Notes on Groups $SO(3)$ and $SU(2)$

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1 Group \( SO(3) \)

1.1 Algebra \( so(3) \)

The generators \( X_i = X^i, i = 1, 2, 3 \), of the algebra \( so(3) \) are 3×3 real antisymmetric matrices defined by

\[
X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

or

\[
(X_i)^k_j = -\varepsilon^k_{ij} = \varepsilon_{ikj}
\]

Raising and lowering a vector index does not change anything; it is done just for convenience of notation. These matrices have a number of properties

\[
X_iX_j = E_{ji} - I\delta_{ij},
\]

and

\[
\varepsilon_{ijk}X^k = E_{ij} - E_{ji}
\]

where

\[
(E_{ji})^{kl} = \delta_j^k\delta_i^l
\]

Therefore, they satisfy the algebra

\[
[X_i, X_j] = -\varepsilon^{k}_{ij}X_k
\]

The Casimir operator is defined by

\[
X^2 = X^iX_i
\]

It is equal to

\[
X^2 = -2I
\]

and commutes with all generators

\[
[X_i, X^2] = 0.
\]
A general element of the group $SO(3)$ in canonical coordinates $y^i$ has the form

$$\exp (X(y)) = I \cos r + P(1 - \cos r) + X(y)\frac{\sin r}{r},$$

(1.10)

$$= P + \Pi \cos r + X(y)\frac{\sin r}{r},$$

(1.11)

where

$$r = \sqrt{y^iy^i}, \quad \theta^i = \frac{y^i}{r}$$

(1.12)

$X(y) = X_iy^i$, and the matrices $\Pi$ and $P$ are defined by

$$P^i_j = \delta^i_j \theta_j$$

(1.13)

$$\Pi^i_j = \delta^i_j - \theta^i \theta_j$$

(1.14)

In particular, for $r = \pi$ we have

$$\exp (X(y)) = P - \Pi$$

(1.15)

Notice that

$$\text{tr } P = 1, \quad \text{tr } \Pi = 2$$

(1.16)

and, therefore,

$$\text{tr } \exp (X(y)) = 1 + 2 \cos r.$$ 

(1.17)

Also,

$$\det \frac{\sinh X(y)}{X(y)} = \left(\frac{\sin r}{r}\right)^2$$

(1.18)

### 1.2 Representations of $SO(3)$

Let $X_i$ be the generators of $SO(3)$ in a general irreducible representation satisfying the algebra

$$[X_i, X_j] = -\epsilon_{ijk}X_k$$

(1.19)

They are determined by symmetric traceless tensors of type $(j, j)$ (with $j$ a positive integer)

$$(X_i)^{l_1...l_j}_{k_1...k_j} = -j\epsilon^{l_1}_{i k_1 \delta^l_{k_2} \cdots \delta^l_{k_j}}$$

(1.20)

Then

$$X^2 = -j(j + 1)I$$

(1.21)
where
\[ I^{i_1\ldots i_{k_1}k_2\ldots k_j} = \delta^{(i_1} \cdots \delta^{i_j)}_{(k_1} \cdots k_j)} \] (1.22)

Another basis of generators is \((J_+, J_-, J_3)\), where
\[ J_+ = -iX_1 + X_2 \] (1.23)
\[ J_- = -iX_1 - X_2 \] (1.24)
\[ J_3 = -iX_3 \] (1.25)
so that
\[ J_+^3 = J_- \quad \text{and} \quad J_3^3 = J_3. \] (1.26)

They satisfy the commutation relations
\[ [J_+, J_-] = 2J_3, \quad [J_3, J_+] = J_+, \quad [J_3, J_-] = -J_- \] (1.27)

The Casimir operator in this basis is
\[ X^2 = -J_+J_- - J_3^2 + J_3 = -J_-J_+ - J_3^2 - J_3 \] (1.28)

From the commutation relations we have
\[ J_3J_+ = J_+(J_3 + 1) \] (1.29)
\[ J_3J_- = J_-(J_3 - 1) \] (1.30)
\[ J_-J_+ = -X^2 - J_3^2 - J_3 \] (1.31)
\[ J_+J_- = -X^2 - J_3^2 + J_3 \] (1.32)

Since \(J_3\) commutes with \(X^2\) they can be diagonalized simultaneously. Since they are both Hermitian, they have real eigenvalues. Notice also that since \(X_i\) are anti-Hermitian, then
\[ X^2 \leq 0 \] (1.33)

Let \(|\lambda, m\rangle\) be the basis in which both operators are diagonal, that is,
\[ X^2|\lambda, m\rangle = -\lambda|\lambda, m\rangle \] (1.34)
\[ J_3|\lambda, m\rangle = m|\lambda, m\rangle \] (1.35)

There are many ways to show that the eigenvalues \(m\) of the operator \(J_3\) are integers. Now, since
\[ X_1^2 + X_2^2 = X_3^2 = X_3^2 + J_3^2 \leq 0 \] (1.36)
then the eigenvalues of $J_3$ are bounded by a maximal integer $j$,

$$|m| \leq j$$  \hspace{1cm} (1.37)

that is, $m$ takes $(2j + 1)$ values

$$-j, -j + 1, \cdots, -1, 0, 1, \cdots, j - 1, j$$  \hspace{1cm} (1.38)

From the commutation relations above we have

$$J_+|\lambda, m\rangle = \text{const} |\lambda, m + 1\rangle$$  \hspace{1cm} (1.39)
$$J_-|\lambda, m\rangle = \text{const} |\lambda, m - 1\rangle$$  \hspace{1cm} (1.40)

Let $|\lambda, 0\rangle$ be the eigenvector corresponding to the eigenvalue $m = 0$, that is

$$J_3|\lambda, 0\rangle = 0.$$  \hspace{1cm} (1.41)

Then for positive $m > 0$

$$|\lambda, m\rangle = a_{\lambda m} J^m_+ |\lambda, 0\rangle$$  \hspace{1cm} (1.42)
$$|\lambda, -m\rangle = a_{\lambda, -m} J^m_- |\lambda, -m\rangle$$  \hspace{1cm} (1.43)

Then

$$J_+|\lambda, j\rangle = 0$$  \hspace{1cm} (1.44)
$$J_-|\lambda, -j\rangle = 0$$  \hspace{1cm} (1.45)

Therefore,

$$0 = J_- J_+ |\lambda, j\rangle = \left(-X^2 - J_3^2 - J_3^3\right) |\lambda, j\rangle$$  \hspace{1cm} (1.46)
$$0 = J_+ J_- |\lambda, -j\rangle = \left(-X^2 - J_3^2 + J_3^3\right) |\lambda, -j\rangle$$  \hspace{1cm} (1.47)

Thus,

$$\left(\lambda^2 - j^2 - j\right) |\lambda, j\rangle = 0$$  \hspace{1cm} (1.48)

So, the eigenvalues of the Casimir operator $X^2$ are labeled by a non-negative integer $j \geq 0$,

$$\lambda_j = j(j + 1).$$  \hspace{1cm} (1.49)
These eigenvalues have the multiplicity

$$d_j = 2j + 1.$$  \hfill (1.50)

In particular, for $j = 1$ we get $\lambda_1 = 2$ as expected before.

From now on we will denote the basis in which the operators $X^2$ and $J_3$ are diagonal by

$$|\lambda_j, m\rangle = \left| \begin{array}{c} j \\ m \end{array} \right|.$$  \hfill (1.51)

The matrix elements of the generators $J_\pm$ can be computed as follows. First of all,

$$\langle m | J_3 | m' \rangle = \delta_{mm'}$$  \hfill (1.52)

We have

$$j(j + 1) = \langle m | J_+ | m-1 \rangle \langle j-1 | J_- | j \rangle + m^2 - m$$  \hfill (1.53)

and

$$\langle m | J_+ | m-1 \rangle = \langle m-1 | J_- | m \rangle$$  \hfill (1.54)

This gives

$$\langle m | J_+ | m' \rangle = \delta_{m',m-1} \sqrt{(j + m)(j - m + 1)}$$  \hfill (1.55)

$$\langle m | J_- | m' \rangle = \delta_{m',m+1} \sqrt{(j - m)(j + m + 1)}$$  \hfill (1.56)

The matrix elements of the generators $X_i$ are

$$\langle m | X_1 | m' \rangle = \delta_{m',m-1} \frac{i}{2} \sqrt{(j + m)(j - m + 1)}$$

$$+ \delta_{m',m+1} \frac{i}{2} \sqrt{(j - m)(j + m + 1)}$$  \hfill (1.57)

$$\langle m | X_2 | m' \rangle = \delta_{m',m-1} \frac{1}{2} \sqrt{(j + m)(j - m + 1)}$$

$$- \delta_{m',m+1} \frac{1}{2} \sqrt{(j - m)(j + m + 1)}$$  \hfill (1.58)

$$\langle m | X_3 | m' \rangle = \delta_{m',im}$$  \hfill (1.59)
Thus, irreducible representations of $SO(3)$ are labeled by a non-negative integer $j \geq 0$. The matrices of the representation $j$ in canonical coordinates $y^i$ will be denoted by

$$D^j_{mm'}(y) = \left\langle \begin{array}{c} j \\ m \\ \end{array} \left| \exp [X(y)] \right| \begin{array}{c} j \\ m' \\ \end{array} \right\rangle$$ (1.60)

where $X(y) = X_i y^i$.

The characters of the irreducible representations are

$$\chi^j(y) = \sum_{m=-j}^{j} D^j_{mm}(y) = \sum_{m=-j}^{j} e^{imr} = 1 + 2 \sum_{m=1}^{j} \cos(mr)$$ (1.61)

2 Group $SU(2)$

2.1 Algebra $su(2)$

Pauli matrices $\sigma_i = \sigma^i$, $i = 1, 2, 3$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (2.1)

They have the following properties

$$\sigma^\dagger_i = \sigma_i$$ (2.2)

$$\sigma_i^2 = I$$ (2.3)

$$\det \sigma_i = -1$$ (2.4)

$$\text{tr} \sigma_i = 0.$$ (2.5)

$$\sigma_1 \sigma_2 = i \sigma_3$$ (2.6)

$$\sigma_2 \sigma_3 = i \sigma_1$$ (2.7)

$$\sigma_3 \sigma_1 = i \sigma_2$$ (2.8)

$$\sigma_1 \sigma_2 \sigma_3 = iI$$ (2.9)

which can be written in the form

$$\sigma_i \sigma_j = \delta_{ij} I + i \epsilon_{ijk} \sigma_k.$$ (2.11)
They satisfy the following commutation relations

\[ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \]  

(2.12)

The traces of products are

\[ \text{tr} \sigma_i \sigma_j = 2\delta_{ij} \]  

(2.13)

\[ \text{tr} \sigma_i \sigma_j \sigma_k = 2i\epsilon_{ijk} \]  

(2.14)

Also, there holds

\[ \sigma_i \sigma_i = 3I. \]  

(2.15)

The Pauli matrices satisfy the anti-commutation relations

\[ \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}I \]  

(2.16)

Therefore, Pauli matrices form a representation of Clifford algebra in 3 dimensions and are, in fact, Dirac matrices \( \sigma_i = (\sigma_i^A) \), where \( A, B = 1, 2 \), in 3 dimensions. The spinor metric \( E = (E_{AB}) \) is defined by

\[
E = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]  

(2.17)

Notice that

\[ E = i\sigma_2 \]  

(2.18)

and the inverse metric \( E^{-1} = (E^{AB}) \) is

\[ E^{-1} = -E \]  

(2.19)

Then

\[ \sigma_i^T = -E\sigma_i E^{-1} \]  

(2.20)

This allows to define the matrices \( E\sigma_i = (\sigma_i^{AB}) \) with covariant indices

\[ E\sigma_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(2.21)

\[ E\sigma_2 = iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \]  

(2.22)

\[ E\sigma_3 = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \]  

(2.23)
Notice that the matrices $E\sigma_i$ are all symmetric

$$(E\sigma_i)^T = E\sigma_i \tag{2.24}$$

Pauli matrices satisfy the following completeness relation

$$\sigma_i^A B \sigma_i^C D = 2\delta^A D \delta^C B - \delta^A B \delta^C D \tag{2.25}$$

which implies

$$\sigma_i^{(A} (B \sigma_i^{C)} D) = \delta^{(A} (B \delta^{C)} D) \tag{2.26}$$

This gives

$$\sigma_{iAB} \sigma_{iCD} = 2E_{AD} E_{CB} - E_{AB} E_{CD} \tag{2.27}$$

In a more symmetric form

$$\sigma_{iAB} \sigma_{iCD} = E_{AD} E_{CB} + E_{BD} E_{CA} \tag{2.28}$$

A spinor is a two dimensional complex vector $\psi = (\psi^A), A = 1, 2$ in the spinor space $\mathbb{C}^2$. We use the metric $E$ to lower the index to get the covariant components

$$\psi_A = E_{AB} \psi^B \tag{2.29}$$

which defines the cospinor $\tilde{\psi} = (\psi_A)$

$$\tilde{\psi} = E\psi \tag{2.30}$$

We also define the conjugated spinor $\bar{\psi} = (\bar{\psi}_A)$ by

$$\bar{\psi}_A = \psi^A \tag{2.31}$$

On the complex spinor space there are two invariant bilinear forms

$$\langle \psi, \varphi \rangle = \tilde{\psi} \varphi = \psi^A \varphi^A \tag{2.32}$$

and

$$\tilde{\psi} \varphi = \psi_A \varphi^A = E_{AB} \psi^B \varphi^A = \psi^1 \varphi^2 - \psi^2 \varphi^1 \tag{2.33}$$

Note that

$$\langle \psi, \psi \rangle = \tilde{\psi} \psi = \psi^{A*} \psi^A = |\psi^1|^2 + |\psi^2|^2 \tag{2.34}$$

and

$$\tilde{\psi} \psi = \psi_A \psi^A = 0 \tag{2.35}$$
The matrices

\[ X_i = \frac{i}{2} \sigma_i \] (2.36)

are the generators of the Lie algebra \( su(2) \) with the commutation relations

\[ [X_i, X_j] = -\varepsilon_{ijk} X_k \] (2.37)

The Casimir operator is

\[ X^2 = X_i X^i \] (2.38)

It commutes with all generators

\[ [X_i, X^2] = 0. \] (2.39)

There holds

\[ X^2 = -\frac{3}{4} I \] (2.40)

\[ \text{tr } X_i X_j = -\frac{1}{2} \delta_{ij} \] (2.41)

Notice that

\[ X^2 = -j(j+1)I \] (2.42)

with \( j = \frac{1}{2} \), which determines the fundamental representation of \( SU(2) \).

An arbitrary element of the group \( SU(2) \) in canonical coordinates \( y^i \) has the form

\[ \exp \left( \frac{i}{2} \sigma_j y^j \right) = I \cos \frac{r}{2} + i \sigma_j \theta^j \sin \frac{r}{2}, \] (2.43)

This can be written in the form

\[ \exp (X(y)) = I \cos(r/2) + X(y) \frac{\sin(r/2)}{r/2} \] (2.44)

where \( X(y) = X_i y^i \). In particular, for \( r = 2\pi \)

\[ \exp (X(y)) = -I. \] (2.45)

Obviously,

\[ \text{tr } \exp (X(y)) = 2 \cos(r/2) \] (2.46)
2.2 Representations of $S U(2)$

We define symmetric contravariant spinors $\psi^{A_1\ldots A_{2j}}$ of rank $2j$. Each index here can be lowered by the metric $E_{AB}$. The number of independent components of such spinor is precisely

$$\sum_{i=0}^{2j} 1 = (2j + 1) \quad (2.47)$$

Let $X_i$ be the generators of $S U(2)$ in a general irreducible representation satisfying the algebra

$$[X_i, X_j] = -\varepsilon_{ijk} X_k \quad (2.48)$$

They are determined by symmetric spinors of type $(2j, 2j)$

$$(X_k)^{A_1\ldots A_{2j} B_1\ldots B_{2j}} = ij \sigma_k^{(A_1(B_1 \delta_{A_2}^{B_2} \cdots \delta_{A_{2j}}^{B_{2j})}} \quad (2.49)$$

Then

$$X^2 = -j(j + 1)I \quad (2.50)$$

where

$$I^{A_1\ldots A_{2j)}_{B_1\ldots B_{2j}}} = \delta^{(A_1 \cdots A_{2j)}}_{(B_1 \cdots B_{2j})} \quad (2.51)$$

Another basis of generators is

$$J_+ = -i X_1 + X_2 \quad (2.52)$$
$$J_- = -i X_1 - X_2 \quad (2.53)$$
$$J_3 = -i X_3 \quad (2.54)$$

They have all the properties of the generators of $S O(3)$ discussed above except that the operator $J_3$ has integer eigenvalues $m$. One can show that they can be either integers or half-integers taking $(2j + 1)$ values

$$-j, -j + 1, \ldots, j - 1, j, \quad |m| \leq j \quad (2.55)$$

with $j$ being a positive integer or half-integer. If $j$ is integer, then all eigenvalues $m$ of $J_3$ are integers including 0. If $j$ is a half-integer, then all eigenvalues $m$ are half-integers and 0 is excluded.

We choose a basis in which the operators $X^2$ and $J_3$ are diagonal. Let $\psi = \psi^{A_1\ldots A_{2j}}$ be a symmetric spinor of rank $2j$. Let $e_1 = (e_1^A)$ and $e_2 = (e_2^A)$ be the basis cospinors defined by

$$e_1^A = \delta_1^A, \quad e_2^A = \delta_2^A. \quad (2.56)$$
Let
\[ e_{j,m}^{A_1\ldots A_{jm}B_1\ldots B_{jm}} = e_1^{(A_1} \ldots e_1^{A_{jm}e_2^{B_1} \ldots e_2^{B_{jm})} } \]
\[ = \delta_{1}^{(A_1} \ldots \delta_{1}^{A_{jm}} \delta_{2}^{B_1} \ldots \delta_{2}^{B_{jm})} \] \hspace{1cm} (2.57)

Then this is an eigenspinor of the operator \( J_3 \) with the eigenvalue \( m \)
\[ J_3 e_{j,m} = me_{j,m} \] \hspace{1cm} (2.58)

Then the spinors
\[ \begin{vmatrix} j \\ m \end{vmatrix} = \left( \frac{(2j)!}{(j+m)!(j-m)!} \right)^{1/2} e_{j,m} \] \hspace{1cm} (2.59)
form an orthonormal basis in the space of symmetric spinors of rank \( 2j \).

The matrix elements of the generators \( X_i \) are given by exactly the same formulas
\[ \langle j \big| X_1 \big| j' \rangle = \delta_{m',m-1} \frac{i}{2} \sqrt{(j+m)(j-m+1)} \]
\[ + \delta_{m',m+1} \frac{i}{2} \sqrt{(j-m)(j+m+1)} \] \hspace{1cm} (2.60)
\[ \langle j \big| X_2 \big| j' \rangle = \delta_{m',m-1} \frac{1}{2} \sqrt{(j+m)(j-m+1)} \]
\[ - \delta_{m',m+1} \frac{1}{2} \sqrt{(j-m)(j+m+1)} \] \hspace{1cm} (2.61)
\[ \langle j \big| X_3 \big| j' \rangle = \delta_{m'm} \] \hspace{1cm} (2.62)

but now \( j \) and \( m \) are either both integers or half-integers.

Thus, irreducible representations of \( SU(2) \) are labeled by a non-negative half-integer \( j \geq 0 \). The matrices of the representation \( j \) in canonical coordinates \( y^j \) are
\[ D^j_{m'm}(y) = \left( j \big| \begin{vmatrix} m \\ \exp[X(y)] \end{vmatrix} \big| j' \right) \] \hspace{1cm} (2.63)
where \( X(y) = X_i y^i \). The representations of \( SU(2) \) with integer \( j \) are at the same time representations of \( SO(3) \). However, representations with half-integer \( j \) do not give representations of \( SO(3) \), since for \( r = 2\pi \)
\[ D^j_{m'm}(y) = -\delta_{m'm} \] \hspace{1cm} (2.64)
The characters of the irreducible representations are: for integer \( j \)

\[
\chi^j(y) = \sum_{m=-j}^j D^j_m(y) = \sum_{m=-j}^j e^{imr} = 1 + 2 \sum_{m=1}^j \cos(mr)
\]  

(2.65)

and for half-integer \( j \)

\[
\chi^j(y) = \sum_{k=\frac{1}{2}+j}^{\frac{1}{2}+j} D^j_{\frac{1}{2}+k,\frac{1}{2}+k}(y) = \sum_{k=\frac{1}{2}+j}^{\frac{1}{2}+j} \exp \left[ i \left( \frac{1}{2} + k \right) r \right] = 2 \sum_{k=1}^j \cos \left[ \left( \frac{1}{2} + k \right) r \right]
\]

(2.66)

Now, let \( X_i \) and \( Y_j \) be the generators of two representations and let

\[
G_i = X_i \otimes I_Y + I_X \otimes Y_i
\]

(2.67)

Then \( G_i \) generate the product representation \( X \otimes Y \) and

\[
\exp[G(y)] = \exp[X(y)] \exp[Y(y)]
\]

(2.68)

Therefore, the character of the product representation factorizes

\[
\chi^G(y) = \chi^X(y) \chi^Y(y)
\]

(2.69)

### 2.3 Double Covering Homomorphism \( SU(2) \to SO(3) \)

Recall that the matrices \( E \sigma_i = (\sigma_{i AB}) \) are symmetric and the matrix \( E = (E_{AB}) \) is anti-symmetric. Let \( \psi^{AB} \) be a symmetric spinor. Then one can form a vector

\[
A_i = \sigma_{i BA} \psi^{AB} = \text{tr} (E \sigma_i \psi)
\]

(2.70)

Conversely, given a vector \( A_i \) we get a symmetric spinor of rank 2 by

\[
\psi_{AB} = \frac{1}{2} A^i \sigma_{i AB}
\]

(2.71)

or

\[
\psi = \frac{1}{2} A^i \sigma_i E^{-1}
\]

(2.72)

More generally, given a symmetric spinor of rank \( 2j \) we can associate to it a symmetric tensor of rank \( j \) by

\[
A_{i_1 \cdots i_j} = \sigma_{i_1 A_1 B_1} \cdots \sigma_{i_j A_j B_j} \psi^{A_1 \cdots A_j B_1 \cdots B_j}
\]

(2.73)
One can show that this tensor is traceless. Indeed by using the eq. (2.74) and the fact that the spinor is symmetric we see that the tensor $A$ is traceless. The inverse transformation is defined by

$$
\psi_{A_1\ldots A_j B_1\ldots B_j} = \frac{1}{2} A^{i_1\ldots i_j} \sigma_{i_1 A_1 B_1} \cdots \sigma_{i_j A_j B_j}
$$

(2.74)

The double covering homomorphism

$$
\Lambda : SU(2) \rightarrow SO(3)
$$

(2.75)
is defined as follows. Let $U \in SU(2)$. Then the matrices $U \sigma_i U^{-1}$ satisfy all the properties of the Pauli matrices and are therefore in the algebra $su(2)$. Therefore,

$$
U \sigma_i U^{-1} = \Lambda^i_j(U) \sigma_j.
$$

(2.76)

That is,

$$
\Lambda^i_j(U) = \frac{1}{2} \text{tr} \sigma^i U \sigma_j U^{-1}
$$

(2.77)
The matrix $\Lambda(U)$ depends on the matrix $U$ and is, in fact, in $SO(3)$. For infinitesimal transformations

$$
U = \exp(X_i y^i) = I + \frac{i}{2} \sigma_k y^k + \cdots
$$

(2.78)
the matrix $\Lambda$ is

$$
\Lambda_{ij} = \delta_{ij} + \epsilon_{ijk} y^k + \cdots = \left[ \exp(X_i y^i) \right]_{ij}
$$

(2.79)
It is easy to see that

$$
\Lambda(I) = \Lambda(-I) = I.
$$

(2.80)
So, in short

$$
\Lambda(\exp(X_i y^i)) = \exp(\Lambda(X_i) y^i)
$$

(2.81)
where

$$
[\Lambda(\sigma_k)]^i_j = 2i \epsilon^i_{kj}
$$

(2.82)

3 Invariant Vector Fields

Let $y^i$ be the canonical coordinates on the group $SU(2)$ ranging over $(-\pi, \pi)$. The position of the indices on the coordinates will be irrelevant, that is, $y_i = y^i$. We introduce the radial coordinate $r = |y| = \sqrt{y_0 y^0}$, so that the geodesic distance
from the origin is equal to \( r \), and the angular coordinates (the coordinates on \( S^2 \)) \( \theta^i = \hat{y}^i / r \).

Let us consider the fundamental representation of the group \( SU(2) \) with generators \( T_a = i \sigma_a \), where \( \sigma_a \) are the Pauli matrices satisfying the algebra

\[
[T_a, T_b] = -2 \epsilon^{abc} T_c. \tag{3.83}
\]

Let \( T(y) = T_i y^i \). Then

\[
T(y)T(p) = -(y, p) - T(y \wedge p) \tag{3.84}
\]

where \( (y, p) = y_i p^i \), \( (y \wedge p)_k = \epsilon^{ijk} y^i p^j \) and

\[
[T(y)]^2 = -|y|^2 I \tag{3.85}
\]

Therefore,

\[
\exp[rT(\hat{y})] = \cos r + T(\hat{y}) \sin r \tag{3.86}
\]

where \( \hat{y} \) is a unit vector. Next, we compute

\[
\exp[T(F(r\hat{q}, \rho \hat{p}))] = \cos r \cos \rho - (\hat{q}, \hat{p}) \sin r \sin \rho \\
+ \cos r \sin \rho T(\hat{p}) + \cos \rho \sin rT(\hat{q}) \\
- \sin r\sin \rho T(\hat{q} \wedge \hat{p}) \tag{3.87}
\]

This defines the group multiplication map \( F(q, p) \) in canonical coordinates.

This map has a number of important properties. In particular, \( F(0, p) = F(p, 0) = p \) and \( F(p, -p) = 0 \). Another obvious but very useful property of the group map is that if \( q = F(\omega, p) \), then \( \omega = F(q, -p) \) and \( p = F(-\omega, q) \). Also, there is the associativity property

\[
F(\omega, F(p, q)) = F(F(\omega, p), q) \tag{3.88}
\]

and the inverse property

\[
F(-p, -\omega) = -F(\omega, p). \tag{3.89}
\]

This map defines all the properties of the group. We introduce the matrices

\[
L^a_{\ b}(x) = \frac{\partial}{\partial y^b} F^a(y, x) \bigg|_{y=0}, \tag{3.90}
\]

\[
R^a_{\ b}(x) = \frac{\partial}{\partial y^b} F^a(x, y) \bigg|_{y=0}. \tag{3.91}
\]
Let $X$ and $Y$ be the inverses of the matrices $L$ and $R$, that is,
\[ LX = XL = RY = YR = I. \] (3.92)
Also let
\[ D = XR, \quad D^{-1} = YL \] (3.93)
so that
\[ L = RD^{-1}, \quad X = DY \] (3.94)
Let $C_a$ be the matrices of the adjoint representation defined by
\[ (C_a)^b_c = -2\epsilon^{b}_{ac} \] (3.95)
and $C = C(x)$ be the matrix defined by
\[ C = C_a x^a. \] (3.96)
Then one can show that
\[ X = \frac{\exp(C) - I}{C}, \quad Y = \frac{I - \exp(-C)}{C}, \] (3.97)
\[ L = \frac{C}{\exp(C) - I}, \quad R = \frac{C}{I - \exp(-C)}. \] (3.98)
It is easy to see also that
\[ x^a = L^a_b x^b = R^a_b x^b = X^a_b x^b = Y^a_b x^b. \] (3.99)
and
\[ X = Y^T, \quad L = R^T \] (3.100)
and
\[ XY = YX = \left( \frac{\sinh(C/2)}{C/2} \right)^2, \] (3.101)
\[ LR = RL = \left( \frac{C/2}{\sinh(C/2)} \right)^2, \] (3.102)
as well as
\[ D = XR = RX = \exp(C), \quad D^T = YL = LY = \exp(-C). \] (3.103)
Note that
\[ C^2 = -4r^2 \Pi, \quad (3.104) \]
where \( r = |x| \) and
\[ \Pi'_{ij} = \delta_{ij} - \theta_i \theta_j \quad (3.105) \]
with \( \theta^i = x^i / r \). Therefore,
\[ C^{2n} = (-1)^n (2r)^{2n} \Pi, \quad C^{2n+1} = (-1)^n (2r)^{2n} C \quad (3.106) \]
Therefore, the eigenvalues of the matrix \( C \) are \( (2ir, -2ir, 0) \), and for any analytic function
\[ f(C) = f(0)(I - \Pi) + \Pi \frac{1}{2} [f(2ir) + f(-2ir)] + \frac{1}{4ir} C [f(2ir) - f(-2ir)] \]
and, therefore,
\[ \text{tr} f(C) = f(0) + f(2ir) + f(-2ir) \quad (3.107) \]
\[ \det f(C) = f(0) f(2ir) f(-2ir) \quad (3.108) \]
This enables one to compute
\[ Y = I - \Pi + \frac{\sin r}{r} \cos r \Pi - \frac{1}{2} \frac{\sin^2 r}{r^2} C \quad (3.109) \]
\[ X = I - \Pi + \frac{\sin r}{r} \cos r \Pi + \frac{1}{2} \frac{\sin^2 r}{r^2} C \quad (3.110) \]
\[ R = I - \Pi + r \cot r \Pi + \frac{1}{2} C \quad (3.111) \]
\[ L = I - \Pi + r \cot r \Pi - \frac{1}{2} C \quad (3.112) \]
\[ D = I - \Pi + \Pi \cos(2r) + \frac{\sin(2r)}{2r} C \quad (3.113) \]
Then one can show that
\[ \frac{\partial}{\partial x^\mu} F^a_{\gamma}(x, y) = L^a_{\gamma}(F(x, y)) X^\gamma_{\mu}(x), \quad (3.114) \]
\[ \frac{\partial}{\partial \xi^a} F^a_{\gamma}(y, x) = R^a_{\gamma}(F(y, x)) Y^\gamma_{\mu}(x). \quad (3.115) \]
These matrices satisfy important identities
\[ \partial_{\mu} X^a_{\nu} - \partial_{\nu} X^a_{\mu} = -2 \varepsilon^a_{\nu bc} X^b_{\mu} X^c_{\nu}, \quad (3.116) \]
\[ \partial_{\mu} Y^a_{\nu} - \partial_{\nu} Y^a_{\mu} = 2 \varepsilon^a_{\nu bc} Y^b_{\mu} Y^c_{\nu}. \quad (3.117) \]
and
\[
\begin{align*}
L^\mu_a \partial_\mu L^\nu_b - L^\mu_b \partial_\mu L^\nu_a &= 2 \varepsilon_{ab} L^\nu_c, \\
R^\mu_a \partial_\mu R^\nu_b - R^\mu_b \partial_\mu R^\nu_a &= -2 \varepsilon_{ab} R^\nu_c.
\end{align*}
\] (3.118)
(3.119)

Let us define the one-forms
\[
\sigma_-^a = X^a_\mu(x) dx^\mu, \quad \sigma_+^a = Y^a_\mu(x) dx^\mu.
\]
They satisfy important identities
\[
\begin{align*}
d\sigma_-^a &= -\varepsilon_{cb} \sigma_-^b \wedge \sigma_-^c, \\
d\sigma_+^a &= \varepsilon_{cb} \sigma_+^b \wedge \sigma_+^c.
\end{align*}
\] (3.120)
(3.121)

Next, we define the vector fields.
\[
K_-^a = L^\mu_a(x) \partial_\mu, \quad K_+^a = R^\mu_a(x) \partial_\mu.
\]
These are the right-invariant and the left-invariant vector fields on the group. They are related by
\[
K_-^a = D_{ab} K_+^b
\] (3.122)

They form two representations of the group $SU(2)$ and satisfy the algebra
\[
\begin{align*}
[K_+^a, K_+^b] &= -2 \varepsilon_{ab} K_+^c, \\
[K_-^a, K_-^b] &= 2 \varepsilon_{ab} K_-^c, \\
[K_+^a, K_-^b] &= 0
\end{align*}
\] (3.123)
(3.124)
(3.125)

That is, the left-invariant vector fields commute with the right-invariant ones.

Now, let $T_a$ be the generators of some representation of the group $SU(2)$ satisfying the same algebra. Then, of course,
\[
\exp[T(\omega)] \exp[T(p)] = \exp[T(F(\omega, p))]
\] (3.126)

First, one can derive a useful commutation formula
\[
\exp[T(\omega)] T_b \exp[-T(\omega)] = D_{ab}^c T_a,
\] (3.127)
where the matrix $D = (D_{ab}^c)$ is defined by $D = \exp C$.

More generally,
\[
\exp[T(\omega)] \exp[T(p)] \exp[-T(\omega)] = \exp[T(D(\omega)p)]
\] (3.128)
Next, one can show that
\[ \exp[-T(\omega)] \frac{\partial}{\partial \omega^i} \exp[T(\omega)] = Y^a_i T_a, \]
(3.129)

This means that
\[ \frac{\partial}{\partial \omega^i} \exp[T(\omega)] = Y^a_i \exp[T(\omega)] T_a = Y^a_i T_a \exp[T(\omega)], \]
(3.130)

The Killing vectors have important properties
\[ K^+_a \exp[T(\omega)] = (\exp[T(\omega)]) T_a \]
(3.131)
\[ K^-_a \exp[T(\omega)] = T_a(\exp[T(\omega)]) \]
(3.132)

This immediately leads to further important equations
\[ \exp[K^+(p)] \exp[T(\omega)] = \exp[T(\omega)] \exp[T(p)] \]
(3.133)

and
\[ \exp[K^-(q)] \exp[T(\omega)] = \exp[T(q)] \exp[T(\omega)] \]
(3.134)

where \( K^\pm(p) = p^a K^\pm_a \). More generally,
\[ \exp[K^+(p) + K^-(q)] \exp[T(\omega)] = \exp[T(q)] \exp[T(\omega)] \exp[T(p)] \]
(3.135)

and, therefore,
\[ \exp[K^+(p) + K^-(q)] \exp[T(\omega)] \bigg|_{\omega=0} = \exp[T(q)] \exp[T(p)] = \exp[T(F(q, p))] \]
(3.136)

In particular, for the adjoint representation this formulas take the form
\[ DC_b = D^a_b C_a D \]
(3.137)
\[ K^+_a D = DC_a \]
(3.138)
\[ K^-_a D = C_a D \]
(3.139)
\[ \exp[K^+(p)] D(x) = D(x) D(p) \]
(3.140)
\[ \exp[K^-(q)] D(x) = D(q) D(x) \]
(3.141)
\[ \exp[K^+(p) + K^-(q)] D(x) = D(q) D(x) D(p) \]
(3.142)
Then for a scalar function \( f(\omega) \) the action of the right-invariant and left-invariant vector fields is simply

\[
\exp[K^+(p)]f(\omega) = f(F(\omega, p)), \quad \exp[K^-(q)]f(\omega) = f(F(q, \omega)).
\] (3.143)

Since the action of these vector fields commutes, we have more generally

\[
\exp[K^+(p) + K^-(q)]f(\omega) = f(F(q, F(\omega, p))). \quad (3.144)
\]

Therefore,

\[
\exp[K^+(p) + K^-(q)]f(\omega) \bigg|_{\omega=0} = f(F(q)). \quad (3.145)
\]

The bi-invariant metric is defined by

\[
\begin{align*}
g_{\alpha\beta} &= \delta_{ab} Y^a_\alpha Y^b_\beta = \delta_{ab} X^a_\alpha X^b_\beta \quad (3.146) \\
g^{\alpha\beta} &= \delta^{ab} L^a_\alpha L^b_\beta = \delta^{ab} R^a_\alpha R^b_\beta, \quad (3.147)
\end{align*}
\]

that is,

\[
g = \left(\frac{\sinh[C/2]}{C/2}\right)^2 = I - \Pi + \frac{\sin^2 r}{r^2} \Pi. \quad (3.148)
\]

The Riemannian volume elements of the metric \( g \) is defined as usual

\[
d\text{vol} = g^{1/2}(x)dx^1 \wedge dx^2 \wedge dx^3 \quad (3.149)
\]

where

\[
g^{1/2} = (\det g_{\mu\nu})^{1/2} = \frac{\sin^2 r}{r^2}. \quad (3.150)
\]

Thus the bi-invariant volume element of the group is

\[
d\text{vol}(x) = \frac{\sin^2 r}{r^2} dx = \sin^2 r \, dr \, d\theta \quad (3.151)
\]

where \( dx = dx^1 dx^2 dx^3 \), and \( d\theta \) is the volume element on \( S^2 \). The invariance of the volume element means that for any fixed \( p \)

\[
d\text{vol}(x) = d\text{vol}(F(x, p)) = d\text{vol}(F(p, x)) \quad (3.152)
\]

Notice also, that \( |F(p, q)| \) is nothing but the geodesic distance between the points \( p \) and \( q \).
We will always use the orthonormal basis of the right-invariant vector fields $K_a^+$ and one-forms $\sigma^a_+$ and denote the covariant derivative with respect to $K_a^+$ simply by $\nabla_a$. The Levi-Civita connection $\nabla$ of the bi-invariant metric in the right-invariant basis is defined by

$$\nabla_a K_b^+ = \epsilon^c_{\ b a} K_c^+$$

so that the coefficients of the affine connection are

$$\omega^a_{\ bc} = \sigma^a_+ (\nabla_c K_b^+) = \epsilon^a_{\ bc}$$

so that for an arbitrary vector

$$\nabla_a V = \left( (K_a^+ V^b) + \epsilon^b_{\ ca} V^c \right) K_b^+$$

Then

$$\nabla_a K_d^- = M^b_{\ da} K_b^+$$

where

$$M^b_{\ da} = K_a^+ D^b_d + \epsilon^b_{\ ca} D^c_d$$

$$= -2\epsilon^b_{\ ca} D^c_d + \epsilon^b_{\ ca} D^c_d = -\epsilon^b_{\ ca} D^c_d$$

so that

$$\nabla_a K_d^- = -\epsilon^b_{\ ca} D^c_d K_b^+$$

Now, the curvature tensor is

$$R^a_{\ bcd} = -\epsilon^a_{\ fj} \epsilon^d_{\ fb}$$

The Ricci curvature tensor and the scalar curvature are

$$R_{ab} = 2\delta_{ab} , \quad R = 6.$$  

The scalar Laplacian is

$$\Delta_0 = \delta^{ab} K_a^- K_b^- = \delta^{ab} K_a^+ K_b^+$$

One can show that

$$\Delta_0 \frac{|\omega|}{\sin |\omega|} = \frac{|\omega|}{\sin |\omega|}$$

Further,

$$\Delta_0 \exp[T(\omega)] = (\exp[T(\omega)]) T^2 = T^2 \exp[T(\omega)]$$
where $T^2 = T_a T^a$. This means that

$$\exp[t\Delta_0^2] \exp[T(\omega)] = \exp(tT^2) \exp[T(\omega)]$$  

(3.164)

The Killing vector fields $K^\pm_a$ are divergence free, which means that they are anti-self-adjoint with respect to the invariant measure $d\text{vol}(\omega)$. That is, the Laplacian $\Delta_0$ is self-adjoint with respect to the same measure as well.

Let $\nabla$ be the total connection on a vector bundle $V$ realizing the representation $G$. Let us define the one-forms

$$\mathcal{A} = \sigma^a G_a = G_aY^a d\mathcal{x}^\mu.$$  

(3.165)

and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. Then, by using the above properties of the right-invariant one-forms we compute

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{ab} \sigma^a_+ \wedge \sigma^b_+ = \frac{1}{2} \mathcal{F}_{ab} Y^a Y^b_{\mu} d\mathcal{x}^\mu \wedge d\mathcal{x}^\nu$$  

(3.166)

where

$$\mathcal{F}_{ab} = \varepsilon^{ce}_{ab} G_c.$$  

(3.167)

Thus $\mathcal{A}$ is the Yang-Mills curvature on the vector bundle $V$ with covariantly constant curvature.

The covariant derivative of a section of the bundle $V$ is then (recall that $\nabla_a = \nabla_{K^a}$)

$$\nabla_a \varphi = (K^+_a + G_a) \varphi$$  

(3.168)

and the covariant derivative of a section of the endomorphism bundle $\text{End}(V)$ along the right-invariant basis is then

$$\nabla_a Q = K^+_a Q + [G_a, Q]$$  

(3.169)

Therefore, we find

$$\nabla_a G_b = \varepsilon^e_{ba} G_c + [G_a, G_b] = 0.$$  

(3.170)

Then the derivatives along the left-invariant vector fields are

$$\nabla_{K^-} \varphi = (K^-_a + B_a) \varphi$$  

(3.171)

$$\nabla_{K^-} Q = K^-_a Q + [B_a, Q]$$  

(3.172)

where

$$B_a = D^b_a G_b.$$  

(3.173)
The Laplacian takes the form

\[ \Delta = \nabla_a \nabla^a = (K_+ + G_a)(K_+ + G_a) = \Delta_0 + 2G^aK_+ + G^2 \]  

(3.174)

where \( G^2 = G_iG^i \).

We want to rewrite the Laplacian in terms of Casimir operators of some representations of the group \( SU(2) \). The covariant derivatives \( \nabla_a \) do not form a representation of the algebra \( SU(2) \). The operators that do are the covariant Lie derivatives. The covariant Lie derivatives along a Killing vector \( \xi \) of sections of this vector bundle are defined by

\[ L_\xi = \nabla_\xi - \frac{1}{2} \sigma^a (\nabla_b \xi) \epsilon_{abc} G^c \]  

(3.175)

By denoting the Lie derivatives along the Killing vectors \( K_\pm \) by \( \mathcal{K}_\pm \) this gives for the right-invariant and the left-invariant bases

\[ \mathcal{K}_+ = L_{K_+} = \nabla_{K_+} + G_a = K_+ + 2G_a . \]  

(3.176)

\[ \mathcal{K}_- = L_{K_-} = \nabla_{K_-} - B_a = K_- . \]  

(3.177)

It is easy to see that these operators form the same algebra \( SU(2) \times SU(2) \)

\[ [\mathcal{K}_a^+, \mathcal{K}_b^+] = -2\epsilon_{abc} \mathcal{K}_c^+ \]  

(3.178)

\[ [\mathcal{K}_a^-, \mathcal{K}_b^+] = 2\epsilon_{abc} \mathcal{K}_c^- \]  

(3.179)

\[ [\mathcal{K}_a^+, \mathcal{K}_b^-] = 0 \]  

(3.180)

Now, the Laplacian is given now by the sums of the Casimir operators

\[ \Delta = \mathcal{K}^2 - G^2 \]  

(3.181)

where

\[ \mathcal{K}^2 = \frac{1}{2} \mathcal{K}_+^2 + \frac{1}{2} \mathcal{K}_-^2 \]  

(3.182)

and \( \mathcal{K}_\pm = \mathcal{K}_a^\pm \mathcal{K}_a^\pm \).

## 4 Heat Kernel on \( S^3 \)

Our goal is to evaluate the heat kernel diagonal. Since it is constant we can evaluate it at any point, say, at the origin. We use the geodesic coordinates \( y^i \) defined
above. That is why, we will evaluate the heat kernel when one point is fixed at the origin. We will denote it simply by $U(t, y)$. We obviously have

$$\exp(t\Delta) = \exp(-tG^2) \exp(tK^2).$$

(4.1)

Therefore, the heat kernel is equal to

$$U(t, y) = \exp(-tG^2) \Psi\left(\frac{t}{2}, y\right)$$

(4.2)

where $\Psi(t, y) = \exp(2tK^2)\delta(y)$ is the heat kernel of the operator $2K^2$.

### 4.1 Scalar Heat Kernel

Let

$$\Phi(t, \omega) = (4\pi t)^{-3/2} e^{\frac{1}{t}} \sum_{n=-\infty}^{\infty} \frac{|\omega| + 2\pi n}{\sin |\omega|} \exp\left(-\frac{(|\omega| + 2\pi n)^2}{4t}\right)$$

(4.3)

One can show by direct calculation that this function satisfies the equation

$$\partial_t \Phi = \Delta_0^2 \Phi$$

(4.4)

with the initial condition

$$\Phi(0, \omega) = \delta_{S^3}(\omega)$$

(4.5)

Therefore, this is the scalar heat kernel on the group $SU(2)$ and, therefore, on $S^3$.

Now, let us compute the integral

$$\Psi(t) = \int_{SU(2)} d\text{vol}(\omega) \Phi(t, \omega) \exp[T(\omega)]$$

(4.6)

Obviously, $\Psi(0) = 1$. Next, we have

$$\partial_t \Psi = \int_{SU(2)} d\text{vol}(\omega) \exp[T(\omega)] \Delta_0 \Phi(t, \omega)$$

(4.7)

Now, by integrating by parts we get

$$\partial_t \Psi = \int_{SU(2)} D\omega \Phi(t, \omega) \Delta_0 \exp[T(\omega)]$$

(4.8)

which gives

$$\partial_t \Psi = \Psi T^2$$

(4.9)
Thus, we obtain a very important equation

$$\exp(t T^2) = \int_{SU(2)} d\text{vol}(\omega) \Phi(t, \omega) \exp[T(\omega)]$$

(4.10)

This is true for any representation of $SU(2)$.

We derive two corollaries. First, we get

$$\exp(t T^2) = \int_{SU(2)} d\text{vol}(\omega) \Phi(t, \omega) \exp[T(p)] \exp[T(\omega)] \exp[-T(p)]$$

(4.11)

which means that for any $p$

$$\Phi(t, \omega) = \Phi(t, F(-p, F(\omega, p)).$$

(4.12)

or

$$\Phi(t, F(p, q)) = \Phi(t, F(q, p)),$$

(4.13)

Also, we see that

$$\exp[(t + s) T^2] = \int\int_{S^3 \times S^3} d\text{vol}(q) d\text{vol}(p) \Phi(t, q) \Phi(s, p) \exp[T(q)] \exp[T(p)]$$

(4.14)

By changing the variables $z = F(q, p)$ and $p \mapsto -p$ we obtain

$$\exp[(t + s) T^2] = \int_{SU(2)} d\text{vol}(z) \int_{SU(2)} d\text{vol}(p) \Phi(t, F(p, z)) \Phi(s, p) \exp[T(z)]$$

(4.15)

which means that

$$\int_{SU(2)} d\text{vol}(p) \Phi(t, F(p, z)) \Phi(s, p) = \Phi(t + s, z)$$

(4.16)

In particular, for $z = 0$ and $s = t$ this becomes

$$\int_{SU(2)} d\text{vol}(p) \Phi(t, p) \Phi(t, p) = \Phi(t, 0)$$

(4.17)