Chapter 3

Banach Spaces

3.1 Normed Spaces

• Function Spaces

• The vector space \( F(X, E) \) of all functions \( f : X \to E \) from a set \( X \) into a vector space \( E \).

• Let \( \Omega \subset \mathbb{R}^n \) be an open subset in \( \mathbb{R}^n \). The space of all functions from \( \Omega \) into \( \mathbb{C} \) is a vector space. The following are subspaces of this vector space:

1. \( P(\Omega) \) (polynomials)
2. \( C(\Omega) \) (continuous functions \( f : \Omega \to \mathbb{C} \))
3. \( C^k(\Omega) \) (functions with continuous partial derivatives of order \( k \))
4. \( C^\infty(\Omega) \) (smooth functions)
5. \( C^\infty(\Omega, V) \) (vector-valued functions \( \varphi : \Omega \to V \) taking values in a vector space \( V \))
6. \( L^p(\Omega, \mu, V), \ p \geq 1 \) (\( p \)-integrable vector-valued functions)

\[
\|\varphi\|_p = \left( \int_{\Omega} dx \mu(x) \| \varphi(x) \|_V^p \right)^{1/p} < \infty
\]

• Show that

\[ C(\Omega) = L^\infty(\Omega) \]
• **Sequence Spaces**

Let \( \mathbb{N} \) be the set of positive integers. The space \( F(\mathbb{N}, \mathbb{F}) \) of all functions from \( \mathbb{N} \) into \( \mathbb{F} \) is the vector space of sequences of scalars. The following are subspaces of this vector space:

1. \( l_0 \) (sequences of complex numbers with zero tails, that is, sequences containing only finitely many non-zero elements),
2. convergent sequences,
3. bounded sequences,
4. \( l^p \), \( p \geq 1 \) (sequences with finite \( p \)-norm)

\[
\| x \|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty
\]

**Proof:** Use Minkowski inequality.

**Theorem 3.1.1 Minkowski’s Inequality** Let \( p \geq 1 \). Then for any two sequences of complex numbers

\[
\| x + y \|_p \leq \| x \|_p + \| y \|_p
\]

Without proof.

• **Cartesian Product of Vector Spaces.** Let \( \{E_j\}_{j=1}^{n} \) be a collection of vector spaces over a field \( \mathbb{F} \). The **Cartesian product** (or product) of vector spaces \( E_j \) is the space

\[
E = E_1 \times \cdots \times E_n = \{ (x_1, \ldots, x_n) \mid x_j \in E_j, 1 \leq j \leq n \}.
\]

• **Normed Space.** A vector space with a norm is called a **normed space**.

• One can define different norms on the same vector space.

• A normed space is a pair \( (E, \| \cdot \|) \), where \( E \) is a vector space and \( \| \cdot \| \) is a norm on \( E \).

• Some vector spaces have standard norms.
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- A vector subspace of a normed space is a normed space with the same norm.

- **Norm of Uniform Convergence.** Let $\Omega \subset \mathbb{R}^n$ be a closed bounded subset of $\mathbb{R}^n$ and $C(\Omega)$ be the space of continuous functions on $\Omega$. Norm in $C(\Omega)$

$$\| f \|_\infty = \max_{x \in \Omega} |f(x)|$$

- For the $l^p$-norm as $p \to \infty$

$$\| x \|_p \to \| x \|_\infty = \sup_{n \in \mathbb{Z}} |x_n|$$

- The norm can be used to define convergence.

- **Convergence in a Normed Space.** Let $(E, \| \cdot \|)$ be a normed space and $(x_n)$ be a sequence of vectors in $E$. The sequence $(x_n)$ converges to $x \in E$ if for every $\varepsilon > 0$ there exists a positive integer $M \in \mathbb{N}$ such that for every $n \geq M$ we have

$$\| x_n - x \| < \varepsilon.$$  

Then we write $x = \lim_{n \to \infty} x_n$ or $x_n \to x$.

- $x_n \to x$ simply means that $\| x_n - x \| \to 0$ in $\mathbb{R}$.

- **Properties of convergence in normed space.**

  - A convergent sequence has a unique limit.
  
  - If $x_n \to x$ and $\lambda_n \to \lambda$, then $\lambda_n x_n \to \lambda x$.
  
  - If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.

- Not every convergence in a vector space can be defined by a norm.

- **Uniform Convergence.** Let $C(\Omega)$ be the space of all continuous functions on a closed bounded set $\Omega \subset \mathbb{R}^n$ and let $(f_n) \in C(\Omega)$ be a sequence of functions in $C(\Omega)$. The sequence $(f_n)$ converges uniformly to $f$ if for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon) \in \mathbb{N}$ such that for all $x \in \Omega$ and for all $n \geq M$ we have

$$|f(x) - f_n(x)| < \varepsilon.$$
The norm of uniform convergence defines the uniform convergence, i.e. the sequence \( (f_n) \) converges uniformly to \( f \) if and only if
\[
\| f_n - f \|_\infty = \max_{x \in \Omega} |f_n(x) - f(x)| \to 0.
\]

**Pointwise Convergence.** Let \( C([0, 1]) \) be the space of continuous functions on the interval \([0, 1]\) and let \( (f_n) \) be a sequence of functions in \( C([0, 1]) \). The sequence \( (f_n) \) **converges pointwise** to \( f \) if for all \( x \in [0, 1] \) and for every \( \varepsilon > 0 \) there exists a positive integer \( M = M(\varepsilon, x) \in \mathbb{N} \) such that for all \( n \geq M \) we have
\[
|f(x) - f_n(x)| < \varepsilon.
\]
The pointwise convergence simply means that for every \( x \in [0, 1] \) the sequence \( (f_n(x)) \) converges to \( f(x) \), i.e.
\[
f_n(x) \to f(x) \quad \text{ or } \quad |f_n(x) - f(x)| \to 0.
\]

**There is no norm on \( C([0, 1]) \) which defines the pointwise convergence.**

**Proof:** (by contradiction). Construct a sequence \( (f_n) \) of functions such that
1. \( \| f_n \| = 1 \) for all \( n \in \mathbb{N} \) and
2. \( f_n(x) \to 0 \) as \( n \to \infty \) \( \forall x \in [0, 1] \).

**Equivalence of Norms.** Two norms on the same vector space \( E \) are **equivalent** if they define the same convergence.

That is, the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent if for any sequence \( (x_n) \) in \( E \) and \( x \in E \),
\[
\| x_n - x \|_1 \to 0 \quad \text{ if and only if } \quad \| x_n - x \|_2 \to 0.
\]

**Example.** \( \mathbb{R}^2 \).

**Theorem 3.1.2** Two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) in a vector space \( E \) are equivalent if and only if there exist positive real numbers \( \alpha, \beta \in \mathbb{R}_+ \) such that
\[
\alpha \| x \|_1 \leq \| x \|_2 \leq \beta \| x \|_1 \quad \text{ for all } \quad x \in E.
\]

**Proof:** No proof.
Every normed space \((E, \| \cdot \|)\) is a metric space \((E, d)\) with the metric
\[
d(x, y) = \| x - y \|.
\]

A metric space \((E, d)\) is a set \(E\) with a metric \(d\). A metric \(d\) on a set \(E\) is a function \(d : E \times E \rightarrow \mathbb{R}\) satisfying the following axioms: \(\forall x, y, z \in E\)
1. \(d(x, y) \geq 0\),
2. \(d(x, y) = 0\) if and only if \(x = y\),
3. \(d(x, z) \leq d(x, y) + d(y, z)\).

The convergence defined by the norm \(\| \cdot \|\) is the same as the convergence defined by the metric \(d(x, y) = \| x - y \|\).

The metric defines a topology in \(E\) (open and closed sets).

The basic topological notions can be defined without a metric.

A topological space \((E, T)\) is a set \(E\) with a topology \(T\). A topology \(T\) on a set \(E\) is a collection \(T\) of subsets of \(E\) (called open sets) that contains \(E\) and \(\emptyset\) and is closed under union and finite intersection.

Open Balls, Closed Balls, Spheres. Let \(E\) be a normed space, \(x \in E\) and \(r \in \mathbb{R}_+\) a positive real number. We define the following sets:

Open ball
\[
B(x, r) = \{ y \in E \mid \| x - y \| < r \}
\]

Closed ball
\[
\bar{B}(x, r) = \{ y \in E \mid \| x - y \| \leq r \}
\]

Sphere
\[
S(x, r) = \{ y \in E \mid \| x - y \| = r \}
\]

Here \(x\) is the center and \(r\) is the radius.

Examples. \(\mathbb{R}^2, C([0, 1]), \| \cdot \|_{\infty}\).

Open and Closed Sets. A subset \(S \subseteq E\) of a normed space \(E\) is open if for every \(x \in S\) there exist \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subseteq S\).

A subset \(S \subseteq E\) of a normed space \(E\) is closed if its complement \(E \setminus S\) is open.
• **Example.** Let $\Omega$ be a closed bounded set in $\mathbb{R}^n$ and $C(\Omega)$ be the space of continuous functions on $\Omega$ with the norm of uniform convergence $\| \cdot \|_\infty$. Let $x_0 \in \Omega$. The set

$$\{ g \in C(\Omega) \mid g(x) > 0, \forall x \in \Omega \}$$

is open $C(\Omega)$, and the sets

$$\{ g \in C(\Omega) \mid g(x) \geq 0, \forall x \in \Omega \}$$

and

$$\{ g \in C(\Omega) \mid g(x_0) = 0 \}$$

are closed in $C(\Omega)$.

• **Theorem 3.1.3**

1. The union of any number of open sets is open.
2. The intersection of a finite number of open sets is open.
3. The union of a finite number of closed sets is closed.
4. The intersection of any number of closed sets is closed.
5. The empty set and the whole space are both open and closed.

*Proof:* Exercise.

• **Theorem 3.1.4** A subset $S$ of a normed space $E$ is closed if and only if every sequence of elements of $S$ convergent in $E$ has its limit in $S$.

*Proof:* No proof.

• **Closure.** Let $S$ be a subset of a normed space $E$. The closure of $S$ (denoted by $\bar{S}$ or $\text{cl} S$) is the intersection of all closed sets containing $S$.

- The closure of a set is a closed set.
- The closure of a set is the smallest closed set which contains $S$.

• **Theorem 3.1.5** Let $S$ be a subset of a normed space $E$. The closure of $S$ is the set of limits of all convergent sequences of elements of $S$.

*Proof:* Exercise.

• **Weierstrass Approximation Theorem.** The closure of the set of all polynomials on $[a, b]$ is the whole space $C([a, b])$. 

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- **Dense Subsets.** A subset $S$ of a normed space $E$ is **dense** in $E$ if $\text{cl } S = E$.

- **Examples.**
  1. The set of all polynomials on $[a, b]$ is dense in $C([a, b])$.
  2. The set of all sequences of complex numbers which have only a finite number of nonzero terms is dense in $l^p$ for any $p \geq 1$.

- **Theorem 3.1.6** Let $S$ be a subset of a normed space $E$. The following conditions are equivalent:
  1. $S$ is dense in $E$.
  2. Every element of $E$ is a limit of a convergent sequence in $S$.
  3. Every nonempty open subset of $E$ contains an element of $S$.

  **Proof:** Exercise.

- **Compact Sets.** A subset $S$ of a normed space $E$ is **compact** in $E$ if every sequence in $S$ contains a convergent subsequence whose limit belongs to $S$.

- **Examples.** $\mathbb{R}^n$, $\mathbb{C}^n$

- **Theorem 3.1.7** Compact sets are closed and bounded.

  **Proof:** No proof.

- **Noncompact Closed and Bounded Set.** Let $C([0, 1])$ be the space of continuous functions on $[0, 1]$. The closed unit ball $\bar{B}(0, 1)$ is a closed and bounded set. Let $x_n(t) = t^n \in \bar{B}(0, 1)$ be the sequence of functions of unit norm. Then $(x_n)$ does not have a convergent subsequence. So, the closed unit ball $\bar{B}(0, 1)$ is not compact.