The Ricci curvature tensor of a bi-invariant metric for semi-simple group is proportional to the metric

\[ \text{Ric} = \frac{1}{4}g , \quad (3.211) \]

and, therefore, the scalar curvature is

\[ R = \frac{n}{4} , \quad (3.212) \]

3.12 **SL(2, C) and Its Subgroups: SL(2, \mathbb{R}), SU(2), SU(1, 1) and SO(1, 2)**

The group SL(2, \mathbb{C}) is the group of non-degenerate complex matrices with unit determinant. Of course, it has the subgroup SL(2, \mathbb{R}) of real non-degenerate matrices with unit determinant, as well as the subgroup SU(2) of complex unitary matrices with unit determinant. The Lie algebra of SL(2, \mathbb{C}) consists of traceless complex matrices, while the Lie algebra of SL(2, \mathbb{R}) consists of traceless real matrices and the Lie algebra of SU(2) consists of traceless complex anti-Hermitian matrices. Obviously, the dimensions of these groups are

\[ \dim \text{SL}(2, \mathbb{C}) = 6 , \quad \dim \text{SL}(2, \mathbb{R}) = \dim \text{SU}(2) = 3 . \quad (3.213) \]

In the Lie algebra of SL(2, \mathbb{C}) we can choose the following basis

\[ \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} , \quad (3.214) \]

\[ \Sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \Sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \Sigma_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \quad (3.215) \]

so that the matrices \( \sigma_j \) are anti-Hermitian and the matrices \( \Sigma_k \) are Hermitian, that is,

\[ \sigma_j^\dagger = -\sigma_j , \quad \Sigma_k^\dagger = \Sigma_k . \quad (3.216) \]
The Lie algebra formed by these matrices has the form

\[
\begin{align*}
[\sigma_i, \sigma_j] &= \varepsilon_{ijk} \sigma_k, \\
[\sigma_i, \Sigma_j] &= \varepsilon_{ijk} \Sigma_k, \\
[\Sigma_i, \Sigma_j] &= -\varepsilon_{ijk} \sigma_k,
\end{align*}
\] (3.217)

where \( \varepsilon_{ijk} \) is the Levi-Civita symbol. More explicitly,

\[
\begin{align*}
[\sigma_1, \sigma_2] &= \sigma_3, & [\sigma_2, \sigma_3] &= \sigma_1, & [\sigma_3, \sigma_1] &= \sigma_2, \\
[\sigma_1, \Sigma_2] &= \Sigma_3, & [\sigma_2, \Sigma_3] &= \Sigma_1, & [\sigma_3, \Sigma_1] &= \Sigma_2, \\
[\Sigma_1, \sigma_2] &= \Sigma_3, & [\Sigma_2, \sigma_3] &= \Sigma_1, & [\Sigma_3, \sigma_1] &= \Sigma_2, \\
[\Sigma_1, \Sigma_2] &= -\sigma_3, & [\Sigma_2, \Sigma_3] &= -\sigma_1, & [\Sigma_3, \Sigma_1] &= -\sigma_2.
\end{align*}
\] (3.220)

Now, we immediately see that this algebra has a subalgebra formed by the matrices \( \sigma_1, \sigma_2 \) and \( \sigma_3 \). This is the Lie algebra of SU(2).

Another subalgebra is formed by the real traceless matrices

\[
\begin{align*}
B_1 &= \Sigma_1, & B_2 &= \sigma_2, & B_3 &= \Sigma_3.
\end{align*}
\] (3.224)

It has the form

\[
\begin{align*}
[B_1, B_2] &= B_3, & [B_2, B_3] &= B_1, & [B_3, B_1] &= -B_2.
\end{align*}
\] (3.225)

This is the Lie algebra of SL(2, \( \mathbb{R} \)). We see that it is very similar to the Lie algebra of SU(2). The only difference is the sign of the last commutator.

The third subalgebra is formed by the matrices

\[
\begin{align*}
A_1 &= \sigma_3, & A_2 &= \Sigma_1, & A_3 &= \Sigma_2,
\end{align*}
\] (3.226)

it has the form

\[
\begin{align*}
\end{align*}
\] (3.227)

This is the algebra of complex traceless matrices that satisfy the identity

\[
A_i^\dagger = -\eta A_i \eta^{-1},
\] (3.228)

where \( \eta \) is the matrix

\[
\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (3.229)
The corresponding group SU(1, 1) is the group of complex matrices with unit determinant that preserve the bilinear form \( \eta \) in the sense
\[
U^\dagger = \eta U^{-1} \eta^{-1}.
\] (3.230)

This algebra is isomorphic to the Lie algebra of the group SO(1, 2).

In the following we will consider these algebras simultaneously. We introduce parameters \((\lambda_i) = (\lambda_1, \lambda_2, \lambda_3)\) such that the algebra is given by
\[
[\mathcal{G}_1, \mathcal{G}_2] = \lambda_3 \mathcal{G}_3, \quad [\mathcal{G}_2, \mathcal{G}_3] = \lambda_1 \mathcal{G}_1, \quad [\mathcal{G}_3, \mathcal{G}_1] = \lambda_2 \mathcal{G}_2.
\] (3.231)

Then for \((\lambda_i) = (1, 1, 1)\) this is the algebra of SU(2), while for \((\lambda_i) = (1, -1, 1)\) this is the algebra of SL(2, \(\mathbb{R}\)), and for \((\lambda_i) = (-1, 1, 1)\) this is the algebra of SU(1, 1).

Notice also that if one of the parameters is equal to zero, this describes the solvable Lie algebra of the group \(E(2)\) of motions of the \(\mathbb{R}^2\), for example, if \((\lambda_i) = (1, 0, 1)\) this is the Lie algebra of the group of motions of the \(xz\)-plane in \(\mathbb{R}^3\). In this case, \(G_2\) generates rotations around the \(y\)-axis and \(G_1\) and \(G_3\) generate translations along the \(x\)-axis and the \(z\)-axis. If two parameters are equal to zero, then this is the nilpotent Heisenberg algebra \(H_3(\mathbb{R})\). Finally, if all three parameters are equal to zero then this is just an Abelian algebra of \(\mathbb{R}^3\). We will only consider the case when all parameters are not equal to zero.

The structure constants of the group define the generators of the adjoint representation
\[
C_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\lambda_2 \\
0 & \lambda_3 & 0
\end{pmatrix}, \quad
C_2 = \begin{pmatrix}
0 & 0 & \lambda_1 \\
0 & 0 & 0 \\
-\lambda_3 & 0 & 0
\end{pmatrix}, \quad
C_3 = \begin{pmatrix}
0 & -\lambda_1 & 0 \\
\lambda_2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\] (3.232)

We see that the matrices \(C_a\) are traceless, which follows, of course, from the fact that the group is uni-modular. Moreover, for \((\lambda_i)\) all matrices \(C_a\) are anti-symmetric.

Let \(x_a\) be the canonical coordinates on the group. Then
\[
C(x) = C_a x^a = \begin{pmatrix}
0 & -\lambda_1 x_3 & \lambda_1 x_2 \\
\lambda_2 x_3 & 0 & -\lambda_2 x_1 \\
-\lambda_3 x_2 & \lambda_3 x_1 & 0
\end{pmatrix}.
\] (3.233)
This enables us to compute the Cartan metric $\gamma = (\gamma_{ab})$. We obtain

$$\gamma = -\frac{1}{2}(\text{tr } C_a C_b) = \begin{pmatrix}
\lambda_2 \lambda_3 & 0 & 0 \\
0 & \lambda_1 \lambda_3 & 0 \\
0 & 0 & \lambda_1 \lambda_2
\end{pmatrix}.$$  \hfill (3.234)

We see that if all parameters $\lambda_i$ are not equal to zero, then this metric is non-degenerate, and, therefore, the groups $\text{SL}(2, \mathbb{R})$, $\text{SU}(2)$ and $\text{SU}(1,1)$ are semisimple.

We parametrize the parameters $\lambda_i$ by complex parameters $\mu_i$ such that

$$\lambda_i = \mu_i^2 ,$$  \hfill (3.235)

and introduce the parameters

$$\omega_1 = \mu_2 \mu_3 , \quad \omega_2 = \mu_1 \mu_3 , \quad \omega_3 = \mu_1 \mu_2 .$$  \hfill (3.236)

Then

$$\lambda_1 = \frac{\omega_2 \omega_3}{\omega_1} , \quad \lambda_2 = \frac{\omega_1 \omega_3}{\omega_2} , \quad \lambda_3 = \frac{\omega_1 \omega_2}{\omega_3} ,$$  \hfill (3.237)

and the Cartan metric takes the form

$$\gamma = \begin{pmatrix}
\omega_1^2 & 0 & 0 \\
0 & \omega_2^2 & 0 \\
0 & 0 & \omega_3^2
\end{pmatrix}.$$  \hfill (3.238)

This suggest rescaling the coordinates by

$$y_i = \omega_i x_i , \quad \text{(no summation!)}$$  \hfill (3.239)

Notice that these coordinates are complex, in general. Then

$$C(x) = \begin{pmatrix}
0 & -\frac{\omega_1}{\omega_2} y_3 & \frac{\omega_1}{\omega_2} y_2 \\
\frac{\omega_1}{\omega_2} y_3 & 0 & -\frac{\omega_1}{\omega_2} y_1 \\
-\frac{\omega_1}{\omega_3} y_2 & \frac{\omega_1}{\omega_3} y_1 & 0
\end{pmatrix}.$$  \hfill (3.240)
Now, let $\Omega$ be a square root of the Cartan metric defined by

$$\Omega = \begin{pmatrix}
\omega_1 & 0 & 0 \\
0 & \omega_2 & 0 \\
0 & 0 & \omega_3
\end{pmatrix}, \quad (3.241)$$

so that

$$\Omega^2 = \gamma. \quad (3.242)$$

Then it is easy to check that

$$C(x) = \Omega S(x)\Omega^{-1}, \quad (3.243)$$

where $S(x)$ is anti-symmetric matrix

$$S(x) = \begin{pmatrix}
0 & -y_3 & y_2 \\
y_3 & 0 & -y_1 \\
y_2 & y_1 & 0
\end{pmatrix}. \quad (3.244)$$

To rewrite it in a more compact and more familiar form, we introduce the generators $T_a$ of the group SU(2) in adjoint representation defined by

$$(T_a)_{bc} = \varepsilon_{bac}, \quad (3.245)$$

or, more explicitly,

$$T_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad T_3 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (3.246)$$

Then the matrix $S(x)$ takes the form

$$S(x) = T_a y_a, \quad (3.247)$$

or

$$S_{ab}(x) = -\varepsilon_{abc} y_c. \quad (3.248)$$
Therefore, the matrix $C$ is now
\[ C_{ab}(x) = -\Omega_{ad}\epsilon_{def}\Omega^{-1}_{fb} y^e. \] (3.249)

One can show that the products of the matrices $T_a$ form the Lie algebra
\[ [T_a, T_b] = \epsilon_{abc} T_c, \] (3.250)

and, moreover, their products are
\[ (T_a T_b)_{cd} = -\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}. \] (3.251)

This form of the matrix $S(x)$ and the matrix $C(x)$ greatly simplifies the calculations. Indeed, one can easily compute the square of the matrix $S$
\[ S^2 = -y^2 P, \] (3.252)

where
\[ y^2 = y_1^2 + y_2^2 + y_3^2 = \gamma_{ab} x_a x_b = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2, \] (3.253)

and $P$ is a projection on the plane orthogonal to vector $y^a$ defined by
\[ P_{ab}(x) = \delta_{ab} - \frac{y_a y_b}{y^2}, \] (3.254)

and satisfying the equations
\[ P^2 = P, \quad PS = SP = S, \quad \text{tr} P = 2. \] (3.255)

Therefore,
\[ S^{2n} = (-y^2)^n P, \quad S^{2n+1} = (-y^2)^n S. \] (3.256)

Thus, for any analytic function of $S$ one can compute
\[ f(S) = f(0)\mathbb{I} + \left\{ \frac{1}{2} [f(r) + f(-r)] - f(0) \right\} P + \frac{1}{2r} [f(r) - f(-r)] S, \] (3.257)

where $r = \sqrt{-y^2}$. By using this equation one can compute now any analytic function of the matrix $C(x)$; we get
\[ f(C) = f(0)\mathbb{I} + \left\{ \frac{1}{2} [f(r) + f(-r)] - f(0) \right\} \Pi + \frac{1}{2r} [f(r) - f(-r)] C, \] (3.258)
where \( \Pi \) is another projection defined by
\[
\Pi = \Omega P \Omega^{-1}.
\]
(3.259)

It satisfies the identities
\[
\Pi^2 = \Pi, \quad C \Pi = \Pi C = C, \quad \text{tr} \, \Pi = 2.
\]
(3.260)

Notice that the projection \( \Pi \) could be determined by the square of the matrix \( C \),
\[
C^2 = -y^2 \Pi.
\]
(3.261)

Notice that the matrix \( C(x) \) and the invariant \( y^2 \) are real; therefore, the projection \( \Pi \) is also real.

Let us introduce rescale coordinates according to
\[
\tilde{x}_a = \omega_a^2 x_a = \lambda_a x_a, \quad \text{(no summation!)},
\]
(3.262)
so that
\[
y^2 = \tilde{x}_a x_a.
\]
(3.263)

Then the projection \( \Pi \) has the form
\[
\Pi_{ab} = \delta_{ab} - \frac{1}{y^2} \tilde{x}_a x_b.
\]
(3.264)

Now, we can compute everything in canonical coordinates. The matrix \( X \) determining the left-invariant one-forms has the form
\[
X = \frac{\exp C - \mathbb{I}}{C} = \mathbb{I} + \left( \frac{\sinh r}{r} - 1 \right) \Pi + \frac{\cosh r - 1}{r^2} C,
\]
(3.265)
and its inverse, \( L = X^{-1} \), determining the left-invariant vector fields is
\[
L = \frac{C}{\exp C - \mathbb{I}} = \mathbb{I} + \left[ \frac{r}{2} \coth \left( \frac{r}{2} \right) - 1 \right] \Pi - \frac{1}{2} C.
\]
(3.266)

The matrix \( Y \) determining the one-forms and the matrix \( R = Y^{-1} \) determining the right-invariant vector fields are obtained by just changing the sign of \( x \), which is equivalent to changing the sign of \( C \), that is,
\[
Y = \frac{\mathbb{I} - \exp(-C)}{C} = \mathbb{I} + \left( \frac{\sinh r}{r} - 1 \right) \Pi - \frac{\cosh r - 1}{r^2} C,
\]
(3.267)
\[
R = \frac{C}{\mathbb{I} - \exp(-C)} = \mathbb{I} + \left[ \frac{r}{2} \coth \left( \frac{r}{2} \right) - 1 \right] \Pi + \frac{1}{2} C.
\]
(3.268)

Now, one can obtain the metric, the connection, and the curvature.