- Every orbit of any group action is a homogeneous space.
- Let a group $G$ act transitively on a set $M$. Let $x_0 \in M$ be a fixed point in a homogeneous space $M$.
- For any $x \in M$, let
  \[ K_x = \{ g \in G \mid gx = x_0 \}. \]
  Then
  \[ H = K_{x_0} \]
  is the stabilizer of $x_0$.
- Then $K_x$ is the right coset of $H$.
- Thus, there is a one-to-one correspondence between $M$ and the coset space $G/H$.

### 3.5 Free Groups

- A subset $X = \{ x_i \}_{i=1}^n \subset G$ of a group $G$ is called a **free set of generators** of $G$ if any element of $G$ different of the identity $e$ can be uniquely written as the product of finitely many elements from $X$
  \[ g = x_1^{i_1} \cdots x_n^{i_n} \]
  with some non-zero integer $i_k \in \mathbb{Z}$; here it is assumed that no adjacent elements are equal, however, non adjacent elements can be equal.
3.5. **FREE GROUPS**

- If a group $G$ has a free set of generators it is called a **free group**.
- Given a set $X$ one can construct a free group $G$ whose free set of generators is exactly $X$.
- The generators are called **letters** and the products are called **words**.
- If all powers are non-zero and the adjacent elements are distinct the word is called a **reduced word**.
- A word with no letters is called an **empty word** and denoted by 1.
- The product of words is defined naturally (juxtaposition and reduction).
- The set of all reduced words forms a free group called the **free group generated by** $X$.
- An arbitrary group $G$ can be specified by the generators and some **relations** of the form
  \[ r = 1 \]
  where
  \[ r = x_{k_1}^{i_1} \cdots x_{k_n}^{i_n} \]
- Let $G$ be a group and $X$ be a subset of $G$.
- Let $F[X]$ be the free group generated by $X$.
- Suppose that any element of $G$ can be written as
  \[ g = x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} \]
  in a non-unique way, that is, $G$ is not free.
- Then there is a natural surjective homomorphism
  \[ \varphi : F[X] \to G \]
  This is not an isomorphism. However, since $\text{Ker} \varphi$ is a normal subgroup of $F[X]$ there is an isomorphism
  \[ F[X]/\text{Ker} \varphi \cong G \]
• Thus, a group is completely determined by the set of generators \( X \) and \( \text{Ker} \varphi \), that is, relations.

• The set of generators and the set of relations form a \textbf{presentation} of the group \( G \).

• The presentation is said to be \textbf{finite} if the set of generators and the set of relations are finite sets.

• The group \( G \) is said to be \textbf{finitely presented} if it has a finite presentation.

• The following notation (called a \textbf{presentation of} \( G \)) is used:

\[
(x_1, \ldots, x_p; r_1, \ldots, r_q)
\]

denotes a group generated by generators \( x_i \) with relations \( r_j \).

• \textbf{Examples}. The presentation of some Abelian groups:

\[
\mathbb{Z} = \{ x^n \mid n \in \mathbb{Z} \} \simeq (x; \emptyset)
\]

\[
\mathbb{Z}_2 = \{ x^n \mid n \in \mathbb{Z}_2 \} \simeq (x; x^2)
\]

\[
\mathbb{Z} \oplus \mathbb{Z} = \{ x^n y^m \mid n, m \in \mathbb{Z} \} \simeq (x, y; xyx^{-1} y^{-1})
\]

\[
\mathbb{Z} \oplus \mathbb{Z}_2 = \{ x^n y^m \mid n \in \mathbb{Z}, m \in \mathbb{Z}_2 \} \simeq (x, y; xyx^{-1} y^{-1}, y^2)
\]

The group

\[
(x, y; xyxy^{-1})
\]

is non-Abelian. It is isomorphic to the group

\[
(a, b; a^2 = b^2)
\]

• Let \( G \) be a group with a representation

\[
G = (x_i; r_j)
\]

The \textbf{commutator subgroup} of \( G \) is a subgroup \( F \) generated by the elements of the form \( x_i x_j x_i^{-1} x_j^{-1} \).

• The quotient of the group \( G \) by its commutator subgroup \( F \) has additional relations

\[
G/F = (x_i; r_j, x_i x_j x_i^{-1} x_j^{-1})
\]
3.6 Abelian Groups

3.6.1 Groups

- A group is a set $G$ with a binary operation, $\cdot$, called the group multiplication, that is,
  1. associative,
  2. has an identity element,
  3. every element has an inverse.

- A group is Abelian if the group operation is commutative.

- For an Abelian group the group operation is called addition and is denoted by $+$.
  The identity element is called zero and denoted by 0.
  The inverse element of an element $g \in G$ is denoted by $(-g)$.

- Let $G$ and $E$ be Abelian groups. A map $F : G \to E$ is called a homomorphism if for any $g, g' \in G$,
  \[ F(g +_G g') = F(g) +_E F(g'), \]
  where $+_G$ and $+_E$ are the group operations in $G$ and $E$ respectively.

- In particular,
  \[ F(0_G) = 0_E, \quad \text{and} \quad F(-g) = -F(g). \]

- Let $F : G \to E$ be a homomorphism of an Abelian group $G$ into an Abelian group $E$.
  The set of elements of $G$ mapped to the identity element of $E$ is denoted by
  \[ \text{Ker } F = \{ g \in G \mid F(g) = 0_E \}, \]
  where $0_E$ is the identity element of $E$, and called the kernel of the homomorphism $F$.

- The image of the homomorphism $F : G \to E$ is the set
  \[ \text{Im } F = \{ h \in E \mid h = F(g) \text{ for some } g \in G \}. \]
• A homomorphism \( F : G \to E \) of a group \( G \) into a group \( E \) is called an **isomorphism** if it is a bijection.

• A subset \( H \) of a group \( G \) is called a **subgroup** of \( G \) if it contains the identity element and is closed under the group operation.

• For any homomorphism \( F : G \to E \) the kernel \( \text{Ker} \ F \) is a subgroup of \( G \).

• Let \( G \) be an Abelian group and \( H \) be a subgroup of \( G \). We say that two elements of \( G \) are **equivalent** if they differ by an element of \( H \).

• Let \( g \in G \) be an element of \( G \). Then the set of all elements of \( G \) equivalent to \( g \), denoted by \([g] = g + H\), is an **equivalence class** of \( g \) called a **coset**.

• The set of cosets is denoted by \( G/H \).

• An element \( g \) used to describe a coset \([g]\) is called a **representative**.

• The set of cosets \( G/H \) is an Abelian group, called the **quotient group**, with the addition defined by

\[
[g] + [g'] = [g + g'].
\]

• The projection map \( \pi : G \to G/H \) defined by

\[
\pi(g) = [g]
\]

is a homomorphism.

**Theorem 3.6.1 Fundamental Theorem of Homomorphisms.** Let \( G \) and \( F \) be groups. Let \( F : G \to F \) be a homomorphism. Then

\[
G/ \text{Ker} \ F \cong \text{Im} \ F.
\]

**Proof:**

1. Since

\[
F(g + \text{Ker} \ F) = F(g).
\]
Theorem 3.6.2 Let $G$ and $E$ be Abelian groups, and $H \subset G$ and $N \subset E$ be their subgroups so that $G/H$ and $E/N$ are the quotient groups. Let $F : G \to E$ be a homomorphism such that the image of the subgroup $H$ of $G$ is the subgroup $N$ of $E$, that is,

$$F(H) = N.$$

Then the homomorphism $F$ induces a homomorphism of the quotient groups

$$F_* : G/H \to E/N.$$

Proof:

1. Let

$$\pi : E \to E/N$$

be the projection homomorphism defined by, for any $x \in E$

$$\pi(x) = [x] = x + N.$$

2. Then

$$F_* = \pi \circ F : G/H \to E/N$$

is a homomorphism.

\[\blacksquare\]

A field is a set $K$ with two binary operations, addition, $+$, and multiplication, $\cdot$, that satisfy the following conditions:

1. both addition and multiplication are associative,
2. both addition and multiplication are commutative,
3. both addition and multiplication have identity elements, the additive identity $0$ and the multiplicative identity $1$, such that $0 \neq 1,$
4. the multiplication is distributive with respect to addition,
5. every element has an additive inverse,
6. every nonzero element has a multiplicative inverse.

In particular, any field is an Abelian group with respect to addition.
A vector space over a field \( K \) consists of a set \( E \), whose elements are called vectors, and the field \( K \), whose elements are called scalars, with two operations: vector addition, \( + : E \times E \to E \), and multiplication by scalars, \( \cdot : K \times E \to E \), that satisfy the following conditions:

1. the vector addition is associative and commutative,
2. there is an additive identity, called the zero vector,
3. every vector has an additive inverse, called the opposite vector,
4. for any \( v \in E, a, b \in K \),
   \[(a)\] \( a(bv) = (ab)v \),
   \[(b)\] \( (a + b)v = av + bv \),
   \[(c)\] \( a(u + v) = au + av \),
   \[(d)\] \( 1v = v \).

Let \( E \) be a vector space and \( F \) be a vector subspace of \( E \). Then \( E/F \) is a vector space.

Let \( E \) be an inner product vector space and \( F \) be its subspace. Then

\[ E/F = F^\perp \]

is the orthogonal complement of \( F \) in \( E \), and

\[ \pi : E \to F^\perp \]

is the orthogonal projection to \( F^\perp \).

### 3.6.2 Finitely Generated and Free Abelian Groups

- Let \( G \) be an Abelian group. Let \( g_1, \ldots, g_r \in G \) be some elements of \( G \) and

\[ H = \left\{ \sum_{k=1}^{r} n_k g_k \mid g_k \in G, n_k \in \mathbb{Z} \right\} \]

be a set of linear combinations of \( g_k \).

- Then \( H \) is a subgroup of \( G \). The elements \( g_k \) are called the generators of \( H \) and \( H \) is said to be generated by \( g_k \).
• If a group $G$ is generated by finitely many elements of $G$, then $G$ is called a **finitely generated group**.

• The elements $g_1, \ldots, g_r$ are **linearly independent** if for any integer coefficients $n_1, \ldots, n_r$ the linear combination

\[
\sum_{k=1}^{r} n_k g_k \neq 0
\]

is not equal to zero.

• A finitely generated group $G$ is called a **free Abelian group of rank** $r$ if it is generated by $r$ linearly independent elements.

### 3.6.3 Cyclic Groups

• An Abelian group generated by one element is called a **cyclic group**.

• Infinite cyclic groups.

• Finite cyclic group.

---

**Theorem 3.6.3 Fundamental Theorem of Finitely Generated Abelian Groups.** Let $G$ be a finitely generated Abelian group with $m$ generators. Then $G$ is isomorphic to the direct sum of cyclic groups,

\[
G \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p},
\]

where $m = r + p$. The number $r$ is called the **rank** of $G$.

**Proof:**

1. 

• The product $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ is a free Abelian group of rank $r$.

• The product $\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p}$ is called the **torsion** of the subgroup of $G$.  

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*topicsdiffgeom.tex; December 4, 2014; 13:01; p. 83*
Chapter 4

Homology Theory

4.1 Simplicial Homology Groups

4.1.1 Simplicial Complexes

- Let \( p_0, p_1, \ldots, p_k \) be \( k+1 \) points in \( \mathbb{R}^n \), with \( k \leq n \). We identify points in \( \mathbb{R}^n \) with the vectors that point to them.

- Assume that they are independent, that is, do not lie in a \((k-1)\)-dimensional hyperplane, or that the vectors \( v_{ij} = p_j - p_i \) are linearly independent.

- The \( k \)-simplex \( \sigma_k = \langle p_0, p_1, \ldots, p_k \rangle \) is the compact subset of \( \mathbb{R}^n \) defined by

\[
\sigma_k = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=0}^{k} c_i p_i, \text{ with } c_i \geq 0, \sum_{i=0}^{k} c_i = 1 \right\}
\]

- For any \( j, 0 \leq j \leq k \), a subset of \( j+1 \) points defines a \( j \)-simplex called the \( j \)-face.

- A 0-simplex is a point, called a vertex.

- A 1-simplex is a line segment, called an edge.

- A 2-simplex is the interior of a triangle.

- The 3-simplex is a tetrahedron.

- A simplicial complex is a set \( K \) of finitely many simplexes such that:
– every face of every simplex of $K$ belongs to $K$,
– the intersection of any two simplexes in $K$ is either empty or is a common face.

• A subset $|K|$ of $\mathbb{R}^n$ which is the union of all simplexes in a complex $K$ is called a **polyhedron**.

• A simplicial complex $K$ and a homeomorphism

$$F : |K| \rightarrow X$$

to a topological space $X$ is called a **triangulation** of $X$.

• A topological space $X$ is called **triangulable** if there is a triangulation of $X$.

• An unoriented $k$-simplex $\langle p_0, p_1, \ldots, p_k \rangle$ can be oriented as follows. An **oriented $k$-simplex** $(p_0, p_1, \ldots, p_k)$ changes sign under a permutation of any two points. Let $\varphi$ be a permutation of points $\{p_0, p_1, \ldots, p_k\}$. Then

$$(p_{\varphi(0)}, p_{\varphi(1)}, \ldots, p_{\varphi(k)}) = (\text{sign } \varphi)(p_0, p_1, \ldots, p_k),$$

where $\text{sign } \varphi$ is the parity of the permutation $\varphi$.

### 4.1.2 Simplicial Homology Groups

• Let $K$ be an $n$-dimensional simplicial complex.

• Let $N_p$ is the number of $p$-simplexes in $K$.

• A **$p$-chains** is a formal sum

$$c = \sum_{i=1}^{N_p} c_i \sigma_{p,i}$$

where $\sigma_{p,i}$ are $p$-simplexes in $K$ and $c_i \in \mathbb{Z}$.

• **Remark.** We can define chains over any Abelian group, for example, $\mathbb{R}$ or $\mathbb{Z}_2$.

• This allows to define the Abelian group structure: addition, zero, opposite.
4.1. SIMPLICIAL HOMOLOGY GROUPS

- The $p$-chain group $C_p(K)$ of $K$ is a free Abelian group generated by the oriented $k$-simplexes of $K$,

$$C_p(K) \cong \bigoplus_{i=1}^{N_p} \mathbb{Z}$$

- By definition $C_p(K) = 0$ for $p > n$.

- The boundary operator is a homomorphism

$$\partial_p : C_p(K) \to C_{p-1}(K)$$

defined as follows.

- The boundary of an oriented $p$-simplex $\sigma_p = (p_0, p_1, \ldots, p_p)$ is a $(p-1)$-chain defined by

$$\partial_p \sigma_p = \sum_{i=0}^{p} (-1)^i (p_0, p_1, \ldots, \hat{p}_i, \ldots, p_k)$$

where $\hat{p}_i$ is omitted.

- The boundary of a $p$-chain is defined by linearity.

- The chain complex is a sequence of free Abelian groups and homomorphisms

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

where $i : \hookrightarrow C_n(K)$ is the inclusion map.

- A $p$-chain $z$ such that

$$\partial_p z = 0$$

is called a $p$-cycle.

- The $p$-cycles form a free Abelian subgroup of $C_p(K)$ called the $p$-cycle group

$$Z_p(K) = \text{Ker} \partial_p$$

- A $p$-chain $b$ such that

$$b = \partial_{p+1} c$$

for some $(p+1)$-chain $c$, is called a $p$-boundary.
• The $p$-boundaries form a free Abelian subgroup of $C_p(K)$ called the $p$-boundary group

$$B_p(K) = \text{Im} \partial_{p+1}$$

• **Proposition.** The boundary of a boundary vanishes, that is,

$$\partial_p \partial_{p+1} = 0$$

• **Corollary.** Every boundary is a cycle, that is,

$$B_p(K) \subset Z_p(K)$$

• The $p$-homology group $H_p(K)$ is defined by

$$H_p(K) = Z_p(K)/B_p(K)$$

It is not necessarily free Abelian.

• We say that two $p$-cycles are **homologous** if they differ by a boundary.

• Homology is an equivalence relation.

• The equivalence classes of the homology are called **homology classes**.

• The homology groups are the sets of homology classes.

• **Theorem.** Homology groups are topological invariants. In particular,

  – The homology groups of different triangulations of the same topological space are isomorphic.

  – The homology groups of any triangulations of homeomorphic topological spaces are isomorphic.

• Therefore, the homology groups of a triangulable topological space (which is not necessarily a polyhedron) are defined to be the homology groups of some triangulation.

• **Spheres.**

$$H_0(S^1) = H_1(S^1) = \mathbb{Z},$$

$$H_0(S^2) = H_2(S^2) = \mathbb{Z}, \quad H_1(S^2) = 0.$$
• **Theorem.** For any connected simplicial complex $K$

$$H_0(K) = \mathbb{Z}.$$ 

• **Möbius Strip.**

$$H_0(K) = \mathbb{Z}, \quad H_1(K) = \mathbb{Z}, \quad H_2(K) = 0.$$ 

• **Real Projective Space $\mathbb{R}P^2$.**

$$H_0(\mathbb{R}P^2) = \mathbb{Z}, \quad H_1(\mathbb{R}P^2) = \mathbb{Z}_2, \quad H_2(\mathbb{R}P^2) = 0.$$ 

• The homology group over $\mathbb{Z}$ is not necessarily free Abelian group but may include the torsion.

• **Torus $T^2$.**

$$H_0(T^2) = H_2(T^2) = \mathbb{Z}, \quad H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}.$$ 

• **Surface $\Sigma_g$ of genus $g$.**

$$H_0(\Sigma_g) = H_2(\Sigma_g) = \mathbb{Z}, \quad H_1(\Sigma_g) = \bigoplus_{i=1}^{2g} \mathbb{Z}.$$ 

• **Klein Bottle $K^2$.**

$$H_0(K^2) = \mathbb{Z}, \quad H_2(K^2) = 0, \quad H_1(K^2) = \mathbb{Z} \oplus \mathbb{Z}_2.$$ 

• **Theorem.** The homology groups of a disconnected simplicial complex are equal to the direct sum of the homology groups of its connected components.

• **Corollary.** If a complex $K$ has $m$ connected components, then

$$H_0(K) = \bigoplus_{i=1}^{m} \mathbb{Z}.$$ 

• **Corollary.** For a complex $K$

$$H_0(K) = \mathbb{Z}$$ 

if and only if $K$ is connected.
• A general homology group over \( \mathbb{Z} \) has the form

\[
H_p(K) = \bigoplus_{i=1}^{m} \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}
\]

• The number of generators of \( H_p \) counts the number of \((p + 1)\) dimensional holes in the polyhedron \(|K|\).

• The torsion subgroup \( \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k} \) measures the twisting in the polyhedron \(|K|\).

• The homology groups over \( \mathbb{R} \) or \( \mathbb{Z}_2 \) do not have torsion.

• The homology groups \( H_p(K, \mathbb{R}) \) are finite-dimensional vector spaces.

• The dimension of the vector spaces \( H_p(K, \mathbb{R}) \) are called Betti numbers

\[
b_p(K) = \dim H_p(K, \mathbb{R})
\]

• The Betti numbers are equal to the ranks of the free Abelian parts of the homology groups over \( \mathbb{Z} \).

• The Euler characteristic of a simplicial complex \( K \) with \( N_p \) \( p \)-simplexes is an integer defined by

\[
\chi(K) = \sum_{p=0}^{n} (-1)^p \dim C_p(K, \mathbb{R}) = \sum_{p=0}^{n} (-1)^p N_p.
\]

• Theorem. The Euler characteristic of a simplicial complex \( K \) is equal to

\[
\chi(K) = \sum_{p=0}^{n} (-1)^p \dim H_p(K, \mathbb{R}) = \sum_{p=0}^{n} (-1)^p b_p(K).
\]

• The Euler characteristic is a topological invariant.

• The Euler characteristic of a topological space does not depend on the triangulation, so, it can be defined for any triangulation.
4.2 Singular Chains

- The **standard Euclidean** \( p \)-**simplex** in \( \mathbb{R}^p \) is the convex set \( \Delta_p \subset \mathbb{R}^p \) generated by an ordered \((p + 1)\)-tuple \((P_0, P_1, \ldots, P_p)\) of points in \( \mathbb{R}^p \)

\[
P_0 = (0, \ldots, 0), \quad P_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 1, \ldots, p,
\]

all of whose components are 0 except for the \( i \)-th component, which is equal to 1.

- We use the following notation

\[
\Delta_p = (P_0, P_1, \ldots, P_p).
\]

- Let \( M \) be an \( n \)-dimensional manifold. A **singular** \( p \)-**simplex** in \( M \) is a differentiable map

\[
\sigma_p : \Delta_p \to M.
\]

- By slightly abusing notation we denote the image \( \sigma_p(\Delta_p) \) of \( \Delta_p \) in \( M \) under the map \( \sigma_p \) just by \( \sigma_p \).

- Let \( \alpha \) be a \( p \)-form on \( M \) and \( \sigma_p \) be a \( p \)-simplex in \( M \). We define the integral of \( \alpha \) over \( \sigma_p \) by

\[
\int_{\sigma_p} \alpha = \int_{\Delta_p} \sigma_p^* \alpha.
\]

- The **k-th face** of a standard \( p \)-simplex \( \Delta_p = (P_0, P_1, \ldots, P_p) \) (a face opposite to the vertex \( P_k \)) is the convex set in \( \mathbb{R}^p \) generated by an ordered \( p \)-tuple of points in \( \mathbb{R}^p \)

\[
\Delta^{(k)}_{p-1} = (P_0, \ldots, \hat{P}_k, \ldots, P_p)
\]

where the \( k \)-th point \( P_k \) is omitted.

- Since these points lie in \( \mathbb{R}^p \) and not in \( \mathbb{R}^{p-1} \) a face is not a standard \((p - 1)\)-simplex. It can be rather described as a singular simplex in \( \mathbb{R}^p \) defined by the unique affine map

\[
f_k : \Delta_{p-1} \to \Delta_p
\]

whose image in \( \mathbb{R}^p \) is exactly \( \Delta^{(k)}_p \).
• Such a map is uniquely defined as follows. Let $\Delta_{p-1} = (Q_0,\ldots,Q_{p-1})$, where $Q_i \in \mathbb{R}^{p-1}$, be the standard $(p-1)$-simplex in $\mathbb{R}^{p-1}$ and $\Delta_p = (P_0,\ldots,P_p)$, where $P_i \in \mathbb{R}^p$, be the standard $p$-simplex in $\mathbb{R}^p$. Then

$$f_k(Q_i) = P_i \quad \text{for } i = 0,\ldots,k-1$$

and

$$f_k(Q_i) = P_{i+1} \quad \text{for } i = k,\ldots,p-1.$$

• If $Q$ is a point in $\mathbb{R}^{p-1}$ with coordinates $(x^i) = (x^1,\ldots,x^{p-1})$, then $P = f_k(Q)$ is the point in $\mathbb{R}^p$ with coordinates $(y^\mu) = (y^1,\ldots,y^p)$, where

$$y^\mu = A^\mu_{(k)j}x^j + b^\mu_{(k)}.$$

The above requirements fix the matrix $A_{(k)}$ and the vector $b_{(k)}$ uniquely.

• Exercise. Find the maps $f_k$.

• Let $M$ and $V$ be manifolds and $\varphi : V \to M$ be a differentiable map. Let $\sigma_p : \Delta_p \to V$ be a singular $p$-simplex in $V$.

• Then the composition map

$$\varphi \circ \sigma_p : \Delta_p \to M$$

defines a singular $p$-simplex in $M$.

• Thus, the composition

$$\sigma_p \circ f_k : \Delta_{p-1} \to M$$

defines a singular $(p-1)$-simplex in $M$, which is the $k$-th face of the singular $p$-simplex $\sigma_p$ in $M$.

• The boundary $\partial \Delta_p$ of the standard $p$-simplex $\Delta_p$ is defined as follows. For $p > 0$, we define

$$\partial(P_0,P_1,\ldots,P_p) = \sum_{k=0}^p (-1)^k (P_0,\ldots,\hat{P}_k,\ldots,P_p),$$

that is, the boundary is the formal sum

$$\partial \Delta_p = \sum_{k=0}^p (-1)^k \Delta_{p-1}^{(k)}.$$

For $p = 0$, we define

$$\partial \Delta_0 = 0.$$
4.2. SINGULAR CHAINS

- **Examples.**
  - The boundary of a standard simplex is not a simplex, but an integer \((p − 1)\)-chain.

- Let \(G\) be an Abelian group. A **singular \(p\)-chain on \(M\) with coefficients in \(G\)** is a finite formal sum

  \[ c_p = \sum_{k=1}^{r} g_k \sigma^k_p \]

  of singular simplexes \(\sigma^k_p : \Delta_p \to M\) with coefficients \(g_k\) which are elements of the group \(G\).

- **Examples.**
  - Let \(S_p(M)\) be the set of all singular \(p\)-simplexes in \(M\). Then, a \(p\)-chain is a function

    \[ c_p : S_p(M) \to G, \]

    such that *its value is not equal to zero only for finitely many simplexes*. These simplexes are exactly \(\sigma^k_p\) listed in the formal sum, and the values of the function \(c_p\) are exactly the coefficients of the formal sum, that is

    \[ c_p(\sigma^k_p) = g_k. \]

  - Alternatively, a \(p\)-chain can be thought of as a finite subset of \(S_p(M) \times G\), that is, a finite set of ordered pairs

    \[ c_p = \{(\sigma^k_p, g_k)\}_{k=1}^{r} \]

  - The notation of a \(p\)-chain as a sum, or as a function, is useful since we can define the addition of \(p\)-chains simply as addition of corresponding functions. If two \(p\)-chains \(c_p\) and \(c'_p\) have the same \(p\)-simplex \(\sigma_p\) in both of them, then in the sum \(c_p + c'_p\) we add the corresponding group elements \(g\) and \(g'\). That is, if

    \[ c_p = g_\sigma_p + \ldots, \quad \text{and} \quad c'_p = g'_\sigma_p + \ldots, \]

    then

    \[ c_p + c'_p = (g + g')\sigma_p + \ldots. \]
We denote the identity element of $G$ by 0 and define the **zero $p$-chain** simply by

$$0_p = 0.$$ 

Then the set of all singular $p$-chains on $M$ with coefficients in $G$ forms an Abelian group, called the **singular $p$-chain group** of $M$ with coefficients in $G$ and denoted by $C_p(M; G)$.

A chain with integer coefficients, when $G = \mathbb{Z}$, is called an **integer chain**.

The standard $p$-simplex $\Delta_p$ is an integer $p$-chain in $\mathbb{R}^p$ with only one term: $c_p = 1 \cdot \Delta_p$. That is, it is an element of $C_p(\mathbb{R}^p; \mathbb{Z})$.

The boundary of the standard $p$-simplex in $\mathbb{R}^p$, 

$$\partial \Delta_p = \sum_{k=0}^p (-1)^k \Delta_{p-1}^k$$

is an integer $(p - 1)$-chain in $\mathbb{R}^p$, that is, an element of $C_{p-1}(\mathbb{R}^p; \mathbb{Z})$.

Let $M$ and $V$ be closed manifolds. Let $F : M \to V$ be a map of $M$ into $V$ and $\sigma_p : \Delta_p \to M$ be a singular $p$-simplex in $M$. Then the composition $F \circ \sigma_p : \Delta_p \to V$ is a singular $p$-simplex in $V$. We denote it by

$$F \ast \sigma_p = F \circ \sigma_p.$$ 

**The induced chain homomorphism**

$$F_* : C_p(M; G) \to C_p(V; G)$$

is defined by: for any $g_k \in G$ and $\sigma_p^k \in S_p(M)$,

$$F_* \left( \sum_{k=1}^r g_k \sigma_p^k \right) = \sum_{k=1}^r g_k F_* \sigma_p^k$$

Let $F : M \to V$ and $E : V \to W$ be two maps of manifolds and $F_* : C_p(M; G) \to C_p(V; G)$ and $E_* : C_p(V; G) \to C_p(W; G)$ be the corresponding induced chain homomorphisms. Then

$$(E \circ F)_* = E_* \circ F_*.$$
4.2. SINGULAR CHAINS

Let $\sigma_p : \Delta_p \to M$ be a singular $p$-simplex in $M$. Then its boundary $\partial \sigma_p$ is the integer $(p - 1)$-chain in $M$ defined by

$$\partial \sigma_p = \sigma_p^* (\partial \Delta_p).$$

In more detail

$$\partial \sigma_p = \sigma_p^* (\partial \Delta_p) = \sum_{k=0}^{p} (-1)^k \sigma_p^* (\Delta_p^{(k)}) = \sum_{k=0}^{p} (-1)^k (\sigma_p \circ f_k).$$

Recall that $\sigma_p^* (\Delta_p^{(k)}) = (\sigma_p \circ f_k)$ is the $k$-th face of the singular $p$-simplex $\sigma_p$.

That is, the boundary of the image of $\Delta_p$ is the image of the boundary of $\Delta_p$.

The boundary of any singular $p$-chain with coefficients in $G$ is defined by, for any $g_k \in G$, $\sigma_p^k \in S_p(M)$,

$$\partial \left( \sum_{k=1}^{r} g_k \sigma_p^k \right) = \sum_{k=1}^{r} g_k \partial \sigma_p^k.$$

This leads to the boundary homomorphism

$$\partial : C_p(M; G) \to C_{p-1}(M; G).$$

Let $F : M \to V$ be a map, $\sigma_p$ be a singular $p$-simplex in $M$ and $F_* \sigma_p$ be the induced singular $p$-simplex in $V$. Then

$$\partial (F_* \sigma_p) = \partial (F \circ \sigma_p) = (F \circ \sigma_p)_* (\partial \Delta_p) = (F_* \circ \sigma_p^*) (\partial \Delta_p) = F_* [\sigma_p^* (\partial \Delta_p)] = F_*(\partial \sigma_p)$$

More generally, let

$$c_p = \sum_{k=1}^{r} g_k \sigma_p^k$$

be a $p$-chain on $M$ and $F_* c_p$ be the induced $p$-chain on $V$. 
Then
\[
\partial(F_*c_p) = F_*(\partial c_p)
\]

Therefore,
\[
\partial \circ F_* = F_* \circ \partial,
\]
in other words, the boundary of an image is the image of the boundary.

Thus, we obtain a commutative diagram
\[
\begin{array}{ccc}
F_* & : & C_p(M; G) \to C_p(V; G) \\
\partial \downarrow & & \partial \downarrow \\
C_{p-1}(M; G) & \to & C_{p-1}(V; G) \\
\end{array}
\]
which is a fancy way to say that for any \( p \)-chain \( c_p \in C_p(M; G) \) on \( M \) we have
\[
F_*(\partial c_p) = \partial(F_*c_p).
\]

\textbf{Theorem 4.2.1} The boundary of a boundary is zero, that is,
\[
\partial^2 = 0.
\]

\textit{Proof:}

1. For a standard \( p \)-simplex \( \Delta_p \) we have
\[
\partial \partial \Delta_p = \sum_{k=0}^{p} (-1)^k \partial \Delta_p^{(k)}
\]
\[
= \sum_{k=0}^{p} (-1)^k \partial(P_0, \ldots, \hat{P}_k, \ldots, P_p)
\]
\[
= \sum_{k=0}^{p} (-1)^k \sum_{j=0}^{k-1} (-1)^j(P_0, \ldots, \hat{P}_j, \ldots, \hat{P}_k, \ldots, P_p)
\]
\[
+ \sum_{k=0}^{p} (-1)^k \sum_{j=k+1}^{p} (-1)^j(P_0, \ldots, \hat{P}_k, \ldots, \hat{P}_j, \ldots, P_p)
\]
\[
= 0
\]
because of the pairwise cancellation.
4.2. SINGULAR CHAINS

2. Then, for a singular $p$-simplex $\sigma_p$

$$\partial \partial \sigma_p = \partial \left[ \sigma_p \left( \partial \Delta_p \right) \right] = \sigma_p \partial \left( \partial \Delta_p \right) = \sigma_p (0) = 0$$

4.2.1 Examples

- Cylinder.
- Möbius Band.
4.3  Singular Homology Groups

4.3.1  Cycles, Boundaries and Homology Groups

- We can define the singular $p$-chains with coefficients in a field $K$.

- Furthermore, we can define the multiplication of $p$-chains by elements of the field $K$, called the scalars by, for any $a, b_i \in K$,

$$a \left( \sum_{i=1}^{r} b_i \sigma^i_p \right) = \sum_{i=1}^{r} ab_i \sigma^i_p.$$

- The chain groups $C_p(M, K)$ with coefficients in a field $K$ become infinite-dimensional vector spaces.

- In this case the boundary homomorphism becomes a linear transformation (operator) in a vector space.

- Let $M$ be a manifold and $G$ an Abelian group. A singular $p$-chain $z_p$ in $M$ whose boundary is 0 is called a singular $p$-cycle.

- The set of all $p$-cycles in $M$

$$Z_p(M; G) = \{ z_p \in C_p(M; G) \mid \partial z_p = 0 \}$$

is a subgroup of the chain group $C_p(M; G)$ called the $p$-cycle group.

- Obviously the $p$-cycle group is the kernel of the boundary homomorphism

$$Z_p(M; G) = \text{Ker} \partial_p,$$

where

$$\partial_p : C_p(M; G) \rightarrow C_{p-1}(M; G).$$

- In the case, when $G = K$ is a field, then $Z_p(M; K)$ is a vector subspace of the vector space $C_p(M; K)$.

- A singular $p$-chain $b_p$ in $M$ that is the boundary of a singular $(p + 1)$-chain is called a $p$-boundary.
• The set of all $p$-boundaries in $M$

$$B_p(M; G) = \{ b_p \in C_p(M; G) \mid b_p = \partial c_{p+1} \text{ for some } c_{p+1} \in C_{p+1}(M; G) \}$$

is a subgroup of the chain group $C_p(M; G)$ called the $p$-boundary group.

• Obviously the $p$-boundary group is the image of the boundary homomorphism

$$B_p(M; G) = \operatorname{Im} \partial_{p+1}.$$  

• Since every $p$-boundary is a $p$-cycle (because of $\partial^2 = 0$) the group $B_p(M; G)$ is a subgroup of $Z_p(M; G)$.

• In the case, when $G = K$ is a field, then $B_p(M; K)$ is a vector subspace of the vector space $Z_p(M; K)$.

• Let $M$ be a manifold and $G$ be an Abelian group. We say that two $p$-cycles are homologous if they differ by a boundary.

• The set of equivalence classes of $p$-cycles homologous to each other, that is, the quotient group

$$H_p(M; G) = Z_p(M; G)/B_p(M; G),$$

is called the $p$-th homology group.

• In the case when the coefficient group $G$ is a field $G = K$, all the groups, $Z_p$, $B_p$ and $H_p$ are vector spaces.

### 4.3.2 Simplicial Homology

• The important fact is that if $M$ is a compact manifold, then the vector space $H_p(M; K)$ is finite dimensional. (This can be proved but we will not do that).

• Let $M$ be a compact $n$-dimensional manifold. Then there is a triangulation of $M$ by finitely many $n$-simplexes diffeomorphic to the standard $n$-simplex $\Delta_n$.

• Thus, $M$ is a union of finitely many $n$-simplexes, which are either disjoint or intersect along common $r$-simplexes with $r = 0, 1, \ldots, n - 1$. 
• The set of all these simplexes form a finite simplicial complex with some coefficient group $G$.

• All the simplicial chain groups $\tilde{C}_p(M; G), \tilde{Z}_p(M; G)$ and $\tilde{B}_p(M; G)$ are finitely generated Abelian groups.

• Therefore, the homology group $\tilde{H}_p(M; G)$ is a finitely generated group.

• Since any simplicial cycle can be described as a singular cycle, there are homomorphisms

\[
\tilde{Z}_p \to Z_p, \quad \tilde{B}_p \to B_p
\]

and the induced homomorphism

\[
\tilde{H}_p \to H_p.
\]

**Theorem 4.3.1** Let $M$ be a compact manifold and $G$ be an Abelian group. Then the singular homology groups are isomorphic to the simplicial homology groups

\[
\tilde{H}_p(M; G) = H_p(M; G)
\]

and are finitely generated Abelian groups.

**Proof**: Nontrivial.

**Corollary 4.3.1** Let $M$ be a compact manifold and $K$ be field. Then the homology groups $H_p(M; K)$ are finite-dimensional vector spaces.

**Proof**: Follows from above theorem.

### 4.3.3 Betti Numbers and Topological Invariants

• Let $M$ be a compact manifold. The dimensions of the real homology groups $H_p(M; \mathbb{R})$ are called the **Betti numbers**, that is,

\[
b_p(M) = \dim H_p(M; \mathbb{R}).
\]

• The Betti number $b_p$ is the maximal number of linearly independent $p$-cycles modulo a boundary.
4.3. SINGULAR HOMOLOGY GROUPS

• Let $M$ and $V$ be manifolds, $G$ be an Abelian group, $F : M \rightarrow V$ be a map and $F_* : C_p(M; G) \rightarrow C_p(V; G)$ be the induced homomorphism of chain groups.

• Since the induced homomorphism $F_*$ commutes with the boundary homomorphism $\partial$, the groups $Z_p(M; G)$ and $B_p(M; G)$ are closed under $F_*$.  

• Therefore, the homomorphism $F_*$ naturally acts on the homology groups
  $$F_* : H_p(M; G) \rightarrow H_p(V; G).$$

• If $F : M \rightarrow V$ is a homeomorphism, then there is the inverse homeomorphism $F^{-1} : V \rightarrow M$ and the inverse induced homomorphism
  $$F^{-1}_* : H_p(V; G) \rightarrow H_p(M; G).$$

• In this case, the induced homomorphism $F_*$ is an isomorphism.

**Theorem 4.3.2** Let $M$ and $V$ be compact homeomorphic manifolds and $G$ be an Abelian group. Then their homology groups are isomorphic, that is, for any $p$

$$H_p(M; K) \cong H_p(V; G).$$

**Proof**: Follows from above.

• Thus, homology groups are **topological invariants**.

**Corollary 4.3.2** Let $M$ and $V$ be compact manifolds. If there is an Abelian group $G$ and an integer $p$ such that their homology groups are not isomorphic, that is,

$$H_p(M; K) \not\cong H_p(V; G),$$

then the manifolds $M$ and $V$ are not homeomorphic to each other.

• **Remark**. The converse is not true.
4.3.4 Some Theorems from Algebraic Topology

- A manifold $M$ is **path-connected** (or just **connected**) if every two points in $M$ can be connected by a piecewise-smooth curve.

- Let $M$ be a manifold and $G$ be an Abelian group.

- A point $p$ in a manifold $M$ is a 0-chain. Since $\partial p = 0$, then each point is a 0-cycle (by definition).

- A smooth map $C : [0, 1] \to M$ defines a singular 1-simplex and

$$\partial C = C(1) - C(0)$$

is a 0-chain.

- More generally, for any $g \in G$, then $gC$ is a singular 1-simplex and

$$\partial(gC) = gC(1) - gC(0).$$

- A piecewise-smooth map $C : [0, 1] \to M$ defines a 1-chain (as a formal sum

$$c_1 = \sum_{k=1}^{r} gC_k$$

of smooth pieces with the same coefficient) and

$$\partial c_1 = gC(1) - gC(0)$$

is a 0-chain.

### Theorem 4.3.3

Let $M$ be a compact connected manifold and $G$ be an Abelian group. Then

$$H_0(M; G) = Gp = \{gp \mid g \in G\},$$

where $p \in M$.

**Proof:**

1. In a connected manifold any two 0-simplexes with the same coefficient are homologous.
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2. Moreover, for any point \( p \in M, g \neq 0 \), and any element \( g \in G \), there is no 1-chain \( c_1 \) such that \( \partial c_1 = gp \).

3. Therefore, a multiple \( gp \) of a single point is not a boundary for any \( g \neq 0 \).

4. Thus, any point \( p \in M \) is a 0-cycle that is not a boundary.

5. Moreover, for any \( g \in G \) and any \( p \in M \) the 0-chain \( gp \) is a 0-cycle that is not a boundary.

\[ \square \]

- In particular,
  \[ H_0(M; \mathbb{Z}) = \{0, \pm p, \pm 2p, \ldots \} = \mathbb{Z}p \]
  and
  \[ H_0(M; \mathbb{R}) = \mathbb{R} \]
  is a one-dimensional vector space.

**Corollary 4.3.3**  Let \( M \) be a compact connected manifold. Then the zero Betti number is equal to

\[ b_0(M) = 1. \]

**Proof:** Follows from above.

\[ \square \]

**Theorem 4.3.4**  Let \( M \) be a compact manifold consisting of \( k \) connected pieces \( M_1, \ldots, M_k \). Then

\[ H_0(M; \mathbb{R}) = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k, \]

where \( p_i \in M_i, i = 1, \ldots, k \), meaning

\[ \mathbb{R}p_1 + \cdots + \mathbb{R}p_k = \left\{ \sum_{i=1}^{k} a_i p_i \mid a_i \in \mathbb{R}, p_i \in M_i, i = 1, \ldots, k \right\}. \]

**Proof:**

\[ \square \]

- In this case
  \[ H_0(M; \mathbb{R}) = \mathbb{R}^k \]
  is a \( k \)-dimensional vector space.
Corollary 4.3.4 Let $M$ be a compact manifold consisting of $k$ connected pieces. Then the zero Betti number is equal to

$$b_0(M) = k.$$ 

Proof: Follows from above.

Let $V$ be a $p$-dimensional oriented closed (compact without boundary) manifold.

Then a triangulation of $V$ defines an integer $p$-cycle, denoted by $[V]$ so that

$$\partial[V] = 0.$$ 

This does not work for non-orientable closed manifolds. A triangulation of a non-orientable closed manifold $V$ gives an integer $p$-chain, which is not a $p$-cycle since

$$\partial[V] \neq 0.$$ 

Example. Klein bottle.

By changing the coefficient group $G$ sometimes one can get a $p$-cycle even for non-orientable manifolds.

Theorem 4.3.5 Let $M$ be an $n$-dimensional compact manifold and $V$ be a closed oriented $p$-dimensional submanifold of $M$. Let $G$ be an Abelian group and $g$ be an element of $G$. Consider a triangulation of $V$ in $p$-triangles and assign to each $p$-triangle in this triangulation of $V$ the same coefficient $g \in G$. Then $g[V]$ defines a $p$-cycle $z_p$ in $H_p(M; G)$.

Proof: Without proof.

Remark. A $p$-cycle is a generalization of the concept of a closed oriented submanifold.

In the case of real homology groups, when $G = \mathbb{R}$, the following theorem is true.
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Theorem 4.3.6  Let $M$ be an $n$-dimensional compact manifold. Every real $p$-cycle $z_p$ in $H_p(M; \mathbb{R})$ is homologous to a finite formal sum

$$z_p \sim \sum_{k=1}^{r} a_k [V_k]$$

of closed oriented $p$-dimensional submanifolds $V_k$ of $M$ with real coefficients $a_k$.

Proof: Nontrivial.

Theorem 4.3.7  Let $M$ be an $n$-dimensional manifold and $G$ be an Abelian group. Let $z_p$ and $z'_p$ be two cycles in $H_p(M; G)$ that can be deformed into each other. Then they are homologous to each other

$$z_p \sim z'_p .$$

Proof:

1. Since the deformation defines a deformation chain $c_{p+1}$ such that

$$\partial c_{p+1} = z'_p - z_p .$$

Proposition 4.3.1  Let $M$ be an $n$-dimensional closed manifold and $G$ be an Abelian group. Then for $p > n$ the singular homology groups $H_p(M; G)$ are trivial

$$H_p(M; G) = 0 .$$

Proof:

1. Singular homology groups are isomorphic to the simplicial homology groups.

2. Since there are no simplicial complexes of dimension $p > n$ then all simplicial homology groups are trivial for $p > n$.
4.3.5 Examples.

• **Sphere** $S^n$.
  
  From the facts that $S^n$ is connected, orientable and closed it follows that
  
  \[
  H_0(S^n; G) = H_n(S^n; G) = G,
  \]
  
  \[
  H_p(S^n; G) = 0, \quad \text{for } p \neq 0, n, 
  \]
  
  \[
  B_0(S^n) = B_n(S^n) = 1,
  \]
  
  \[
  B_p(S^n) = 0, \quad \text{for } p \neq 0, n.
  \]

• **Torus** $T^2$.
  
  \[
  H_0(T^2; G) = H_2(T^2; G) = G,
  \]
  
  \[
  H_1(T^2; G) = GA + GB,
  \]
  
  \[
  B_0(T^2) = 1, \quad B_1(T^2) = 2, \quad B_2(T^2) = 1,
  \]
  
  where $A$ and $B$ are the basic 1-cycles.

• **Klein Bottle** $K^2$.
  
  Since $K^2$ is connected closed non-orientable it follows that
  
  \[
  H_0(K^2; \mathbb{Z}) = \mathbb{Z},
  \]
  
  \[
  H_2(K^2; \mathbb{Z}) = 0,
  \]
  
  \[
  H_1(K^2; \mathbb{Z}) = \mathbb{Z}A + \mathbb{Z}B,
  \]
  
  where $A$ and $B$ are the basic 1-cycles, and
  
  \[
  H_0(K^2; \mathbb{R}) = \mathbb{R},
  \]
  
  \[
  H_2(K^2; \mathbb{R}) = 0,
  \]
  
  \[
  H_1(K^2; \mathbb{R}) = \mathbb{R}A,
  \]
  
  \[
  B_0(K^2) = 1, \quad B_1(K^2) = 1, \quad B_2(K^2) = 0.
  \]

• **Real Projective Plane** $\mathbb{R}P^2$.
  
  $\mathbb{R}P^2$ is connected closed non-orientable.
  
  \[
  H_0(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z},
  \]
  
  \[
  H_2(\mathbb{R}P^2; \mathbb{Z}) = 0,
  \]
  
  \[
  H_1(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2A,
  \]
where $A$ is the basic 1-cycle, and
\[
\begin{align*}
H_0(\mathbb{RP}^2; \mathbb{R}) &= \mathbb{R}, \\
H_2(\mathbb{RP}^2; \mathbb{R}) &= 0, \\
H_1(\mathbb{RP}^2; \mathbb{R}) &= 0, \\
B_0(\mathbb{RP}^2) &= 1, \\
B_1(\mathbb{RP}^2) &= 0, \\
B_2(\mathbb{RP}^2) &= 0.
\end{align*}
\]

- **Torus** $T^3$.

- $T^3$ is a connected closed orientable manifold.

\[
\begin{align*}
H_0(T^3; \mathbb{Z}) &= H_3(T^3; \mathbb{Z}) = \mathbb{Z}, \\
H_1(T^3; \mathbb{Z}) &= \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}C, \\
H_2(T^3; \mathbb{Z}) &= \mathbb{Z}D + \mathbb{Z}E + \mathbb{Z}F,
\end{align*}
\]

where $A$, $B$ and $C$ are basic 1-cycles, and $D$, $E$ and $F$ are basic 2-cycles,
\[
\begin{align*}
B_0(T^3) &= 1, \\
B_1(T^3) &= B_2(T^3) = 3, \\
B_2(T^3) &= 1.
\end{align*}
\]

- **Real Projective Space** $\mathbb{RP}^3$.

- $\mathbb{RP}^3$ is connected closed orientable.

\[
\begin{align*}
H_0(\mathbb{RP}^3; \mathbb{R}) &= H_3(\mathbb{RP}^3; \mathbb{R}) = \mathbb{R}, \\
H_1(\mathbb{RP}^3; \mathbb{R}) &= H_2(\mathbb{RP}^3; \mathbb{R}) = 0, \\
B_0(\mathbb{RP}^3) &= B_3(\mathbb{RP}^3) = 1, \\
B_1(\mathbb{RP}^3) &= B_2(\mathbb{RP}^3) = 0.
\end{align*}
\]
4.4 de Rham Cohomology Groups

- Let $M$ be a manifold and $G = \mathbb{R}$ be the coefficient group.

- Then $C_p(M; \mathbb{R}), Z_p(M; \mathbb{R}), B_p(M; \mathbb{R})$ and $H_p(M; \mathbb{R})$ are vector spaces.

- For simplicity we will denote them in this section simply by $C_p(M), Z_p(M), B_p(M)$ and $H_p(M)$.

- Let $C^p(M) = C^\infty(\Lambda^p(M))$ be the space of smooth $p$-forms on $M$.

- We will call the closed $p$-form $p$-cocyles and the space
  
  $$Z^p(M) = \{ \alpha_p \in C^p(M) \mid d_p \alpha_p = 0 \}$$

  of all closed $p$-forms on $M$, the cocycle group.

- The exact $p$-forms on $M$ are called the $p$-coboundaries and the space
  
  $$B^p(M) = \{ \alpha_p \in Z^p(M) \mid \alpha_p = d_{p-1} \beta_{p-1} \text{ for some } \beta_{p-1} \in C^{p-1}(M) \}$$

  of all exact $p$-forms on $M$ is called the coboundary group.

- Both $Z^p(M)$ and $B^p(M)$ are vector spaces with real coefficients.

- Recall that the exterior derivative is a map
  
  $$d_p : C^p(M) \to C^{p+1}(M)$$

  such that
  
  $$\text{Ker } d_p = Z^p(M)$$

  and
  
  $$\text{Im } d_{p-1} = B^p(M).$$

- Two closed forms are said to be equivalent (or cohomologous) if they differ by an exact form.

- The collection of all equivalence classes of closed forms is the quotient vector space
  
  $$H^p(M) = Z^p(M)/B^p(M)$$

  called the $p$-th de Rham cohomology group.
• de Rham cohomology groups are vector spaces.

• Let
  \[ c_p = \sum_{k=1}^{r} a_k \sigma_p^k \]
  be a real \( p \)-chain in \( M \), and \( \alpha \) be a \( p \)-form on \( M \).

• We define the integral of \( \alpha \) over \( c_p \) by
  \[ \langle \alpha, c_p \rangle = \int_{c_p} \alpha = \sum_{k=1}^{r} a_k \int_{\sigma_p^k} \alpha . \]

• Thus every \( p \)-form on \( M \) defines a linear functional on \( C_p(M) \)
  \[ \alpha : C_p(M) \to \mathbb{R}, \]
  by
  \[ c_p \mapsto \langle \alpha, c_p \rangle . \]

• The space of all \( p \)-forms can be naturally identified with the dual space \( C^*_p(M) \)
  \[ C^p(M) \cong C^*_p(M) . \]

• Furthermore, by Stokes theorem we have for a \((p - 1)\)-form
  \[ \langle d\alpha, c_p \rangle = \langle \alpha, \partial c_p \rangle . \]

• Thus, for every \( p \)-cycle \( z_p \), that is, if \( \partial z_p = 0 \),
  \[ \langle d\alpha, z_p \rangle = 0 , \]
  and for every closed form \( \alpha \), that is, if \( d\alpha = 0 \),
  \[ \langle \alpha, \partial c_p \rangle = 0 . \]

• More generally, let \( \alpha_p \in Z^p(M) \) be a closed \( p \)-form, \( \beta_{p+1} \in C^{p+1}(M) \) be a \((p+1)\)-form, \( z_p \in Z_p(M) \) be a \( p \)-cycle and \( c_{p+1} \in C_{p+1}(M) \) be a \((p+1)\)-chain. Then
  \[ \langle \alpha + d\beta, z_p + \partial c_p \rangle = \langle \alpha, z_p \rangle . \]
Therefore, for every equivalence class \([\alpha_p] \in H^p(M)\) of closed forms we can define a linear functional \(H_p(M) \to \mathbb{R}\) on the space of homology groups by, for any \([z_p] \in H_p(M)\),

\[
\langle [\alpha_p], [z_p] \rangle = \langle \alpha_p, z_p \rangle.
\]

This is well defined since the right hand side does not depend on the choice of representatives in the equivalence classes.

This naturally identifies the space of cycles with the space of cocycles

\[Z^p(M) \cong Z^*_p(M).\]

For a closed \(p\)-form \(\alpha_p\) and a \(p\)-cycle \(z_p\) the value of the functional

\[\langle \alpha, z_p \rangle = \int_{z_p} \alpha_p\]

is called the period of the form \(\alpha_p\) on the cycle \(z_p\).

We conjecture that

\[H^p(M) \cong H^*_p(M).\]

**Proposition 4.4.1.** Let \(M\) be a closed manifold. Then for any linear functional \(\varphi : H_p(M) \to \mathbb{R}\) on homology groups there is a closed \(p\)-form \(\alpha_p\) such that

\[\varphi(z_p) = \langle \alpha_p, z_p \rangle .\]

**Proof:** Difficult.

**Corollary 4.4.1.** Let \(M\) be a closed manifold. Let \(k = b_p(M)\) be the \(p\)-th Betti number. Let \(z_p^{(1)}, \ldots, z_p^{(k)}\), be a basis of \(p\)-cycles in the homology groups \(H_p(M)\) and \(\pi_1, \ldots, \pi_k\) be arbitrary real numbers. Then there is a closed \(p\)-form \(\alpha_p\) such that

\[\langle \alpha_p, z_p^{(i)} \rangle = \pi_i, \quad i = 1, 2, \ldots, k.\]

**Proof:**
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**Proposition 4.4.2.** Let $M$ be a closed manifold. Let $\alpha_p \in Z^p(M)$ be a closed $p$-form on $M$ such that for any $p$-cycle $z_p \in Z_p(M)$

\[
\langle \alpha_p, z_p \rangle = 0.
\]

Then the $p$-form $\alpha_p$ is exact.

**Proof:** Difficult. ■

**Theorem 4.4.1** de Rham Theorem. Let $M$ be a closed manifold. Then the map

\[
H^p(M) \to H^*_p(M),
\]

that associates to each equivalence class $[\alpha_p]$ of closed $p$-forms a linear functional $H_p(M) \to \mathbb{R}$ on the homology group $H_p(M)$ defined by

\[
[z_p] \mapsto \langle \alpha_p, z_p \rangle,
\]

is an isomorphism.

**Proof:** ■

- **Theorem. Poincaré Lemma.** Any closed form is locally exact.
- **Torus $T^2$.**
- **Closed Surfaces in $\mathbb{R}^3$.**
- The wedge product of forms defines a natural product of cohomology classes

\[
\wedge : H^p(M) \times H^q(M) \to H^{p+q}(M)
\]

by

\[
[\alpha] \wedge [\beta] = [\alpha \wedge \beta]
\]

- This enables one to define the **cohomology ring**

\[
H^*(M) = \bigoplus_{p=0}^{n} H^p(M)
\]
Künneth formula. The cohomology groups of the product of two manifolds are
\[ H^p(M_1 \times M_2) = \bigoplus_{r+q=p} H^r(M_1) \otimes H^q(M_2) \]

Corollary. The Betti numbers of the product of two manifolds are
\[ b_p(M_1 \times M_2) = \sum_{r+q=p} b_r(M_1)b_q(M_2) \]

This suggest defining Poincaré polynomial
\[ P_M(t) = \sum_{p=0}^n t^p b_p(M) \]

Note that the Euler characteristic is given by the value at \( t = -1 \)
\[ \chi(M) = P_M(-1) \]

Then the Poincaré polynomial is multiplicative, that is,
\[ P_{M_1 \times M_2}(t) = P_{M_1}(t)P_{M_2}(t) \]

Corollary. The Euler characteristic of the product of two manifolds is multiplicative
\[ \chi(M_1 \times M_2) = \chi(M_1)\chi(M_2) \]

Torus \( T^n = S^1 \times \cdots \times S^1 \). The Betti numbers of the \( n \)-torus are
\[ b_p(T^n) = \binom{n}{p} \]

The Euler characteristic of the torus \( T^n \) vanishes,
\[ \chi(T^n) = 0 \]

Let \( F : M \to V \) be a smooth map. Recall that the pullback commutes with the exterior derivative and preserves the wedge product,
\[ F^*d\alpha = dF^*\alpha, \quad F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta \]
4.4. DE RHAM COHOMOLOGY GROUPS

- Therefore, one can define the pullback of the cohomology groups
  \[ F^* : H^p(V) \to H^p(M) \]
  by
  \[ F^* [\omega] = [F^* \omega] \]
- The pullback preserves the cohomology ring structure by
  \[ F^* ([\alpha] \wedge [\beta]) = [F^* \alpha] \wedge [F^* \beta] \]

4.4.1 Hopf Invariants

- Let \( F : S^3 \to S^2 \) be a smooth map.
- Let \( \omega \) be a 2-form on \( S^2 \) such that
  \[ \int_{S^2} \omega = 1 \]
- Then the pullback of this form to \( S^3 \) is closed
  \[ dF^* \omega = F^* d\omega = 0 \]
- Since \( H^2(S^3) = 0 \) every closed 2-form on \( S^3 \) is exact, that is,
  \[ F^* \omega = d\alpha \]
  for some 1-form \( \alpha \) on \( S^3 \).
- Then the number
  \[ H(F) = \int_{S^3} \alpha \wedge d\alpha \]
  is called the Hopf invariant of the map \( F \).
- More generally, let \( F : M^{2n-1} \to V^n \) be a smooth map.
- Assume that \( H^n(M) = 0 \).
- Let \( \omega \) be a \( n \)-form on \( V \) such that
  \[ \int_V \omega = 1 \]
• Then the pullback of this form to $M$ is closed

$$dF^* \omega = F^* d\omega = 0$$

• Since $H^n(M) = 0$, every closed $n$-form on $M$ is exact, that is,

$$F^* \omega = d\alpha$$

for some $(n-1)$-form $\alpha$ on $M$.

• Then the number

$$H(F) = \int_M \alpha \wedge d\alpha$$

is called the **Hopf invariant** of the map $F$. 
4.5 Harmonic Forms

- Let \((M, g)\) be a closed oriented \(n\)-dimensional Riemannian manifold with a Riemannian metric \(g\) and the Riemannian volume \(n\)-form \(\text{vol}\).
- Let \(\Lambda_p(M)\) be the bundle of \(p\)-forms.
- Recall that there is a natural fiber inner product on \(\Lambda_p\) defined by
  \[
  \langle \alpha, \beta \rangle = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p},
  \]
  and the corresponding fiber norm
  \[
  ||\alpha|| = \sqrt{\langle \alpha, \alpha \rangle}.
  \]
- Also, there is a duality between \(p\)-forms and \((n - p)\)-forms defined by the Hodge star operator
  \[
  * : \Lambda_p \to \Lambda_{n-p},
  \]
  that maps any \(p\)-form \(\alpha\) to a \((n - p)\)-form \(*\alpha\) dual to \(\alpha\) defined as follows.
- For each \(p\)-form \(\alpha\) the form \(*\alpha\) is the unique \((n - p)\)-form such that for any \(p\)-form \(\beta\)
  \[
  \beta \wedge *\alpha = \langle \beta, \alpha \rangle \text{vol}.
  \]
- Note that
  \[
  \alpha \wedge *\beta = \beta \wedge *\alpha
  \]
- In components, this means that
  \[
  (*\alpha)_{i_1 \ldots i_{n-p}} = \frac{1}{p!} E_{j_1 \ldots j_p i_1 \ldots i_{n-p}} \alpha^{j_1 \ldots j_p},
  \]
  where
  \[
  E_{j_1 \ldots j_n} = \sqrt{g} e_{j_1 \ldots j_n}
  \]
- Recall that the Hodge star maps forms to pseudo-forms and vice-versa.
- Recall also that for any \(p\)-form \(\alpha\),
  \[
  *^2 \alpha = (-1)^{p(n-p)} \alpha,
  \]
  meaning that
  \[
  *^{-1} \alpha = (-1)^{p(n-p)} * \alpha.
  \]
• Since $\ast$ commutes with the exterior derivative, the Hodge star $\ast$ defines a
duality between the cohomology groups,

$$\ast : H^p(M) \rightarrow H^{n-p}(M)$$
called the Poincaré duality.

• Therefore, Betti numbers satisfy

$$b_p(M) = b_{n-p}(M).$$

• **Corollary.** The Euler characteristic of an odd-dimensional manifold van-
ishes.

• The **coderivative** is a linear map

$$\delta : \Lambda_p \rightarrow \Lambda_{p-1}$$
defined by

$$\delta = (-1)^{p(p+1)} d \ast .$$

• The exterior derivative and the coderivative satisfy the important conditions

$$d^2 = \delta^2 = 0.$$

• A $p$-form $\alpha$ such that

$$\delta \alpha = 0$$
is called **co-closed**.

• A $p$-form $\alpha$ such that

$$\alpha = \delta \beta$$
for some $(p+1)$ form $\beta$ is called **co-exact**.

• **Exercise.** Show that the coderivative of a $p$-form $\alpha$ is the $(p-1)$-form $\delta \alpha$
with components

$$(\delta \alpha)_{i_1 \ldots i_{p-1}} = \nabla_j \alpha^j_{i_1 \ldots i_{p-1}}$$
Now we define the $L^2$-inner product of $p$-forms by

$$(\alpha, \beta)_{L^2} = \int_M \alpha \wedge \ast \beta = \int_M \langle \alpha, \beta \rangle \text{vol} \, ,$$

and the $L^2$-norm

$$||\alpha||_{L^2} = \sqrt{(\alpha, \alpha)_{L^2}} \, .$$

This makes the space $C^\infty(\Lambda_p(M))$ of smooth $p$-forms an inner-product vector space.

The completion of $C^\infty(\Lambda_p(M))$ in the $L^2$-norm gives the Hilbert space $L^2(\Lambda_p(M))$ of square-integrable $p$-forms.

Let $A : H \to H$ be an operator on a Hilbert space $H$. The adjoint of the operator $A$ with respect to the inner product of the space $H$ is the operator $A^* : H \to H$ defined by, for any $\varphi, \psi \in H$,

$$(A \varphi, \psi) = (\varphi, A^* \psi) \, .$$

**Theorem 4.5.1** Let $M$ be a closed orientable Riemannian manifold. Then the adjoint of the exterior derivative is the negative coderivative

$$d^* = -\delta \, .$$

That is, for any $p-1$-form $\alpha$ and any $p$-form $\beta$,

$$(d\alpha, \beta) = -(\alpha, \delta \beta) \, .$$

**Proof:**

1.

**Theorem 4.5.2** Let $M$ be a compact orientable Riemannian manifold with boundary. Then for any $p$-form $\alpha$ and any $(p+1)$-form $\beta$,

$$(d\alpha, \beta) + (\alpha, \delta \beta) = \int_{\partial M} \alpha \wedge \ast \beta \, .$$

**Proof:**
1. Direct calculation.

- Notice that in case of a manifold with boundary the coderivative is the negative adjoint of the exterior derivative only on the forms that satisfy one of the two types of boundary conditions:
  \[ \alpha|_{\partial M} = 0 \quad \text{or} \quad \ast \alpha|_{\partial M} = 0. \]

- Let \( g \) be the Riemannian metric and \( \nabla \) be the corresponding Levi-Civita connection. It defines a natural connection on the bundle of \( p \)-forms. The **Bochner Laplacian** (or metric Laplacian, or canonical Laplacian, or covariant Laplacian) on \( p \)-forms is the operator
  \[ \Delta : C^\infty(\Lambda_p(M)) \to C^\infty(\Lambda_p(M)) \]
  defined by
  \[ \Delta = g^{ij} \nabla_i \nabla_j. \]

- The **Hodge Laplacian** on \( p \)-forms is the operator
  \[ L : C^\infty(\Lambda_p(M)) \to C^\infty(\Lambda_p(M)) \]
  defined by
  \[ L = d\delta + \delta d = (d + \delta)^2. \]

- The operators \( d \) and \( \delta \) commute with the Hodge Laplacian, i.e.
  \[ dL = Ld, \quad \delta L = L\delta. \]

**Theorem 4.5.3**  
For any \( p \) there holds

\[ L = \Delta - W, \]

where \( W : \Lambda_p \to \Lambda_p \) is an endomorphism on the bundle of \( p \)-forms called the **Weitzenböck endomorphism**.

**Proof:**

1.
4.5. HARMONIC FORMS

- The Hodge star also commutes with both Laplacians
  \[ *L = L*, \quad \Delta * = *\Delta \]

- Therefore, the Weitzenböck endomorphism satisfies
  \[ *W = W*. \]

- Weitzenböck endomorphism is a linear combination of Riemann curvature tensor, that is, when acting on \( p \)-forms \( W \) has the form
  \[ W^{i_1\ldots i_p}_{j_1\ldots j_p} = F^{mni_1\ldots i_p}_{klj_1\ldots j_p} R^{kl}_{mn}, \]
  where \( F^{mni_1\ldots i_p}_{klj_1\ldots j_p} \) is constructed only from the Kronecker symbol \( \delta_j^i \) and the metric \( g_{ij} \) and \( g^{ij} \).

- **Exercise.** Obtain the expression for the Weitzenböck endomorphism for \( p \)-forms. **Hint:** replace partial derivatives by covariant derivatives and use the definition of the curvature.

- Since the Levi-Civita connection is compatible with the metric and it is torsion free it allows to rewrite all formulas for the exterior derivative and the co-derivative in terms of covariant derivatives
  \[
  (d\alpha)_{\mu_1\mu_2\ldots\mu_p} = (p + 1)\nabla_{[\mu_1} \alpha_{\mu_2\ldots\mu_p+1]}, \\
  (\delta\alpha)_{\lambda_1\ldots\lambda_{p-1}} = \nabla^\mu \alpha_{\lambda_1\ldots\lambda_{p-1}}. 
  \]

- Therefore, the Hodge Laplacian on \( p \)-forms has the form
  \[
  (L\omega)_{\alpha_1\ldots\alpha_p} = p \nabla_{[\alpha_1} \nabla^{\beta} \omega_{\beta\alpha_2\ldots\alpha_p]} + (p + 1)\nabla^{\beta} \nabla_{[\beta} \omega_{\alpha_1\ldots\alpha_p]}.
  \]

- This can be easily simplified to
  \[
  (L\omega)_{\alpha_1\ldots\alpha_p} = \Delta \omega_{\alpha_1\ldots\alpha_p} + p[\nabla_{[\alpha_1}, \nabla^{\beta}] \omega_{\beta\alpha_2\ldots\alpha_p}].
  \]

- Now by commuting covariant derivatives and using the Ricci identities we obtain
  \[
  (W\omega)_{\alpha_1\ldots\alpha_p} = p R^{\beta}_{[\alpha_1 \omega_{[\beta\alpha_2\ldots\alpha_p]}} + p \sum_{k=2}^{p} R^\mu_{[\alpha_2\alpha_3\ldots\alpha_k+1]} R^\beta_{\alpha_1 \alpha_k+1\ldots\alpha_p]} \omega_{\beta\alpha_2\ldots\alpha_k+1]} (4.2),
  \]
  where the indices \( \mu \) and \( \beta \) are excluded from anti-symmetrization.
• Now, permuting the indices $\alpha_i$ and using the property of the Riemann tensor

$$R^{\mu}_{\ [\rho}^{\nu} \sigma] = \frac{1}{2} R^{\mu\nu}_{\rho\sigma},$$

we get

$$(W\omega)_{\alpha_1...\alpha_p} = pR^{\nu}_{\ [\alpha_1\omega] [\alpha_2...\alpha_p]} - \frac{p(p-1)}{2} R^{\mu\nu}_{\ [\alpha_1\alpha_2\omega] [\nu\alpha_3...\alpha_p]}.$$  

• Of course, for $p = 0$ the Weitzenböck endomorphism is equal to zero. Also, the second term linear in Riemann curvature is present only for $p \geq 2$.

• Let us list particular formulas for one-forms

$$\langle \beta, W\omega \rangle = \langle \beta, \omega \rangle = \text{Ric}(\beta, \omega),$$

and for two-forms,

$$\langle \beta, \omega \rangle = \frac{1}{p!} \beta^{\alpha_1...\alpha_p} (W\omega)_{\alpha_1...\alpha_p} = \frac{1}{p!} \beta^{\alpha_1...\alpha_p} W_{\rho\sigma}^{\mu\nu} \omega_{\mu\nu\alpha_3...\alpha_p},$$

where

$$W^{\mu\nu}_{\rho\sigma} = pR^{\alpha\mu}_{\ [\nu\alpha]} [\sigma] - \frac{p(p-1)}{2} R^{\mu\nu}_{\rho\sigma}.$$  

• A $p$-form $\alpha$ is called harmonic if

$$L\alpha = 0.$$
### Theorem 4.5.4
Let $M$ be a closed Riemannian manifold. Then a $p$-form $\alpha$ is harmonic if and only if it is closed and coclosed, that is,

$$d\alpha = \delta\alpha = 0.$$ 

**Proof:**

1. Use

$$0 = (L\alpha, \alpha) = \|d\alpha\|^2 + \|\delta\alpha\|^2.$$ 

---

### Theorem 4.5.5
**Hodge Theorem.** Let $M$ be a closed Riemannian manifold. Then:

1. The vector space $H^p(M)$ of harmonic $p$-forms on $M$ is finite-dimensional.

2. The equation $L\alpha = \rho$ has a solution if and only if $\rho$ is orthogonal to $H^p(M)$.

**Proof:**

1. 

---

### Theorem 4.5.6
Let $M$ be a closed Riemannian manifold. Then any $p$-form $\beta$ is a sum of an exact form $d\alpha$, a coexact form $\delta\gamma$ and a harmonic form $h$, that is,

$$\beta = d\alpha + \delta\gamma + h.$$ 

In other words, there is an orthogonal decomposition (called the **Hodge decomposition**)

$$C^\infty(\Lambda_p) = \text{Im } d_{p-1} \oplus \text{Im } \delta_{p+1} \oplus H^p(M).$$

**Proof:**
Corollary 4.5.1 Let $M$ be a closed Riemannian manifold. Then any closed $p$-form $\beta$ is a sum of an exact form $d\alpha$ and a harmonic form $h$, that is,

$$\beta = d\alpha + h.$$ 

\textbf{Proof}:

1. 

Corollary 4.5.2 Let $M$ be a closed Riemannian manifold. Then each de Rham class of cohomologous closed $p$-forms has a unique harmonic representative, that is,

$$H^p(M) \cong \mathcal{H}^p(M)$$

This means that the Betti numbers are equal to the number of linearly independent harmonic forms

$$b_p(M) = \dim \mathcal{H}^p(M).$$

Corollary 4.5.3 Let $M$ be a closed Riemannian manifold. Let $k = b_p(M)$ be the $p$-th Betti number. Let $z_p^{(1)}, \ldots, z_p^{(k)}$ be a basis of real $p$-cycles in the real homology groups $H_p(M)$ and $\pi_1, \ldots, \pi_k$ be arbitrary real numbers. Then there is a unique harmonic $p$-form $h_p$ such that

$$\langle h_p, z_p^{(i)} \rangle = \pi_i, \quad i = 1, 2, \ldots, k.$$ 

\textbf{Proof}:

1. 

The importance of Weitzenböck formulas lies in the integral identity

$$||d\omega||^2_{L^2} + ||\delta\omega||^2_{L^2} = -\langle \omega, L\omega \rangle_{L^2}$$

$$= -\langle \omega, \Delta\omega \rangle_{L^2} + \langle \omega, W\omega \rangle_{L^2}$$

$$= \|
abla\omega\|^2_{L^2} + \langle \omega, W\omega \rangle_{L^2}. \quad (4.10)$$
• Therefore, if $W$ is strictly positive uniformly throughout the manifold so that

$$\langle \omega, W\omega \rangle_{L^2} = \int_M d\text{vol} \langle \omega, W\omega \rangle > 0, \quad (4.11)$$

then the above expression is strictly positive, which means that there are no harmonic forms, therefore, the corresponding Betti number vanishes.

• For example, for one-forms this means that there cannot exist harmonic one-forms on a manifold with strictly positive Ricci curvature.

• The metric $g$ is said to have **positive Ricci curvature** if its Ricci tensor is positive-definite.

**Corollary 4.5.4 Bochner Theorem** Let $M$ be a closed Riemannian manifold with positive Ricci curvature. Then the first Betti number vanishes, i.e.

$$b_1(M) = 0.$$  

That is there are no harmonic 1-forms on $M$.

**Proof:**

1. Let $h$ be a harmonic 1-form. Then

$$0 = \frac{1}{2} \int_M \Delta \langle h, h \rangle \text{vol} = \int_M R^{ij} h_i h_j \text{vol} + \|\nabla h\|^2 \geq 0.$$  

2. Thus $h = 0$.

• In general, positive Ricci curvature and negative sectional curvature work against harmonic forms. If Ricci curvature is strictly positive and sectional curvature is strictly negative then there are no harmonic forms for $1 \leq p \leq n - 1$, that is, their Betti numbers in corresponding dimension are equal to zero

$$b_p(M) = 0 \quad \text{for} \quad p = 1, 2, \ldots, (n - 1). \quad (4.12)$$

• Given two orthonormal tangent vectors $u, v$ at a point of a manifold the **sectional curvature** of a metric $g$ is defined by

$$K(u, v) = R_{ijkl} u^i v^k u^j v^l.$$
The metric \( g \) has **constant sectional curvature** if and only if
\[
R_{ijkl} = \Lambda (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k),
\]
where \( \Lambda \) is a constant.

In this case, the Ricci tensor and the scalar curvature are
\[
R_i^j = (n-1)\Lambda \delta_i^j,
\]
\[
R = n(n-1)\Lambda.
\]

**Theorem.** Let \( M \) be a closed Riemannian manifold with positive constant sectional curvature. Then
\[
B_p(M) = 0 \quad \text{for} \quad p = 1, 2, \ldots, (n-1).
\]

Since there is exactly one harmonic 0-form (a constant) on compact manifolds and the dual \( n \)-form then
\[
B_0(M) = B_n(M) = 1,
\]
and, therefore, the Euler characteristic of such manifolds is
\[
\chi(M) = \sum_{k=0}^{n} (-1)^k B_k(M) = 1 + (-1)^n,
\]
which is equal to 2 for even \( n \) and to zero for odd \( n \).

Thus such manifolds cannot have a rich topology. The sphere \( S^n \) is such a manifold with positive constant curvature. That is why, \( \chi(S^{2n}) = 2 \) and \( \chi(S^{2n+1}) = 0 \).

**Remark.** The elements of the first homology group \( H_1(M, G) \) are equivalence classes of 1-cycles.

The 1-cycles are closed oriented curves (loops) on \( M \).

If a closed curve can be deformed to a point, then it is a boundary of a surface (a 2-simplex).

That is, a closed curve that can be contracted to a point is a trivial 1-cycle.
For any simply connected manifold $M$ and any Abelian group $G$,
Chapter 5

Homotopy Theory

5.1 Homotopy Groups

• Let $X$ and $Y$ be topological spaces and

$$f_0 : X \to Y, \quad f_1 : X \to Y$$

be two continuous maps. Then the map $f_0$ is said to be homotopic to the map $f_1$ if there is a continuous map

$$F : X \times [0, 1] \to Y$$

such that for any $x \in X$

$$F(x, 0) = f_0(x), \quad F(x, 1) = f_1(x).$$

• In other words, if one can continuously deform the map $f_0$ to the map $f_1$.

• Homotopy is an equivalence relation.

• We fix a point $x_0 \in X$. Let $f : X \to Y$ and $y_0 = f(x_0) \in Y$. The homotopy class of $f$ at the basepoint $y_0$ is the set of all maps $g : X \to Y$ homotopic to $f$ such that $g(x_0) = y_0$

$$[f, y_0] = \{ g : X \to Y \mid g \sim f, \ g(x_0) = y_0 \}$$

• The homotopy classes are invariant under homeomorphisms and are topological invariants.
• The set \([X, Y; y_0]\) of all homotopy classes of maps from \(X\) to \(Y\) with the basepoint \(y_0\) is a topological invariant.

• Usually one fixes a standard manifold \(X = S^n\) and studies the homotopy classes of continuous maps \(f : S^n \rightarrow Y\). These are topological invariants of \(Y\).

• The set of homotopy classes of maps \(f : S^n \rightarrow Y\) is called the \(n\)-th homotopy group \(\pi_n(Y)\) of \(Y\),

\[
\pi_n(Y, y_0) = [S^n, Y; y_0]
\]

• The set of homotopy classes of maps \(f : S^1 \rightarrow Y\) is called the fundamental group \(\pi_1(Y, y_0)\) of \(Y\).

5.2 Fundamental Group

• Let \(X\) be a topological space. A path with an initial point \(x_0\) and an endpoint \(x_1\) is a continuous map

\[
\alpha : [0, 1] \rightarrow X
\]

such that

\[
\alpha(0) = x_0, \quad \alpha(1) = x_1.
\]

• A topological space \(X\) is arcwise connected if for any two points \(x, y \in X\) there is a path \(\alpha\) with the initial point \(x\) and the endpoint \(y\).

• A loop with a base point \(x_0\) is a closed path such that

\[
\alpha(0) = \alpha(1) = x_0
\]

• A constant path \(c_{x_0}\) is defined by

\[
c_{x_0}(t) = x_0, \quad t \in [0, 1].
\]

• A topological space is simply connected if any loop in it can be continuously shrunk to a point.
5.2. FUNDAMENTAL GROUP

- The **product** \( \gamma = \alpha \ast \beta \) of two paths, \( \alpha \) and \( \beta \), such that \( \alpha(1) = \beta(0) \), is a path defined by
  \[
  \gamma(t) = \begin{cases}
  \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\
  \beta(2t - 1), & \frac{1}{2} \leq t \leq 1
  \end{cases}
  \]

- The **inverse** \( \alpha^{-1} \) of a path \( \alpha \) is the path defined by,
  \[
  \alpha^{-1}(t) = \alpha(1 - t), \quad 0 \leq t \leq 1.
  \]

- Two loops \( \alpha_0, \alpha_1 \) with a base point \( x_0 \) are said to be **homotopic**, \( \alpha_0 \sim \alpha_1 \), if there is a continuous map \( F : [0, 1] \times [0, 1] \rightarrow X \) (called a **homotopy** between \( \alpha_0 \) and \( \alpha_1 \)) such that
  \[
  F(t, 0) = \alpha_0(t), \quad 0 \leq t \leq 1,
  \]
  \[
  F(t, 1) = \alpha_1(t), \quad 0 \leq t \leq 1,
  \]
  \[
  F(0, s) = F(1, s) = x_0, \quad 0 \leq s \leq 1,
  \]

- **Proposition.** The loop homotopy is an equivalence relation.

- **Proof.**
  - The equivalence classes of loops are denoted by \([\alpha]\) and are called **homotopy classes**.
  - The set \( \pi_1(X, x_0) \) of homotopy classes of loops with base point \( x_0 \) is called the **fundamental group** (or the **first homotopy group**) of \( X \) at \( x_0 \).
  - The product of homotopy classes is defined by
    \[
    [\alpha] \ast [\beta] = [\alpha \ast \beta]
    \]
  - The inverse of the homotopy class is defined by
    \[
    [\alpha]^{-1} = [\alpha^{-1}]
    \]
  - The homotopy class \([c_{x_0}]\) defines the identity element.

- **Proposition.** Let \( \alpha_0, \alpha_1, \beta_0, \beta_1 \) be loops based at \( x_0 \in X \). Suppose that \( \alpha_0 \sim \alpha_1 \) and \( \beta_0 \sim \beta_1 \). Then
  \[
  \alpha_0^{-1} \sim \alpha_1^{-1}
  \]
  \[
  \alpha_0 \ast \beta_0 \sim \alpha_1 \ast \beta_1
  \]
• **Proof.**

• **Proposition.** The product and the inverse of homotopy classes do not depend on the representatives.

• **Proof.**

• **Theorem.** The fundamental group is a group.

• **Proof.** Prove 1) associativity, 2) unit element, 3) inverse.

• A fundamental group is **trivial** if \( \pi_1(X, x_0) = \{e\} \).

• **Theorem.** Let \( X \) be an arcwise connected topological space. Then for any two points \( x_0, x_1 \in X \), \( \pi_1(X, x_0) \) is isomorphic to \( \pi_1(X, x_1) \).

• So, we can just denote it by \( \pi_1(X) \).

• **Proof.**

• Let \( \alpha \) be a loop at \( x_0 \).

• Let \( \gamma \) be a path from \( x_0 \) to \( x_1 \).

• Then \( \gamma^{-1} \ast \alpha \ast \gamma \) is a loop at \( x_1 \).

• This induces a map

\[
\sigma_\gamma : \pi_1(X, x_0) \to \pi_1(X, x_1).
\]

defined by

\[
\sigma_\gamma([\alpha]) = [\gamma^{-1} \ast \alpha \ast \gamma]
\]

• It is easy to see that this map is a homomorphism.

• This map has an inverse

\[
\sigma_{\gamma}^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)
\]

defined by

\[
\sigma_\gamma([\beta]) = [\gamma \ast \beta \ast \gamma^{-1}]
\]

• Therefore, this map is bijective and is an isomorphism.
5.3 Homotopy Type

- Let $X$ and $Y$ be topological spaces. We say that $X$ and $Y$ are of the same **homotopy type** if there are two continuous maps

$$f : X \to Y, \quad g : Y \to X$$

such that the maps

$$f \circ g : Y \to Y, \quad g \circ f : X \to X$$

are homotopic to identity maps,

$$f \circ g \sim \text{id}_Y, \quad g \circ f \sim \text{id}_X.$$  

- The map $f$ is called the **homotopy equivalence** and the map $g$ is the **homotopy inverse**.

- **Proposition.** Having the same homotopy type is an equivalence relation.

- **Proof.**

- **Lemma.** Let $f_i : X \to Y$, $i = 0, 1$, be maps between topological spaces $X$ and $Y$ and $F : X \times [0, 1] \to Y$ be a homotopy between $f_0$ and $f_1$. Let $x_0 \in X$ and $\gamma$ be a path in $Y$ with initial point $f_0(x_0)$ and endpoint $f_1(x_0)$. Let

$$\sigma_\gamma : \pi_1(Y, f_0(x_0)) \to \pi_1(Y, f_1(x_0))$$

be a group isomorphism defined by

$$\sigma_\gamma([\alpha]) = [\gamma^{-1} * \alpha * \gamma].$$

Then there are induced group homomorphisms

$$f_{i,*} : \pi_1(X, x_0) \to \pi_1(Y, f_i(x_0)),$$

defined by

$$f_{i,*}([\alpha], x_0) = ([f(\alpha)], f_i(x_0)),$$

such that

$$f_{1,*} = \sigma_\gamma \circ f_{0,*}. $$
• **Theorem.** Connected topological spaces with the same homotopy type have the same fundamental group.

In other words: Let $X$ and $Y$ be acrwise connected topological spaces with the same homotopy type. Suppose there are maps 

$$f : X \to Y, \quad g : Y \to X,$$

such that 

$$f \circ g \sim \text{id}_Y, \quad g \circ f \sim \text{id}_X.$$

Then 

$$\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)), \quad \pi_1(Y, y_0) \simeq \pi_1(X, g(y_0)),$$

• **Proof.** We have by the Lemma,

$$\sigma_Y \circ (\text{id}_Y)_* = (f \circ g)_*, \quad \sigma_Y^{-1} \circ (\text{id}_X)_* = (g \circ f)_*,$$

• Since $\sigma_Y$ is an isomorphism, so are $(f \circ g)_*$ and $(g \circ f)_*$.

• Since 

$$(f \circ g)_* = f_* \circ g_*, \quad (g \circ f)_* = g_* \circ f_*,$$

the maps $f_*$ and $g_*$ are isomorphisms, which proves the theorem.

• **Corollary.** The fundamental group is a topological invariant.

• Let $R \subset X$ be a subspace of a topological space $X$. Then $R$ is called a **retract** of $X$ if there is a continuous map $f : X \to R$ (called a **retraction**) such that 

$$f|_R = \text{id}_R,$$

that is, 

$$f(x) = x \quad \text{for any } x \in R.$$

**Example.**

• **Proposition.** The retract $R$ and the whole space $X$ have the same fundamental group, for any $x_0 \in R$

$$\pi_1(R, x_0) \simeq \pi_1(X, x_0).$$
The subspace $R$ is called a deformation retract of $X$ if it is a retract of $X$ homotopic to the identity map $\text{id}_X$. That is, there a homotopy

$$H : X \times [0, 1] \to X$$

between the retraction $f$ and the identity map $\text{id}_X$ that leaves the points in $R$ fixed, that is,

$$H(\cdot, 0) = \text{id}_X, \quad H(\cdot, 1) = f,$$

$$H(\cdot, s)|_R = \text{id}_R, \quad s \in [0, 1],$$

or

$$H(x, 0) = x, \quad H(x, 1) = f(x) \in R, \quad \text{for any } x \in X$$

$$H(x, s) = x \quad \text{for any } x \in R \quad \text{and any } s \in [0, 1]$$

Examples.

A topological space $X$ is contractible if a point $x_0 \in X$ is a deformation retract of $X$.

If $X$ is contractible to a point $x_0$ then there is a homotopy $H : X \times [0, 1] \to X$ (called a contraction) such that

$$H(x, 0) = x_0, \quad H(x, 1) = \text{id}_X(x) = x, \quad \text{for any } x \in X$$

$$H(x_0, t) = x_0 \quad \text{for any } t \in [0, 1]$$

Example.

An arcwise connected topological space is called simply connected if it has a trivial fundamental group.

Theorem. A contractible space is simply connected, that is, it has a trivial fundamental group.

Examples.

Theorem. The fundamental group of the circle

$$S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

is isomorphic to $\mathbb{Z}$,

$$\pi_1(S^1) \cong \mathbb{Z}.$$
• One can identify the maps \( f : S^1 \to S^1 \) such that \( f(1) = 1 \) with the maps \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) such that \( \tilde{f}(0) = 0 \) and

\[
\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi n
\]

The integer \( n = \deg f \) is equal to the degree of \( f \).

• Two maps \( f, g : S^1 \to S^1 \) such that \( f(1) = g(1) \) are homotopic if and only if they have the same degree.

• For any \( n \in \mathbb{Z} \) there is a map \( f : S^1 \to S^1 \) with degree \( n \).

• Therefore, there is a bijection, in fact, isomorphism, between \( \pi_1(S^1, 1) \) and \( \mathbb{Z} \).

• **Theorem.** Let \( X \) and \( Y \) be arcwise connected topological spaces and \( x_0 \in X, y_0 \in Y \). Then

\[
\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \oplus \pi_1(Y, y_0)
\]

• **Examples.**

\[
\pi_1(T^n) = \bigoplus_{i=1}^{n} \mathbb{Z}
\]

\[
\pi_1(S^1 \times \mathbb{R}) = \mathbb{Z} \oplus \{e\} \simeq \mathbb{Z}
\]

### 5.4 Fundamental Groups of Polyhedra

• If the space \( X \) is triangulable then one can compute \( \pi_1(X) \) by a routine procedure.

• Let \( X \) be an arcwise connected topological space.

• Let \( f : |K| \to X \) be a triangulation of \( X \), where \( K \) is a simplicial complex and \( |K| \) is a polyhedron \(|K|\).

• Then

\[
\pi_1(X) = \pi_1(|K|)
\]

• There is a systematic way to compute \( \pi_1(|K|) \).
Any loop in $|K|$ is made of 1-simplexes.

An edge path in a simplicial complex $K$ is a sequence of vertices $v_0v_1 \cdots v_k$, such that each consecutive pair $v_i v_{i+1}$ is either a 0-simplex or a 1-simplex.

An edge path is an edge loop at $v_0$ if $v_0 = v_k$.

Two edge loops are equivalent if one is obtained from another by the following operations:

1. The edge path $uvw$ is equivalent to the path $uw$ if $u, v, w$ span a 2-simplex.
2. The edge path $uvu$ is equivalent to the 0-simplex $u$.

The equivalence class of edge loops at a vertex $v$ is denoted by

$$\{vv_1 \cdots v_{k-1}v\}$$

We define the product of equivalence classes of edge loops at a vertex $v$ by

$$\{vv_1 \cdots v_{k-1}v\} \ast \{vu_1 \cdots u_{j-1}v\} = \{vv_1 \cdots v_{k-1}vu_1 \cdots u_{j-1}v\}$$

The unit element is the equivalence class $\{v\}$.

The inverse equivalence class is defined by

$$\{vv_1 \cdots v_{k-1}v\}^{-1} = \{vv_{k-1} \cdots v_1v\}$$

**Theorem.** The set of equivalence classes of edge loops at a vertex $v$ forms a group $E(K, v)$, called the edge group of the simplicial complex $K$.

**Theorem.** The edge group of a simplicial complex $K$ is isomorphic to the fundamental group of the polyhedron $|K|$

$$E(K, v) \cong \pi_1(|K|, v)$$

Let $L$ be a simplicial subcomplex of $K$ such that $L$ contains all vertices of $K$ and the polyhedron $|L|$ is simply connected.

For any path connected simplicial complex $K$ there is such a subcomplex.
• A one-dimensional simply connected simplicial complex is called a tree.

• A tree is called a maximal tree if it is not a proper subset of other trees.

• Proposition. A maximal tree of a simplicial complex contains all its vertices.

• Let \( v_0, v_1, \ldots, v_n \) be vertices of \( K \).

• For each ordered pair of vertices \((v_i, v_j)\) such that \( \langle v_i v_j \rangle \) is a 1-simplex in \( K \) we assign an element \( g_{ij} \) of a group.

• Let \( G(K, L) \) be the group generated by \( g_{ij} \) with the following relations

1. \( g_{ij} = 1 \)
   
   if the \( \langle v_i v_j \rangle \in L \),

2. \( g_{ij} g_{jk} g_{ki} = 1 \)
   
   if \( \langle v_i v_j v_k \rangle \) is a 2-simplex of \( K \) and there are no nontrivial loops around \( v_i v_j v_k \).

• Proposition.

\[ g_{ii} = 1 \]

\[ (g_{ij})^{-1} = g_{ji} \]

• Theorem. The group \( G(K, L) \) is isomorphic to the edge group \( E(K, v) \) and, therefore, to the fundamental group \( \pi_1(|K|, v) \),

\[ G(K, L) \cong E(K, v) \cong \pi_1(|K|, v) \].

• Summary. To compute the fundamental group of \( X \) we need:

1. Find a triangulation \( f : |K| \to X \).
2. Find a simply connected subcomplex \( L \) of \( K \) that contains all vertices of \( K \).
3. Assign a generator \( g_{ij} \) to each 1-simplex \( \langle v_i v_j \rangle \in K - L \) with \( i < j \).
4. Impose a relation

\[ g_{ij} g_{jk} = g_{ik} \]

if there is a 2-simplex \( \langle v_i v_j v_k \rangle \) such that \( i < j < k \).

If two of the vertices \( v_i, v_j, v_k \) form a 1-simplex of \( L \) the generator should be set equal to 1.

5. The fundamental group \( \pi_1(X) \) is isomorphic to the group \( G(K, L) \) generated by \( \{g_{ij}\} \) with the above relations.

- The set \( K^{(p)} \) of all \( i \)-simplexes for \( i = 0, 1, \ldots, p \) of a simplicial complex \( K \) is called a \( p \)-skeleton of \( K \).

- It should be clear that the fundamental group of a polyhedron is determined by its 2-skeleton

\[ \pi_1(|K|) \cong \pi_1(|K^{(2)}|) \]

- For any \( n \geq 2 \), a \((n + 1)\)-simplex \( \sigma_{n+1} \) and its boundary \( \partial \sigma_{n+1} \) have the same 2-skeleton.

- **Examples.**
  - Since \( \sigma_{n+1} \) is contractible and \( \partial \sigma_{n+1} \) is a polyhedron of \( S^n \), we obtain

\[ \pi_1(S^n) = \{e\} \quad n \geq 2 \]

The \( n \)-sphere for \( n \geq 2 \) is simply connected

- A triangulation of the circle is

\[ K = \{v_1, v_2, v_3, \langle v_1 v_2 \rangle, \langle v_1 v_3 \rangle, \langle v_2 v_3 \rangle\} \]

For the circle there is only one generator with no relations

\[ \pi_1(S^1) \cong (x; \emptyset) \cong \mathbb{Z} \]

- A \( n \)-bouquet \( M \) is the one-point union of \( n \) circles.

\[ \pi_1(M) = (x_1, \ldots, x_n; \emptyset) \]

- The 2-disk is simply connected

\[ \pi_1(D^2) \cong \{e\} \]
CHAPTER 5. HOMOTOPY THEORY

• For the 2-torus there are two generators with one relation
  \[ \pi_1(T^2) \simeq (x, y; xyx^{-1}y^{-1}) \simeq \mathbb{Z} \oplus \mathbb{Z} \]

• For the surface of genus \( g \) there are \( 2g \) generators with one relation
  \[ \pi_1(\Sigma_g) \simeq \left\langle x_1, \ldots, x_g, y_1, \ldots, y_g; \prod_{i=1}^{g} x_iy_ix_i^{-1}y_i^{-1} \right\rangle \]

• For the real projective plane there is one generator with one relation
  \[ \pi_1(\mathbb{R}P^2) \simeq (x; x^2) \simeq \mathbb{Z}_2 \]

  The same is true for any \( n > 1 \)
  \[ \pi_1(\mathbb{R}P^n) \simeq (x; x^2) \simeq \mathbb{Z}_2 \]

• Exercises.
  
  • Show that for the Möbius strip \( M \) there is one generator with no relations
    \[ \pi_1(M^2) \simeq (x; \emptyset) \simeq \mathbb{Z} \]

  • Show that for the Klein bottle there are two generators with one relation
    \[ \pi_1(K^2) \simeq (x, y; xyxy^{-1}) \]

• Theorem. Let \( K \) be a connected simplicial complex. Then the first homology group \( H_1(K) \) is isomorphic to the quotient of the fundamental group \( \pi_1(|K|) \) by its commutator subgroup \( F \)
  \[ H_1(K) \simeq \pi_1(|K|)/F. \]

• Corollary. The first homology group of a connected topological space is isomorphic to its fundamental group
  \[ H_1(K) \simeq \pi_1(|K|) \]
  if and only if the fundamental group is commutative.

• Corollary. The first homology groups of two topological spaces \( X \) and \( Y \) of the same homotopy type are the same,
  \[ H_1(X) = H_1(Y) \]
5.5 Higher Homotopy Groups

- Let $I^n$ be the unit cube

$$I^n = [0, 1] \times \cdots \times [0, 1] = \{ t_1, \ldots, t_n \mid t_i \in [0, 1] \}$$

- The boundary of $I^n$ is

$$\partial I^n = \{ t_1, \ldots, t_n \mid t_i \in [0, 1], \ t_j = 0 \text{ or } 1, \ \text{for some } j \}$$

- A map $\alpha : I^n \to X$ is called an $n$-loop at a point $x_0 \in X$ if it maps the boundary $\partial I^n$ to $x_0$.

- A constant $n$-loop is a map

$$c_{x_0}(t_1, \ldots, t_n) = x_0, \quad \text{for any } t_i \in [0, 1].$$

- The cube $I^n$ whose boundary $\partial I^n$ is shrunk to a point is homeomorphic to the sphere

$$I^n/\partial I^n \simeq S^n$$

- Two $n$-loops at $x_0$, $\alpha_0$ and $\alpha_1$, are homotopic if there is a continuous map, called a homotopy between $\alpha_0$ and $\alpha_1$,

$$F : I^n \times [0, 1] \to X$$

such that

$$F(t, 0) = \alpha_0(t), \quad F(t, 1) = \alpha_1(t)$$

$$F(t, s) = x_0 \quad \text{for } t \in \partial I^n, \ s \in [0, 1]$$

where $t = (t_1, \ldots, t_n)$.

- Homotopy is an equivalence relation.

- The corresponding equivalence classes are called the homotopy classes.

- For any $n \geq 1$ the set $\pi_n(X, x_0)$ of all homotopy classes of $n$-loops at $x_0$ is called the $n$-th homotopy group of $X$ at $x_0$. 
• The product of two $n$-loops at $x_0$ is defined by
\[
(\alpha \ast \beta)(t_1, t_2, \ldots, t_n) = \begin{cases} 
\alpha(2t_1, t_2, \ldots, t_n), & 0 \leq t_1 \leq \frac{1}{2} \\
\beta(2t_1 - 1, t_2, \ldots, t_n), & \frac{1}{2} \leq t_1 \leq 1
\end{cases}
\]

• The inverse of an $n$-loop is defined by
\[
\alpha^{-1}(t_1, t_2, \ldots, t_n) = \alpha(1 - t_1, t_2, \ldots, t_n)
\]

• Then
\[
\alpha^{-1} \ast \alpha \sim \alpha \ast \alpha^{-1} \sim c_{x_0}
\]

• **Proposition.** Let $\alpha, \beta, \gamma, \delta$ be $n$-loops at $x_0 \in X$. Assume that $\alpha \sim \beta$ and $\gamma \sim \delta$. Then
  1. $(\alpha \ast \beta) \ast \gamma \sim \alpha \ast (\beta \ast \gamma)$.
  2. $\alpha \ast \alpha^{-1} \sim \alpha^{-1} \ast \alpha \sim c_{x_0}$
  3. $c_{x_0} \ast \alpha \sim \alpha \ast c_{x_0} \sim \alpha$
  4. $\alpha^{-1} \sim \beta^{-1}$
  5. $\alpha \ast \gamma \sim \beta \ast \delta$

• The product of homotopy classes is defined by
\[
[\alpha] \ast [\beta] = [\alpha \ast \beta]
\]

• The inverse of the homotopy classes is defined by
\[
[\alpha]^{-1} = [\alpha^{-1}]
\]

• The identity homotopy class is $[c_{x_0}]$.

• The product and the inverse do not depend on the representatives of the homotopy classes.
5.5. HIGHER HOMOTOPY GROUPS

- The set $\pi_0(X)$ is not a group; it is rather the number of independent connected components of $X$.
- **Theorem.** The $n$-th homotopy groups for $n \geq 1$ are groups.
- **Theorem.** Higher homotopy groups $\pi_n(X, x_0)$, for any $n \geq 2$, are Abelian.
- **Proof.** The main difference between $n = 1$ when $\pi_1(X)$ is non-Abelian and $n > 1$ is that the boundary $\partial I^n$ is disconnected for $n = 1$ and connected for $n > 1$.

- **Theorem.** For any two points $x_0, x_1$ in a connected space $X$ the homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic

$$\pi_n(X, x_0) \simeq \pi_n(X, x_1)$$

- **Proof.** Similar to the case $n = 1$. Use a path joining $x_0$ and $x_1$.

- **Theorem.** If the (connected) spaces $X$ and $Y$ have the same homotopy type then their homotopy groups are isomorphic

$$\pi_n(X) \simeq \pi_n(Y)$$

- **Proof.** Similar to the case $n = 1$.

- **Corollary.** The homotopy groups of a contractible space $X$ are trivial

$$\pi_n(X) \simeq \{e\}$$

- **Proposition.** For any connected topological spaces $X$ and $Y$

$$\pi_n(X \times Y) \simeq \pi_n(X) \oplus \pi_n(Y)$$

- **Proof.** Similar to the case $n = 1$.

- There is no algorithm for computing $\pi_n(X)$ for $n > 1$ even for triangulable spaces.

- **Homotopy groups of spheres.**

$$\pi_k(S^n) = \{0\} \quad k < n.$$ $$\pi_n(S^n) = \mathbb{Z}$$

For $k > n$ the groups $\pi_k(S^n)$ are very complicated.
• **Theorem. (Hopf)**

\[ \pi_3(S^2) = \mathbb{Z}. \]

• **Theorem. (Hurewicz)**

1. The first non-zero homology group \( H_p(X) \) with \( p > 1 \) and the first non-zero homotopy group \( \pi_p(X) \) are isomorphic.

2. If the fundamental group \( \pi_1(X) \) is Abelian, then it is isomorphic to the first homology group \( H_1(X) \).
5.6 Universal Covering Spaces

- A connected topological space $\tilde{X}$ is called the **covering space** of a topological space $X$ if there is a continuous map

$$f : \tilde{X} \to X$$

such that

1. $f$ is surjective,
2. for each $x \in X$ there is a connected open set $U \subset X$ containing $x$ such that the inverse image $f^{-1}(U)$ is a disjoint union of open sets $V_i$ in $\tilde{X}$,

$$f^{-1}(U) = \bigcup_i V_i,$$

each of which is homeomorphic to $U$, that is,

$$f : V_i \to U$$

are homeomorphisms.

- If the covering space $\tilde{X}$ is simply connected, then it is called the **universal covering space** of $X$.

- If the spaces $X$ and $\tilde{X}$ are topological groups and the map $f$ is a group homomorphism, then $\tilde{X}$ is called the **covering group**.

- The universal covering of the circle $S^1$ (or the group $U(1)$) is the real line $\mathbb{R}$.

**Theorem.** Let $\tilde{X}$ be a universal covering space of a connected space $X$ and $f : \tilde{X} \to X$ be the covering map. Let $\tilde{x}_0 \in \tilde{X}$ and $x_0 = f(\tilde{x}_0) \in X$. Then for any $n \geq 2$, the induced homomorphism

$$f_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$$

is an isomorphism.

- As an example,

$$\pi_n(S^1) = \{e\}$$

for any $n \geq 2$. 


• The sphere $S^n$ is a covering space of the real projective space $\mathbb{R}P^n$ for $n \geq 2$.

• For any $n \geq 2$

$$\pi_n(\mathbb{R}P^n) = \{e\}$$

• Examples.

$$\mathbb{R}P^3 = SO(3)$$

•

$$S^3 = SU(2)$$

• Proposition. For any $n \geq 2$

$$\pi_n(SO(3)) = \pi_n(\mathbb{R}P^3) = \pi_n(S^3) = \pi_n(SU(2))$$

• The universal covering group of the special orthogonal group $SO(n)$ is called the spin group.

• Examples.

$$\text{Spin}(3) = SU(2)$$

$$\text{Spin}(4) = SU(2) \times SU(2)$$

$$\text{Spin}(6) = SU(4)$$

• Examples.

$$\pi_n(S^n) = \pi_n(\mathbb{R}P^n) = \mathbb{Z}, \quad n \geq 2$$
Chapter 6

Fibre Bundles

6.1 Fiber Bundle

- A fiber bundle if a topological space that is \textit{locally} a product of two spaces.

- Examples. Cylinder

  \[ M = I \times S^1 \]

  Möbius strip is not a \textit{global} product!

- The spaces which are \textit{global} products are \textit{trivial bundles}.

- A \textbf{fiber bundle} \( E, \pi, F, G, X \) is a collection of the following elements:

  1. A topological space \( E \) called the \textbf{total space},
  2. A topological space \( X \) called the \textbf{base space},
  3. A map

     \[ \pi : E \to X \]

     called the \textbf{projection},
  4. A topological space \( F \) called the \textbf{fiber},
  5. A group \( G \) of homeomorphisms of the fiber \( F \) called the \textbf{structure group},
  6. An open cover \( \{ U_a \} \) of \( X \),

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7. (Local triviality of $E$) A collection of homeomorphisms

$$\varphi_{\alpha} : \pi^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times F$$

satisfying the conditions

$$\pi \circ \varphi_{\alpha}^{-1} = \text{id}_{U_{\alpha}}$$

that is,

$$\pi(\varphi_{\alpha}^{-1}(p, f)) = p, \quad \forall p \in U_{\alpha}, f \in F$$

• Example. M"obius strip.

• Let $U_{\alpha}$ and $U_{\beta}$ be two overlapping charts and $\varphi_{\alpha}, \varphi_{\beta}$ be the corresponding homeomorphisms. Then

$$\varphi_{\alpha} \circ \varphi_{\beta}^{-1} : (U_{\alpha} \cap U_{\beta}) \times F \rightarrow (U_{\alpha} \cap U_{\beta}) \times F$$

• For every fixed $p \in U_{\alpha} \cap U_{\beta}$ this defines the homeomorphisms of $F$ called the transition functions

$$g_{\alpha\beta}(p) = (\varphi_{\alpha} \circ \varphi_{\beta}^{-1})(p, \cdot) : F \rightarrow F$$

by: for any $f \in F$,

$$g_{\alpha\beta}(p)f = (\varphi_{\alpha} \circ \varphi_{\beta}^{-1})(p, f).$$

• The set of all transition functions for all points in all charts forms the group $G$ called the structure group of the fiber bundle $E$.

• Example. M"obius strip.

$$G = \{e, g\}$$

The intersection of the two charts is the disjoint union of two arcs,

$$U_1 \cap U_2 = A \cup B,$$

so that

$$g_{12}(p) = \begin{cases} e, & \text{if } p \in A \\ g, & \text{if } p \in B. \end{cases}$$

$$g_{21} = (g_{12})^{-1}, \quad g_{11} = g_{22} = e$$
6.1. FIBER BUNDLE

- The fiber bundle $E$ is fully determined by the base space $X$, the structure group $G$, the fiber $F$, and the transition functions $g_{\alpha\beta}(x)$.

- Let $\tilde{E}$ be the union of all products

$$\tilde{E} = \bigcup_{\alpha} U_{\alpha} \times F$$

- We define an equivalence relation on $\tilde{E}$ as follows:

$$(p, f) \sim (p', f')$$

where $p \in U_{\alpha}$, $p' \in U_{\beta}$, if

$$p' = p, \quad f' = g_{\alpha\beta}(p)f$$

- Then define the total space $E$ by the quotient

$$E = \tilde{E}/\sim$$

- The projection $\pi : E \to X$ is now defined by

$$\pi([(p, f)]) = p$$

- The homeomorphisms $\varphi_{\alpha} : \pi^{-1}(U_{\alpha}) \to U_{\alpha} \times F$ are now defined by specifying the inverse

$$\varphi_{\alpha}^{-1}(p, f) = [(p, f)]$$

Then we have automatically

$$\pi \circ \varphi_{\alpha}^{-1} = \text{id}_{U_{\alpha}}$$

as needed.

- In practice, one always defines a bundle by specifying the base space $X$, the structure group $G$ and the transition functions $g_{\alpha\beta}$.

- **Example.** Möbius strip.
• The transition functions satisfy the following compatibility relations

\[ g_{\alpha\alpha}(p) = e, \quad \text{for any } p \in U_\alpha, \]
\[ g_{\alpha\beta}(p)g_{\beta\alpha}(p) = e, \quad \text{for any } p \in U_\alpha \cap U_\beta, \]
\[ g_{\alpha\beta}(p)g_{\beta\gamma}(p)g_{\gamma\alpha}(p) = e, \quad \text{for any } p \in U_\alpha \cap U_\beta \cap U_\gamma. \]

• Let \( E \) and \( E' \) be two bundles with the same base space \( X \), the same fiber \( F \) and the same structure group \( G \). Let \( \{ \varphi_\alpha, U_\alpha \} \) and \( \{ \varphi'_\alpha, U_\alpha \} \) be the coverings and the coordinate maps.

• Then the map

\[ \varphi_\alpha \circ \varphi'^{-1}_\alpha : U_\alpha \times F \to U_\alpha \times F \]

defines for any fixed \( p \in U_\alpha \) a homeomorphism

\[ \lambda_\alpha(p) = (\varphi_\alpha \circ \varphi'^{-1}_\alpha)(p, \cdot) : F \to F. \]

• We require that \( \lambda_\alpha(p) \) is an element of the structure group \( G \).

• Then the transition functions of the bundles \( E \) and \( E' \) are related by so called gauge transformations

\[ g'_{\alpha\beta}(p) = \lambda^{-1}_\alpha(p)g_{\alpha\beta}(p)\lambda_\beta(p), \quad p \in U_\alpha \cap U_\beta. \]

• The bundles \( E \) and \( E' \) with the same base space, the same fiber, the same structure group and the transition functions related by such relations are called equivalent.

• The bundle defined with respect to a given set of transition functions is called a coordinate bundle. The bundle is then an equivalent class of coordinate bundles.

• A section of a bundle \( E \) is a continuous map

\[ s : X \to E \]

such that

\[ \pi \circ s = \text{id}_X, \]

that is, for any \( p \in X \)

\[ \pi(s(p)) = p. \]
6.2 Vector Bundles

- A fiber bundle whose typical fiber is a vector space is called a vector bundle.

- A vector bundle is called a real vector bundle of rank \( n \) if \( F = \mathbb{R}^n \) and a complex vector bundle of rank \( n \) if \( F = \mathbb{C}^n \).

- A vector bundle of rank 1 is called a line bundle.

Examples.

- Let \( E \) be a vector bundle over \( X \) with a fiber \( F \) and projection \( \pi \). Its dual bundle \( E^* \) is a vector bundle over \( X \) whose fiber \( F^* \) is the set of linear functionals on \( F \). The projection is defined naturally.

- Let \( E \) be a vector bundle over \( X \) with a fiber \( F \) and \( E' \) is a vector bundle over \( X' \) with a fiber \( F' \). Then the product bundle \( E \times E' \) is a vector bundle over \( X \times X' \) with the fiber \( F \oplus F' \). The projection is defined naturally.

- Let \( E \) and \( E' \) be two vector bundles over the same base space \( X \) with fibers \( F \) and \( F' \). The Whitney sum bundle \( E \oplus E' \) is a vector bundle over \( X \) with fiber \( F \oplus F' \).

- The transition functions of the Whitney sum bundle are

\[
g_{\alpha \beta}(p) = g_{\alpha \beta}^E(p) \oplus g_{\alpha \beta}^{E'}(p)
\]

6.2.1 Tangent Bundle

- The tangent bundle of a manifold \( M \) is the union of all tangent spaces of \( M \)

\[
TM = \bigcap_{p \in M} T_pM
\]

- The base space is \( M \).
• The projection map
\[ \pi : TM \rightarrow M \]
is defined for any \( v_p \in T_p M \) by
\[ \pi(v_p) = p. \]

• The fiber is \( T_p M \) which is diffeomorphic to \( \mathbb{R}^n \), where \( n = \dim M \).

• The diffeomorphisms
\[ \varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n \]
are defined by
\[ \varphi_\alpha(v_p) = (x^i, v_j) \]
where \( x^i \) are local coordinates of the point \( p \) and \( v^j \) are the components of the vector \( v \)
\[ v = v^j \partial_j \]

• The structure group is
\[ G = GL(n, \mathbb{R}) \]

• The transition functions are
\[ g_{\alpha\beta}(p) = J_{\alpha\beta}(p) \]
where
\[ J_{\alpha\beta}(p) = \begin{pmatrix} \frac{\partial x^i_\alpha}{\partial x^j_\beta}(p) \end{pmatrix} \]
is the Jacobian

### 6.2.2 Cotangent Bundle

• The cotangent bundle of a manifold \( M \) is the union of all tangent spaces of \( M \)
\[ T^* M = \bigcap_{p \in M} T^*_p M \]

• The base space is \( M \).
6.2. VECTOR BUNDLES

- The projection map
  \[ \pi : T^*M \to M \]
  is defined for any \( \sigma_p \in T^*_pM \) by
  \[ \pi(\sigma_p) = p \]

- The fiber is \( T^*_pM \) which is diffeomorphic to \( \mathbb{R}^n \), where \( n = \text{dim } M \).

- The diffeomorphisms
  \[ \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \]
  are defined by
  \[ \varphi_\alpha(\sigma_p) = (x^i, \sigma_j) \]
  where \( x^i \) are local coordinates of the point \( p \) and \( \sigma_j \) are the components of the covector \( \sigma \)
  \[ \sigma = \sigma_j dx^j \]

- The structure group is
  \[ G = GL(n, \mathbb{R}) \]

- The transition functions are
  \[ g_{\alpha\beta}(p) = \left( J_{\alpha\beta}^{-1}(p) \right)^T = \left( \frac{\partial x^i_{\beta}}{\partial x^j_\alpha}(p) \right) \]

6.2.3 Tensor Bundles

- The tensor bundle of type \((r, s)\) is the bundle
  \[ T^r_s M = \bigcap_{p \in M} T^r_{s_p} M, \]
  where
  \[ T^r_{s_p} M = T^r_p M \times \cdots \times T^r_p M \times T^s_p M \times \cdots \times T^s_p M \]

- The base space is \( M \).
• The projection map
\[ \pi : TM \to M \]

is defined for any \( A_p \in T^r_s pM \) by
\[ \pi(A_p) = p. \]

• The fiber is \( T^r_s pM \) which is diffeomorphic to \( \mathbb{R}^{n^{rs}} \), where \( n = \dim M \).

• The diffeomorphisms
\[ \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r^s} \]

are defined by
\[ \varphi_\alpha(A_p) = (x^i, A_{k_1\ldots k_r}^{j_1\ldots j_r}) \]
where \( x^i \) are local coordinates of the point \( p \) and \( A_{k_1\ldots k_r}^{j_1\ldots j_r} \) are the components of the tensor \( T \)
\[ A = A_{k_1\ldots k_r}^{j_1\ldots j_r} dx^{k_1} \otimes \cdots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_r} \]

• The structure group is a specific subgroup of
\[ G = GL(n, \mathbb{R}) \subset GL(n^{r+s}, \mathbb{R}) \]

• The transition functions \( g_{\alpha\beta}(p) \) are tensor products of the Jacobian and the inverse transposed Jacobian,
\[ g_{\alpha\beta}(p) = \otimes^r J_{\alpha\beta}(p) \otimes^s (J_{\alpha\beta}^{-1}(p))^T \]

### 6.2.4 Bundles of Differential Forms

• The bundle of \( q \)-forms is the bundle
\[ \Lambda^q M = \bigsqcup_{p \in M} \Lambda^q_p M, \]
where \( \Lambda^q_p M \) is the space of \( q \)-forms at the point \( p \in M \).

• The fiber is \( \Lambda^q_p M \) which is diffeomorphic to \( \mathbb{R}^N \), where \( N = \binom{q}{q} \).
6.2. VECTOR BUNDLES

- The diffeomorphisms
  \[
  \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^N
  \]
  are defined by
  \[
  \varphi_\alpha(\omega_p) = (x^i, \omega_{j_1...j_q})
  \]
  where \(x^i\) are local coordinates of the point \(p\) and \(\omega_{j_1...j_q}\) are the components of the form \(\omega\)
  \[
  \omega = \frac{1}{q!} \omega_{j_1...j_q} dx^{j_1} \otimes \cdots \otimes dx^{j_q}
  \]

- The structure group is a specific subgroup of
  \[
  G = GL(n, \mathbb{R}) \subset GL(n^N, \mathbb{R})
  \]

6.2.5 Normal Bundle

- Let \(M\) be an \(n\)-dimensional submanifold embedded in an \(m\)-dimensional Riemannian manifold \(W\), with \(m > n\).

- Let \(T_p M\) be the tangent space to \(M\) at a point \(p \in M\) and \(N_p M\) be the vector space orthogonal to \(T_p M\) so that there is an orthogonal decomposition of the tangent space
  \[
  T_p W = T_p M \oplus N_p M
  \]

- Then the normal bundle of \(M\) is a vector bundle
  \[
  NM = \bigcup_{p \in M} N_p M
  \]

- The base space is \(M\).

- The projection map
  \[
  \pi : NM \to M
  \]
  is defined for any \(v_p \in N_p M\) by
  \[
  \pi(v_p) = p.
  \]

- The fiber is \(N_p M\) which is diffeomorphic to \(\mathbb{R}^{m-n}\).
• The diffeomorphisms

\[ \varphi \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{m-n} \]

are defined by

\[ \varphi_{\alpha}(v_p) = (x^i, v^\mu) \]

where \( x^i, i = 1, \ldots, n \) are local coordinates of the point \( p \) and \( v^\mu, \mu = n + 1, \ldots, m \) are the components of the tensor \( v_p \).

• **Exercise.** Find the structure group and the transition functions of the normal bundle.
6.3 Principal Bundles

- A fiber bundle over $X$ whose typical fiber is the structure group $G$ itself is called a principal bundle. It is denoted by $P = P(X, G)$ and also called a $G$ bundle.

- The transition functions act on the fiber on the left, by left action of the group $G$ on itself.

- There is also the right action of the group on the fiber.

- The right action commutes with the left action.

- This defines the right action of the group on the principal bundle $P$ (in a local trivialization $\varphi_\alpha$) by

  $$u = \varphi^{-1}_\alpha(p, h) \mapsto ug = \varphi^{-1}_\alpha(p, hg),$$

  where $u \in P$, $p \in M$, $h \in G$.

- The right action of the group on itself is transitive, that is, for any $g_1, g_2 \in G$ there is $h \in G$ such that

  $$g_2 = g_1h$$

- The right action of the group on itself is free, that is, if $gh = g$ for some $g \in G$ then $h = e$ is the identity.

6.3.1 Associated Bundles

- Let $P = P(M, G)$ be a principal bundle and $F$ be a manifold.

- Let the group $G$ act on $F$ on the left

  $$f \mapsto gf.$$ 

- Extend this action to the product $P \times F$ by

  $$(u, f) \mapsto (ug, g^{-1}f)$$
• Define an equivalence relation by

$$(u, f) \sim (ug, g^{-1}f).$$

• Then the associated fiber bundle $E$ is defined by

$$E = P \times F / \sim.$$

• Let $F = V$ be an $N$-dimensional vector space and $\rho$ be an $N$-dimensional representation of the group $G$.

• The associated vector bundle $P \times_{\rho} V$ is defined by identifying the points

$$(u, g) \sim (ug, \rho(g)^{-1}v),$$

where $u \in P$, $g \in G$, $v \in V$.

• The projection is defined naturally and the transition functions are defined by

$$\rho(g_{\alpha\beta}(p)).$$

### 6.3.2 Frame Bundle

• A frame is a basis $e_i$ of the tangent space $T_p M$.

• A coordinate frame is the basis

$$f_i = \partial_i$$

• Each frame can be obtained from a fixed frame by acting by an element of the group $GL(n, \mathbb{R})$,

$$f_i = f^j_i \partial_j,$$

where $f = (f^i_j) \in GL(n, \mathbb{R})$. Therefore, each frame can be identified with an element $f$ of $GL(n, \mathbb{R})$.

• The frame bundle $FM$ of a manifold $M$ is the collection of all frames at all points of $M$. 
6.3. PRINCIPAL BUNDLES

- The fiber at a point \( p \in M \) of the frame bundle \( FM \) is the collection of all frames at a point \( p \in M \). The fiber can be identified with the group \( GL(n, \mathbb{R}) \).

- The base space is \( M \).

- The projection map
  \[ \pi : FM \to M \]
  is defined for any \( p \in M, f \in GL(n, \mathbb{R}) \) by
  \[ \pi(p, f) = p. \]

- The diffeomorphisms
  \[ \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times GL(n, \mathbb{R}) \]
  are defined by
  \[ \varphi_\alpha(p, f) = (x^i, f^i_j) \]
  where \( x^i \) are local coordinates of the point \( p \) and \( f^i_j \) are the components of the matrix \( f \).

- The structure group is
  \[ G = GL(n, \mathbb{R}). \]
  Thus, the frame bundle is a principal bundle.

- The transition functions are
  \[ g_{\alpha\beta}(p) = J_{\alpha\beta}(p) \]

6.3.3 Spin Bundle

- The structure group of the frame bundle is \( GL(n, \mathbb{R}) \).

- If a manifold is orientable then there is an oriented frame bundle with the structure group \( SL(n, \mathbb{R}) \).

- The structure group of the frame bundle, \( GL(n, \mathbb{R}) \), can be reduced to \( O(n) \).
The structure group of the oriented frame bundle, $SL(n, \mathbb{R})$, can be reduced to $SO(n)$.

Let $g_{\alpha\beta}(p)$ be the transition functions of the oriented frame bundle which belong to $SO(n)$.

The group $SO(n)$ is not simply connected; it has a double universal covering group $Spin(n)$, which is simply connected,

$$\varphi : Spin(n) \rightarrow SO(n).$$

A spin structure on $M$ is defined by the transition functions $\tilde{g}_{\alpha\beta}(p)$ satisfying the compatibility conditions and such that

$$\varphi(\tilde{g}_{\alpha\beta}(p)) = g_{\alpha\beta}(p).$$

A manifold may admit many spin structures or none at all.

A manifold $M$ admits a spin structure if the oriented frame bundle with the structure group $SO(n)$ lifts to a $Spin(n)$ bundle over $M$.

If a manifold admits a spin structure then the set of transition functions $\tilde{g}_{\alpha\beta}(p)$ defines the spin bundle $SM$ over $M$.

The principal bundle of the spin bundle is $P(M, Spin(n))$.

One can define the dual spin bundle $S^*M$ and the tensor product of the spin bundles

$$S^rSM = \otimes^rSM \otimes^sS^*M.$$  

Examples.

Magnetic monopole is the principal bundle over $S^2$ with the group $U(1)$, $P(S^2, U(1))$

Hopf Theorem.

$$P(S^2, U(1)) = S^3$$

Instanton is the principal bundle over $S^4$ with the group $SU(2)$, $P(S^4, SU(2))$. 
• **Homogeneous Spaces.** Let $G$ be a Lie group and $H$ be a closed subgroup of $G$. Then the group $G$ is a principal bundle over the base space $M = G/H$ with fiber $H$,

$$P(G/H, H) = G$$

• The elements of $G/H$ are the right cosets

$$[g] = \{gh \mid h \in H\}$$

• The projection $\pi : G \to G/H$ is defined by

$$\pi(g) = [g]$$

• **Spheres.**

$$O(n)/O(n - 1) = SO(n)/SO(n - 1) = S^{n-1}$$

$$U(n)/U(n - 1) = SU(n)/SU(n - 1) = S^{2n-1}$$
6.4 Trivial Bundles

- A bundle $E$ is trivial if it is a product $E = X \times F$.

- Let $E$ be a fiber bundle with the total space $E$, the base space $X$, the projection $\pi$, the fiber $F$, the structure group $G$ and the transition functions $g_{\alpha \beta}(x)$.

- One can always construct a principal bundle with the same base space $X$, the same structure group $G$ and the same transition functions $g_{\alpha \beta}(x)$ but with a different fiber; namely, one replaces the fiber $F$ by the structure group $G$.

- This principal bundle is denoted by $P(E)$ and called the **principal bundle associated with** $E$.

- Two bundles with the same base, the same structure group and the same transition functions (but different fibers) have the same principal bundle.

- **Example.** The principal bundle of the tangent bundle is the frame bundle, $P(TM) = FM$

- **Example.** The principal bundle of the Möbius strip is a double covering of the circle $S^1$. The fiber is the group $G = \{e, g\}$.

- **Theorem.** The bundle $P(E)$ is trivial if and only if it has a section.

- **Proof.**
  - Suppose $P(E)$ has a section $s : X \rightarrow P(E)$.
  - Idea: construct a homeomorphism
    \[ \varphi : P(E) \rightarrow X \times G \]
  - We have $s(p) \in G$ for any $p \in X$.
  - Therefore, for any $g \in G$, $gs(p) \in \pi^{-1}(p)$ is in the fiber over $p$.
  - It is easy to see that all elements in the fiber $\pi^{-1}(p)$ over $p$ have the form $gs(p)$ for some $g$.
  - As $p$ varies, all elements of $P(E)$ have the form $gs(p)$ for some $g \in G$ and some $p \in X$. 


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- Thus, we can define the homeomorphism \( \varphi \) by

  \[ \varphi(g_s(p)) = (p, g) \]

- Therefore, \( P(E) \) is trivial.

- **Proposition.** A bundle \( E \) is trivial if and only if the transition functions have the form

  \[ g_{\alpha\beta}(p) = \lambda^{-1}_\alpha(p)\lambda_\beta(p) \]

  where \( \lambda_\alpha(p) : F \rightarrow F, p \in U_\alpha \), is a homeomorphism of the fiber over \( p \) that belongs to the structure group \( G \).

- **Proof.** Since the bundle \( E \) is trivial there is a homeomorphism

  \[ \varphi : E \rightarrow X \times F \]

- Let \( \{ U_\alpha \} \) be a covering of the base \( X \) and

  \[ \varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F \]

  be the local trivialization maps.

- Let \( p \in U_\alpha \). Then

  \[ \lambda_\alpha(p) = (\varphi \circ \varphi^{-1}_\alpha)(p, \cdot) : F \rightarrow F \]

  are homeomorphisms belonging to \( G \).

- Then

  \[ \lambda_\alpha(p)g_{\alpha\beta}(p)\lambda^{-1}_\beta(p) = \text{id}_F \]

  This proves the statement.

- **Theorem.** The bundle \( E \) is trivial if and only if the principal bundle \( P(E) \) associated with \( E \) is trivial.

- **Proof.**

  - If a bundle \( E \) is trivial then its transition functions have the form

    \[ g_{\alpha\beta}(p) = \lambda^{-1}_\alpha(p)\lambda_\beta(p). \]
• Then there is a trivialization such that the transition functions are the identity,
\[ g'_{\alpha\beta}(p) = \text{id}_F \]

• Therefore, since \( E \) and \( P(E) \) have the same transition functions, the bundle \( P(E) \) is also trivial.

• Corollary. A bundle \( E \) is trivial if and only if its principal bundle \( P(E) \) has a section.

• Example. Möbius strip.

• Remark. A non-trivial bundle \( E \) itself can have a section. It is the principal bundle \( P(E) \) that cannot have a section.

• Example. Möbius strip.

• Exercise. Let \( L \) be a real line bundle over \( S^1 \). Show that the Whitney sum \( L \oplus L \) is a trivial bundle.

### 6.4.1 Parallelizable Manifolds

• A section of the tangent bundle is just a vector field.

• The singularities of a vector field are the points where it vanishes, its zeros.

• A section \( s \) of a frame bundle \( FM \) is a continuous assignment of a basis for the tangent space \( T_pM \) as \( p \) varies over \( M \).

• The singularities of a collection of vector fields \( \{v_1, \ldots, v_k\} \) are the points where it is linearly dependent.

• The largest number \( m \) of linearly independent vector fields on \( M \) is smaller or equal to \( n = \dim M \).

• If \( m = n \) then there are \( n \) linearly independent vector fields forming a basis for the tangent space \( T_pM \) at every \( p \in M \), thus, defining a section \( s \) of the frame bundle \( FM \). In this case, the manifold \( M \) is called **parallelizable**.

• Thus, the existence of a section of the principal bundle \( F(M) = P(TM) \) of the tangent bundle is equivalent to the existence of \( n \) sections of the tangent bundle \( TM \) linearly independent at each point.
• **Examples.** Lie groups are parallelizable.

• A basis at the identity can be transported by the left action of the group to any point giving a section of the frame bundle.

• The circle $S^1 = U(1)$ is parallelizable.

• The torus $T^n = U(1) \times U(1)$ is parallelizable.

• The spheres $S^n$ is parallelizable only for $n$ equal to

\[ 1 = 2^1 - 1, \quad 3 = 2^2 - 1, \quad 7 = 2^3 - 1 \]

• The parallelizability of $S^1 = U(1)$ has to do with the existence of complex numbers,

\[ S^1 = \{ z = (a, b) \in \mathbb{C} \mid a, b \in \mathbb{R}, \ |z| = 1 \} \]

• The parallelizability of $S^3 = SU(2)$ has to do with the existence of quaternions,

\[ S^3 = \{ q = (u, v) \in \mathbb{H} \mid u, v \in \mathbb{C}, \ |q| = 1 \} \]

• The parallelizability of $S^7$ has to do with the existence of octonions,

\[ S^7 = \{ x = (a, b) \in \mathbb{O} \mid a, b \in \mathbb{H}, \ |x| = 1 \} \]
6.5 Reduction of Bundles

6.5.1 Contraction of the Base Space

- **Theorem.** If a base space $X$ of a bundle $E$ is contractible to a point then the bundle $E$ is trivial.

- **Proof.** Below.

- Let $E$ and $E'$ be two bundles. A map

  $$\varphi : E' \to E$$

  is called a **bundle map** if it maps fibers of $E'$ onto fibers of $E$.

- Two bundles $E$ and $E'$ with the same base space $X$, are **equivalent** if there is a homeomorphism

  $$\varphi : E' \to E$$

  that is a bundle map, that is, it maps fibers of $E'$ onto fibers of $E$.

- Let $E$ be a fiber bundle over $X$ with a projection $\pi : E \to X$ and a fiber $F$. Let

  $$\varphi : X' \to X$$

  be a continuous map.

- This defines a new bundle $E' = \varphi^*E$ over $X'$ called the **pullback bundle** with the same typical fiber $F$ as follows.

- The fiber of $\varphi^*E$ over a point $p' \in X'$ is just the fiber of $E$ over the point $\varphi(p') \in X$.

- Thus $\varphi^*E$ is the **disjoint** union of all these fibers equipped with a suitable topology.

- Define a subset $E'$ of $X' \times E$ by

  $$E' = \varphi^*E = \{(p', u) \in X' \times E \mid \varphi(p') = \pi(u)\}$$
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• We define the projection onto the first factor

\[ \pi' : \varphi^* E \to X' \]

by

\[ \pi'(p', u) = p', \quad p' \in X, u \in E \]

and the projection onto the second factor

\[ \varphi_\ast : \varphi^* E \to E \]

by

\[ \varphi_\ast(p', u) = u, \quad p' \in X, u \in E \]

• Then the map \( \varphi_\ast \) is a bundle map such that

\[ \varphi \circ \pi' = \pi \circ \varphi_\ast \]

• If \( X = X' \) and \( \varphi = \text{id}_X \) is the identity map then the bundles \( \varphi^* E \) and \( E \) are equivalent.

• Let \( \{ U_\alpha \} \) be the open cover of \( X \) and the transition functions for the bundle \( E \) be \( g_{\alpha\beta}(p), \ p \in U_\alpha \cap U_\beta. \)

• The coordinate charts in \( X' \) are defined by

\[ U'_\alpha = \varphi^{-1}(U_\alpha) \]

and the transition functions are

\[ g'_{\alpha\beta}(p') = g_{\alpha\beta}(\varphi(p')), \quad p' \in U'_\alpha \cap U'_\beta \]

• Theorem. Let \( E \) be a fiber bundle over \( X \) with fiber \( F \). Let \( \varphi_0, \varphi_1 : X' \to X \) be homotopic maps. Then the bundles \( \varphi_0^* E \) and \( \varphi_1^* E \) are equivalent.

• Proof in textbook.

• Corollary. If the base space \( X \) of a fiber bundle \( E \) is contractible then the bundle \( E \) is trivial.

• Proof. Since \( X \) is contractible the identity map \( \varphi_0 = \text{id}_X \) is homotopic to the constant map \( \varphi_1 : X \to x_0. \)
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• Then the pullback bundle \( \varphi_1^* E \) over a point \( x_0 \) is trivial and the pullback \( \varphi_0^* E \) is equivalent to \( E \). Thus, \( E \) is trivial.

• **Examples.** A bundle \( E \) over a base space \( X = S^1 \times I \) is equivalent to a pullback bundle \( E' \) over \( S^1 \).

• That is, there is a map

\[
\varphi : S^1 \to S^1 \times I
\]

such that the pullback bundle \( E' = \varphi^* E \) over \( S^1 \) is equivalent to \( E \) over \( S^1 \times I \).

### 6.5.2 Reduction of the Structure Group

• **Theorem.** If a fiber \( F \) of bundle \( E \) is contractible then it has a section \( s \).

• **Corollary.** If the structure group \( G \) of a bundle \( E \) is contractible then the bundle \( E \) is trivial.

• **Example.** The structure group of the frame bundle \( F(M) \) is \( GL(n, \mathbb{R}) \).

• The group \( GL(n, \mathbb{R}) \) is not contractible. It has the form

\[
GL(n, \mathbb{R}) = O(n) \times \text{Sym}_+(n),
\]

where \( O(n) \) is the orthogonal group and \( \text{Sym}_+(n) \) is the set of all positive definite symmetric \( n \times n \) matrices.

• **Proposition.** The set \( \text{Sym}_+(n) \) is homeomorphic to the vector space \( \text{Sym}(n) \) of all symmetric matrices, and, therefore, contractible.

• **Proof.** The homeomorphism is given by the exponential map

\[
\exp : \text{Sym}(n) \to \text{Sym}_+(n).
\]

• Therefore, the non-compact group \( GL(n, \mathbb{R}) \) is contractible to the compact group \( O(n) \).

• The reduction of \( GL(n, \mathbb{R}) \) to \( O(n) \) defines a continuous assignment of an orthogonal frame at each point \( p \in X \).
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• That is, the reduction $GL(n, \mathbb{R})$ to $O(n)$ means the existence of a Riemannian metric for $X$.

• If the structure group $GL(n, \mathbb{R})$ may be reduced to an even smaller subgroup $G$ of $GL(n, \mathbb{R})$ then we say that $X$ has a $G$-structure.

• Examples of $G$ Structures. For even $n$ there are two important examples:

  • Almost Hamiltonian (Symplectic) Structure
    
    $$ G = Sp(n, \mathbb{R}), \quad n = 2m \text{ is even} $$

    where $Sp(n, \mathbb{R})$ is the symplectic group. This is a group of $n \times n$ real matrices $A$ that satisfy
    
    $$ A^T J A = J, $$

    where
    
    $$ J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} $$

  • Almost Complex Structure.
    
    $$ G = GL(n/2, \mathbb{C}), \quad n = 2m \text{ is even} $$

• The reduction of the structure group $GL(n, \mathbb{R})$ to $O(n)$ means that the transition functions $g_{\alpha\beta}(p)$ now take values in $O(n)$.

• This means that the transition functions for the tangent bundle and the cotangent bundle are the same, since for any orthogonal matrix $A^{-1} = A^T$.

• Therefore, the bundles $TM$ and $T^*M$ are equivalent.

• More generally, let $P(X, G)$ be a principal fiber bundle with the structure group $G$.

• If $G$ is a connected Lie group then

    $$ G = H \times D, $$

    where $H$ is the maximal compact subgroup of $G$ and $D$ is a set which is topologically a Euclidean space, and, therefore, contractible.

• Therefore, $G$ may be reduced to its maximal compact subgroup $H$. 

• The resulting bundle $P(X, H)$ is much smaller and simpler, but it is equivalent to $P(X, G)$.

• **Example.** Let $M$ be an $n$-dimensional complex manifold with the frame bundle $FM = GL(n, \mathbb{C})$.

• Then $GL(n, \mathbb{C})$ has the maximal compact subgroup $U(n)$

$$GL(n, \mathbb{C}) = U(n) \times \text{Herm}_+(n),$$

where $\text{Herm}_+(n)$ is the set of positive definite Hermitian matrices.

• The set $\text{Herm}_+(n)$ is homeomorphic to the set $\text{Herm}(n)$ of all Hermitian matrices and, therefore, contractible,

$$\exp : \text{Herm}(n) \to \text{Herm}_+(n).$$

• Thus, the group $GL(n, \mathbb{C})$ may be reduced to its maximal compact subgroup $U(n)$.

• The reduction of $GL(n, \mathbb{R})$ to $O(n, p, p)$ is not always possible. There are some **topological obstructions**.

• The reduction of $GL(n, \mathbb{R})$ to $O(n - p, p)$ is not always possible.

• **Example.** The group $GL(2, \mathbb{R})$ is reducible to $O(1, 1)$ over a closed manifold $M$ only when $M$ is either the torus $T^2$ or the Klein bottle $K^2$.

• The sphere $S^2$ does not admit a Lorentzian metric.

• A closed manifold $M$ admits a Lorentzian metric, that is, the structure group $GL(n, \mathbb{R})$ is reducible to $O(n - 1, 1)$ if its Euler characteristic vanishes

$$\chi(M) = 0.$$ 

• A non-orientable manifold $M$ does not admit the oriented frame bundle, that is, the group $GL(n, \mathbb{R})$ cannot be reduced to $SL(n, \mathbb{R})$.

• For a manifold to admit the spin bundle it has to be orientable and satisfy one more topological condition.
6.6. G-Structures

- Let $n = 2m$ be even and
  \[ \mathbb{R}^n = \mathbb{R}_x^m \times \mathbb{R}_y^m \]

- Let $J$ be a $2m \times 2m$ matrix defined by
  \[ J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \]
  where $I_m$ is the unit $m \times m$ matrix.

- Let $B$ be a bilinear anti-symmetric form on $\mathbb{R}^n$ defined by
  \[ B(z, z') = (z, Jz') = \sum_{i=1}^{m} (x^i y^{i'} - y^i x^{i'}) \]
  where $z = (x, y), z' = (x', y') \in \mathbb{R}^n$.

- The symplectic group $SP(n, \mathbb{R})$ is the set of all $n \times n$ real matrices $A$ that leave this bilinear form invariant, that is,
  \[ B(z, z') = B(Az, Az') \]
  or
  \[ (z, Jz') = (Az, JAz') \]
  which simply means
  \[ SP(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid A^T J A = J \}. \]
  Of course, this means that
  \[ \det A = \pm 1. \]

- Let $M$ be a $n$-dimensional manifold.

- Let $\omega$ be a non-degenerate 2-form on $M$ such that
  \[ \det \omega_{ij} = 1. \]

- Then there is a basis in $T^*_p M$ such that the matrix of components takes the canonical form
  \[ \omega_{ik} = J_{ik}. \]
Therefore, if a manifold admits the existence of a non-degenerate 2-form then it has an almost Hamiltonian structure.

An almost Hamiltonian structure is called a **Hamiltonian structure** if the 2-form $\omega$ is closed

$$d\omega = 0.$$ 

Then the manifold $M$ is called a **symplectic manifold** and $\omega$ is called the **symplectic form**.

This condition guarantees that the smooth assignment of bases in $T^*_pM$ such that the symplectic form is reduced to the canonical form.

**Example.** The volume form on the 2-sphere gives a symplectic form

$$\omega = c\text{vol},$$

where $c$ is a normalization constant.

The cotangent bundle $T^*M$ of any manifold $M$ is a symplectic manifold called the **phase space**. The manifold $M$ is called the **configuration space**.

Let $(q^i, p_j)$ be local coordinates on $T^*M$. Then the symplectic form is given by

$$\omega = dp_i \wedge dq^i$$

It is equal to

$$\omega = d\lambda,$$

where

$$\lambda = p_i dq^i$$

is the **Poincaré 1-form**.

**Proposition.** A non-orientable manifold does not admit an almost Hamiltonian structure.

**Proof.** Since the change of orientation of local coordinates changes the sign of the bilinear form $B(z, z')$.

**Proposition.** Even dimensional spheres $S^{2m}$ with $m \geq 2$ do not admit an almost Hamiltonian structure.

**Proof.** Suppose there is a symplectic form $\omega$. 
6.6. $G$-STRUCTURES

- Then the $2m$-form
  \[
  \theta = \omega \wedge \cdots \wedge \omega
  \]
  does not vanish anywhere and, therefore, is a volume form on $S^{2m}$.

- Therefore,
  \[
  \int_{S^{2m}} \theta \neq 0.
  \]

- However, since $H^2(S^{2m}, \mathbb{R}) = 0$ for $m \geq 2$ all closed 2-forms are exact, that is,
  \[
  \omega = d\lambda
  \]
  
- Therefore, the form $\theta$ is exact
  \[
  \theta = d\sigma,
  \]
  where
  \[
  \sigma = \lambda \wedge \omega \wedge \cdots \wedge \omega
  \]

- Thus,
  \[
  \int_{S^{2m}} \theta = 0
  \]
  which is a contradiction.

- **Theorem.** A closed manifold $M$ with zero second homology group $H^2(M, \mathbb{R}) = 0$ does not admit an almost Hamiltonian structure.

- **Almost Complex Structures.** Note that
  \[
  J^2 = -I
  \]

- The space $\mathbb{R}^{2m}$ can be endowed with the structure of the space $\mathbb{C}^m$.

- We define the scalar multiplication of a vector $z \in \mathbb{R}^{2m}$ by a complex number $\lambda = a + ib$ by
  \[
  \lambda z = (a + bJ)z.
  \]

- Let $e_i$, $i = 1, \ldots, m$, be the basis for $\mathbb{C}^m$. Then the basis for $\mathbb{R}^{2m}$ is $(e_i, Je_k)$, $i, k = 1, \ldots, m$. 

• The group $GL(m, \mathbb{C})$ can be defined as a subgroup of $GL(2m, \mathbb{R})$ of all matrices commuting with $J$:

$$GL(m, \mathbb{C}) = \{ A \in GL(2m, \mathbb{R}) \mid AJ = JA \}$$

• For a manifold $M$ to have an almost complex structure it has to be even-dimensional and orientable.

• **Example.** Among even-dimensional spheres $S^{2m}$, $m \geq 1$, only $S^2$ and $S^6$ admit almost complex structures, which has to do with the fact that $S^3$ and $S^7$ are parallelizable.

• Every complex analytic manifold of dimension $m$ always has a natural almost complex structure.

• An almost complex structure is a complex structure if a specific integrability condition is satisfied.

• **Structures on Manifolds.**
  
  – Smooth structure,
  – Riemannian structure,
  – Pseudo-Riemannian structure,
  – Lorentzian structure,
  – orientability,
  – symplectic (almost Hamiltonian) structure,
  – almost complex structure,
  – spin structure.
Chapter 7

Connections on Fiber Bundles

7.1 Connection

7.1.1 Lie Groups

• Let $G$ be a Lie group and $g \in G$. The **left action** $L_g$ and the **right action** $R_g$ of the Lie group are maps

$$L_g : G \to G, \quad R_g : G \to G,$$

defined by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

• The left action and the right action commute

$$L_g R_g = R_g L_g.$$

• The **adjoint action** on the group $G$ is a map

$$\text{ad}_g : G \to G$$

defined by

$$\text{ad}_g(h) = ghg^{-1}.$$

• Of course,

$$\text{ad}_g = L_g R_{g^{-1}} = R_{g^{-1}} L_g.$$

• All these maps are diffeomorphisms.
• The differentials of these maps define linear maps

\[ L_{g*} : T_hG \to T_{gh}G, \quad R_{g*} : T_hG \to T_{hg}G, \]

• The differential of the adjoint action is a linear map called the **adjoint map** and denoted by

\[ \text{Ad}_g = \text{ad}_{g*} : T_hG \to T_{ghg^{-1}}G \]

• Similarly, the pullbacks of these maps map the cotangent spaces

\[ L_g^* : T_{gh}G \to T_h^*G, \quad R_g^* : T_{hg}G \to T_h^*G, \]

\[ \text{ad}_g^* : T_{ghg^{-1}}^*G \to T_h^*G \]

• The tangent space to the group \( G \) at the identity is denoted by

\[ T_eG = \mathfrak{g}. \]

• Note that for \( h = e \) the adjoint map maps the tangent space at the identity \( T_eG = \mathfrak{g} \) to itself,

\[ \text{Ad}_e : \mathfrak{g} \to \mathfrak{g}, \]

defined as follows. Let \( h(t) \) be a curve passing through \( h(0) = e \) and \( \dot{h}(0) = A \). Then

\[ \text{Ad}_e A = \frac{d}{dt} gh(t)g^{-1} \bigg|_{t=0}, \]

which can be formally written as

\[ \text{Ad}_e A = gAg^{-1}. \]

• Similarly, the pullback of the adjoint map for \( h = e \) defines a map

\[ \text{ad}_e^* : T_e^*G \to T_e^*G, \]

defined as follows. Let \( \omega \in T_e^*G \) be a one-form at \( e \). Then \( \text{ad}_e^* \omega \) is a one-form at \( e \) such that for any vector \( A \in T_eG \) at \( e \),

\[ (\text{ad}_e^* \omega)(A) = \omega(\text{Ad}_e A) = \omega(gAg^{-1}) \]

which can be formally written as

\[ \text{ad}_e^* \omega = g^{-1}\omega g. \]
• All these linear maps are bijections.

• A vector field $X \in T_h G$ is called left-invariant (or right-invariant) if

\[ L_g \cdot X_h = X_{gh}, \quad R_g \cdot X_h = X_{hg}. \]

• The left-invariant vector fields form the Lie algebra of $G$, denoted by $\mathfrak{g}$.

• A basis $L_i$ of the left-invariant vector fields satisfies the commutation relations

\[ [L_i, L_j] = C_{ij}^k L_k, \]

where $C_{ij}^k$ are called the structure constants of the Lie group.

• A basis $R_i$ of the right-invariant vector fields forms a Lie algebra

\[ [R_i, R_j] = -C_{ij}^k R_k, \]

that is isomorphic to $\mathfrak{g}$.

• The left-invariant vector fields commute with the right-invariant vector fields

\[ [L_i, R_j] = 0. \]

• There is an isomorphism between the Lie algebra and the tangent space at identity

\[ \mathfrak{g} \cong T_e G. \]

• Every element $A \in \mathfrak{g}$ of the Lie algebra of the group $G$, that is, a vector at the identity $e \in G$, defines a curve $g(t)$ in $G$, called the one-parameter subgroup, such that

\[ g(0) = e, \quad \dot{g}(0) = A. \]

• This defines a map, called the exponential map,

\[ \exp : \mathfrak{g} \to G, \]

and the one-parameter subgroup is denoted by $g(t) = \exp(tA)$.

• In the case when $G$ is a matrix group the exponential map is just the exponential of a matrix.
• The exponential map is injective in the neighborhood of the identity. However, it is not always surjective.

• **Example.** Consider $GL_+(2, \mathbb{R})$, nondegenerate matrices with positive determinant. The Lie algebra $\mathfrak{g}$ consists of all traceless matrices. Show that for any traceless matrix $A \in \mathfrak{g}$ for any $t$,

$$\exp(tA) \neq B,$$

where

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

That is, $B$ is not in the image of the exponential map.

### 7.1.2 Connection

• Let $P = P(M, G)$ be the principal bundle over a manifold $M$ with the structure group $G$.

• Let $u \in P$ and $T_uP$ be the tangent space to $P$ at the point $u$.

• Let $p = \pi(u)$ and $G_p = \pi^{-1}(p)$ be the fiber at $p$.

• A connection defines a canonical orthogonal decomposition

$$T_uP = V_uP \oplus H_uP,$$

where $V_u$ the **vertical subspace** consisting of vectors tangent to the fiber $G_p$ at $u$ and $H_u$ is the **horizontal subspace** consisting of vectors orthogonal to the fiber $G_p$ at $u$.

• The vertical subspace is constructed as follows. Let $A \in \mathfrak{g}$ be an element of the Lie algebra and $g(t) = \exp(tA)$ be the corresponding one-parameter subgroup.

• Recall that there is the right action of the structure group on the principal bundle. In local trivialization it is just

$$R_g : (p, h) \mapsto (p, hg)$$

Therefore, the right action acts along the fiber $G_p$, that is, for any $g \in G$

$$\pi(R_g(u)) = \pi(u) = p$$
• The right action by $g(t)$ defines a curve in $P$ passing through $u$

$$R_{g(t)}(u) = ug(t)$$

Obviously, this curve lies in the fiber $G_p$.

• Next, we define a vector $A^g \in T_uP$ by its action on a smooth function $f$ on $P$

$$(A^g f)(u) = \frac{d}{dt} f(ug(t)) \bigg|_{t=0}$$

• The vector $A^g$ is tangent to the fiber $G_p$ at $u$. Therefore, $A^g \in V_uP$ is in the vertical subspace.

• Next, we extend it smoothly to all of $P$ to get a vector field $A^g$ on $P$ called the fundamental vector field generated by $A$.

• Further, it is easy to see that the map

$$\# : \mathfrak{g} \to V_uP$$

given by $A \mapsto A^g$ is an isomorphism.

• Example. Show that for any $X \in V_uP$

$$\pi_*X = 0$$

• Example. Show that the map $\#$ is a Lie algebra isomorphism, that is, it preserves the Lie brackets, for any $A, B \in \mathfrak{g}$

$$[A, B]^g = [A^g, B^g].$$

• Then the horizontal subspace $H_uP$ is defined as a complement of the vertical subspace $V_uP$ in $T_uP$. This is done with the help of a connection.

• A connection on $P$ is a unique separation of the tangent space $T_uP$ into the vertical subspace $V_uP$ and the horizontal subspace $H_uP$,

$$T_uP = V_uP \oplus H_uP,$$

such that for any $u \in P$ and any $g \in G$,

$$H_{ug}P = R_g H_uP.$$ 

• This defines how the horizontal subspace $H_u$ changes, as $u$ varies in $P$ along the fiber $G_p$ by the right action,

$$u \mapsto u' = ug, \quad g \in G.$$
7.1.3 Connection One-form

- A connection one-form is a Lie algebra valued one-form $\omega \in g \otimes T^*P$, which is a projection of $T_uP$ onto the vertical component $V_uP$ such that: i) for any $A \in g$
  \[ \omega(A^\sharp) = A \]
  and ii)
  \[ R^*_g \omega = \text{Ad}_{g^{-1}} \omega = g^{-1} \omega g, \]

- Now, the horizontal subspace can be defined as the kernel of the connection one-form
  \[ H_uP = \{ X \in T_uP \mid \omega(X) = 0 \}. \]

- Proposition. The horizontal subspaces $H_u$ satisfy
  \[ R_\ast H_uP = H_{ug}P. \]

- Proof. Let $u \in P$ and $X \in H_uP$ so that $\omega(X) = 0$.
  Then
  \[ \omega(R_\ast X) = (R^*_g \omega)(X) = (\text{Ad}_{g^{-1}} \omega)(X) = g^{-1} \omega(X)g = 0 \]
  Therefore, $R_\ast X \in H_{ug}P$.
  On another hand, let $Y \in H_{ug}P$. Then (since $R_\ast$ is a bijection) there is $X \in H_uP$ such that $Y = R_\ast X$.

7.1.4 Horizontal Lift

- Let $\gamma$ be a curve on $M$ such that $\gamma(0) = p$.

- Let $\tilde{\gamma}$ be a curve in $P$ lying above $\gamma$, that is,
  \[ \pi(\tilde{\gamma}(t)) = \gamma(t). \]

- The curve $\tilde{\gamma}$ is called a horizontal lift of $\gamma$ if the tangent vector at any point $u \in \tilde{\gamma}$ is horizontal,
  \[ \tilde{\gamma}(t) \in H_uP. \]
• For a given curve $\gamma$ and a given point $u \in P$ there is a \textit{unique} horizontal lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = u$.

• We declare that fibers are \textbf{parallel transported} along the horizontal lift $\tilde{\gamma}$. That is, for every point $u \in \pi^{-1}(p)$ in the fiber at $p$, we construct the unique horizontal lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = u$; then the parallel transport maps the point $u(t) \in \pi^{-1}(\gamma(t))$ lying above the point $\gamma(t) \in M$.

• Let $(x, g), x \in M, g \in G$, be the local coordinates on the bundle $P$.

• Let $A$ be a 1-form taking values in the Lie algebra $\mathfrak{g}$ defined by

\[ A = A_i dx^i \]

and $\omega$ be a 1-form defined by

\[
\omega = g^{-1}(d + A)g = g^{-1}dg + g^{-1}Ag
\]

\[
= g^{-1}_a (\frac{\partial g_{\alpha\beta}}{\partial x^i} + A_{i\alpha\gamma}g_{\gamma\beta}) dx^i
\]

• Let

\[ N = \dim G. \]

Then

\[ \dim H_uP = n, \quad \dim V_uP = N. \]

• Let the basis for the horizontal subspace $H_uP$ be

\[ h_i = \frac{\partial}{\partial x^i} + B_{i\alpha\beta} \frac{\partial}{\partial g_{\alpha\beta}} \]

where $i = 1, \ldots, n, \alpha,\beta = 1, \ldots, N$.

• We require that for any tangent vector $X \in H_uP$

\[ \langle \omega, X \rangle = 0. \]

• Let $X \in T_uP$ be a vector field; then

\[ X = X_{\alpha\beta} \frac{\partial}{\partial g_{\alpha\beta}} + X^i \left( \frac{\partial}{\partial x^i} + B_{i\alpha\gamma} \frac{\partial}{\partial g_{\alpha\beta}} \right) \]
• Then

\[
\langle \omega, X \rangle = g^{-1}_{\alpha\beta} \left( X_{\alpha\beta} + X^i B_{i\alpha\beta} + A_{i\alpha\gamma} g_{\gamma\beta} X^i \right)
\]

• Now, if \( X \in H_u P \) then \( X_{\alpha\beta} = 0 \). Therefore,

\[
B_{i\alpha\beta} = -A_{i\alpha\gamma} g_{\gamma\beta}
\]

• This determines the basis vectors of the horizontal subspace \( H_u P \) to be

\[
h_i = \frac{\partial}{\partial x^i} - A_{i\alpha\gamma} g_{\gamma\beta} \frac{\partial}{\partial g_{\alpha\beta}}
\]

• Exercise. Show that

\[
H_{u'P} = R_{g'} H_u P
\]

that is,

\[
h_i(x, g g') = R_{g'} h_i(x, g)
\]
Bibliography


Notation

Add all common notation

\( f : X \to Y \)  \hspace{1em} \text{mapping (function) from } X \text{ to } Y

\( f(X) \)  \hspace{1em} \text{range of } f

\( \chi_A \)  \hspace{1em} \text{characteristic function of the set } A

\( \emptyset \)  \hspace{1em} \text{empty set}

\( \mathbb{N} \)  \hspace{1em} \text{set of natural numbers (positive integers)}

\( \mathbb{Z} \)  \hspace{1em} \text{set of integer numbers}

\( \mathbb{Q} \)  \hspace{1em} \text{set of rational numbers}

\( \mathbb{R} \)  \hspace{1em} \text{set of real numbers}

\( \mathbb{R}_+ \)  \hspace{1em} \text{set of positive real numbers}

\( \mathbb{C} \)  \hspace{1em} \text{set of complex numbers}