The Poincaré and Lorentz groups are typical examples of continuous groups. That is why we give below a very brief description of the theory of continuous groups.

1 Abstract Group

An abstract group $G$ is a set of elements $g$ for which

i. an associative composition law, called the multiplication, is given so that for each ordered pair of elements $(g_1, g_2)$ another element $g_1g_2$, called their product, is associated

\[(g_1, g_2) \rightarrow g_1g_2, \quad (1)\]

and

\[g_1(g_2g_3) = (g_1g_2)g_3, \quad (2)\]

ii. there exists an element $e$, called the unit element (or identity element), such that for any $g$

\[ge = eg = g, \quad (3)\]

iii. an operation, called the inversion is given, i.e., with each element $g$ its inverse $g^{-1}$ is associated,

\[g \rightarrow g^{-1}, \quad (4)\]

such that

\[g^{-1}g = gg^{-1} = e. \quad (5)\]
**Basic Notions**

1. If for any elements \( g_1 \) and \( g_2 \) of the group \( G \)

\[
g_1 g_2 = g_2 g_1, \tag{6}
\]

then the group is called Abelian or commutative. Otherwise it is non-Abelian.

2. The number of elements of the group \( G \) is called the order of \( G \) and denoted by \( |G| \).

3. The group of finite order is called the finite group. Otherwise it is infinite.

4. Infinite group can be discrete and continuous. If the elements \( g \) of a group \( G \) can be enumerated with the help of a discrete index, \( g_i \), \( (i = 1, 2, \ldots) \), the group \( G \) is called discrete. Otherwise it is continuous. Finite groups are obviously always discrete.

5. A subset \( H \) of elements of the group \( G \) is called a subgroup if it itself is a group with the same composition law. This means that the identity element \( e \) of \( G \), the products of any elements of \( H \) as well as their inverses belong to \( H \). One says that \( H \) is closed under the multiplication and the inversion laws.

6. A subgroup \( H \) is called proper subgroup if it consists of more than just the unit element but does not coincide with the whole group itself.

**2 Continuous Groups**

1. The elements of a general continuous group can be parametrized by a set of continuous real parameters

\[
g = g(\lambda), \quad (\lambda = (\lambda^a), \; a = 1, 2, \ldots). \tag{7}
\]

If the set of continuous parameters is finite, i.e., \( a = 1, 2, \ldots, p \), the group is called finite dimensional, the number of the parameters being the dimension of the group \( \dim G = p \). Otherwise, the group is infinitely-dimensional.

2. If the parameters \( f^a(\lambda, \mu) \) of the product of two elements

\[
g(\lambda)g(\mu) = g(f(\lambda, \mu)) \tag{8}
\]

are analytic functions of the parameters of the factors, i.e. the functions \( f^a(\lambda, \mu) \) possess derivatives of all orders with respect to all arguments, and, similarly, the parameters \( \lambda^a(\lambda) \) of the inverse element \( g(\lambda) = g^{-1}(\lambda) \) are analytic functions of the parameters \( \lambda \) then the continuous group is called Lie group.
3. The continuous parameters $\lambda^a$ are called coordinates on the Lie group. For a finite-dimensional Lie group $G$ the coordinates $\lambda^a$ vary in some region of the Euclidean space $\mathbb{R}^p$, $p$ being the dimension of the group. If the domain of variation of the coordinates is finite, or compact, i.e. $|\lambda^a| < \infty$, the group is said to be compact (for more precise definition see the bibliography).

4. A curve (or path) $g = g(\tau)$, $0 \leq \tau \leq 1$, on a Lie group $G$ is a mapping
\[
\tau \in [0,1] \rightarrow g(\tau) \in G,
\]
where $\tau$ is a real parameter. The one-parameter subset $\{g(\tau)\}$ of the group $G$ itself is called the curve too. A curve $g(\tau)$ is continuous if the coordinates $\lambda^a(\tau)$ of the element $g(\tau)$ are continuous functions of the parameter $\tau$. We will call the continuous curves just curves.

5. One says that two elements $g_0$ and $g_1$ are connected by a curve $g(\tau)$ if
\[
g(0) = g_0, \quad g(1) = g_1.
\]

6. If $g(0) = g(1) = g$ the curve is called closed curve, or the loop, going through the element $g$. The loop consisting only from one element $g$ is called the null loop at $g$.

7. A subset $H$ of the group $G$ is called arcwise connected (or connected) if every two elements of $H$ can be connected by a continuous curve.

8. A component of an element $g$ of a Lie group $G$ is the union of all connected subsets of $G$ containing the element $g$.

9. The component $G_1$ of the identity element of the group $G$ is called the proper connected component of the group $G$.

10. A general Lie group $G$ consists of many connected components $G_i$, which are disconnected from each other. Each connected component $G_i$ is obtained from the proper subgroup $G_1$ by applying some discrete transformation $\gamma_i$ of a discrete subgroup $\Gamma$. Thus a Lie group is a direct product of the proper subgroup and some discrete subgroup
\[
G = G_1 \times \Gamma,
\]
where $G_1$ is the proper group and $\Gamma$ the discrete subgroup.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{general_continuous_group.png}
\caption{General continuous group}
\end{figure}

**Lorentz Group as Example** The general Lorentz group $\mathcal{L} \simeq O(1,d-1)$ has four connected components. The role of the component of the identity $G_1$ plays the proper orthochronous Lorentz group $\mathcal{L}_1 \simeq SO(1,d-1)$. The discrete subgroup $\Gamma$ is the finite group of reflections of the time and one space coordinate

$$\Gamma = \{1, T, P, TP\} \quad TP = PT.$$  \hfill (12)

$$T^2 = P^2 = (TP)^2 = 1.$$  \hfill (13)

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{general_lorentz_group.png}
\caption{General Lorentz group}
\end{figure}
11. Two curves \( g(\tau) \) and \( g'(\tau) \) connecting the elements \( g_0 \) and \( g_1 \) are said to be \textit{homotopic} if there exists a continuous deformation of one curve into another, which leaves the end points \( g_0 \) and \( g_1 \) unaltered, i.e., there exist a \textit{continuous} function \( h(\tau, s) \) of two parameters \( \tau \) and \( s \) such that
\[
\begin{align*}
    h(0, s) &= g_0, & h(1, s) &= g_1, \\
    h(\tau, 0) &= g(\tau), & h(\tau, 1) &= g'(\tau).
\end{align*}
\]

12. A Lie group is said to be \textit{simply connected} if every loop is homotopic to the null loop, i.e., every loop is contractable to one point.

13. All loops at an element \( g \) are classified into the so called \textit{homotopy classes}.

14. A Lie group is said to be \textit{n-connected} if it has \( n \) homotopy classes at each element.

3 \hspace{1em} \textbf{Invariant Subgroups}

1. A parametrized curve \( g = g(\tau), \ (a \leq \tau \leq b) \), on the Lie group \( G \) is called the \textit{one-parameter subgroup} of the Lie group \( G \) if
\[
\begin{align*}
    g(0) &= e, & g(\tau_1 + \tau_2) &= g(\tau_1)g(\tau_2), \\
    g(-\tau) &= (g(\tau))^{-1}.
\end{align*}
\]

2. Let \( H \) be a subgroup of \( G \). The \textit{orbit} \( gH \) of \( H \) through an element \( g \) in \( G \) is the set of all elements \( gh \) with \( h \) in \( H \)
\[
gH = \{ gh : \ h \in H \}.
\]

3. If \( H \) is a proper subgroup, the orbits of \( H \) in \( G \) are called the \textit{left cosets} of \( H \) in \( G \). The set of all left cosets is denoted by \( G/H \). Analogously it is defined the set \( H \setminus G \) of all \textit{right cosets}
\[
Hg = \{ hg : \ h \in H \}.
\]

4. A subgroup \( H \) of a group \( G \) is said to be \textit{normal} or \textit{invariant subgroup} of \( G \) if for any \( g \) in \( G \) and any \( h \) in \( H \) the element \( ghg^{-1} \), called the \textit{conjugate element}, is again in \( H \)
\[
gH = Hg, \quad \text{or} \quad gHg^{-1} \subset H.
\]

In other words the normal subgroup is closed under the conjugation and the left and right cosets of \( H \) in \( G \) coincide with each other.

5. \textbf{Theorem.} The set of left cosets \( G/H \) of a subgroup \( H \) in \( G \) is itself a group if \( H \) is normal subgroup.
6. **Theorem.** The component $G_1$ of the identity $e$ of a Lie group $G$, i.e. the connected component containing the identity element, is a closed invariant subgroup of $G$.

7. A set $C(G)$ of all elements of a group $G$ which commute with all elements of $G$ is called the *center* of the group

$$C(G) = \{ g \in G : g'g = gg' \ \forall g' \in G \}. \quad (22)$$

8. **Theorem.** The center $C(G)$ is an Abelian normal subgroup of $G$. Thus $G/C(G)$ is itself a group.

9. **Theorem.** If $G$ is a connected Lie group and $H$ is an invariant discrete subgroup, then $H$ is *central*, i.e. it is a subgroup of the center.

10. A Lie group is said to be *simple* if it has no proper, connected invariant Lie subgroup. It might, however, contain a discrete invariant subgroup.

11. A Lie group is said to be *semisimple* if it contains no proper invariant connected Abelian Lie subgroup.

4 **Homomorphisms**

1. A mapping

$$\varphi : G \rightarrow G' \quad (23)$$

which preserves the group multiplication, i.e.

$$\varphi(gg') = \varphi(g)\varphi(g') \quad (24)$$

is called *homomorphism*.

2. Note that several elements of $G$ may have the same image in $G'$. The set of all elements of $G$ which are mapped to the identity element of $G'$ is called the *kernel* of the homomorphism $\varphi$:

$$\text{Ker } \varphi = \{ g \in G : \varphi(g) = e' \in G' \}. \quad (25)$$

3. **Theorem.** The kernel of a homomorphism $\varphi$ is a *normal subgroup* of $G$, i.e. for any $g$ in $G$ and any $h$ in Ker $\varphi$ the element $ghg^{-1}$ is again in Ker $\varphi$; in other words

$$g(\text{Ker } \varphi)g^{-1} = \text{Ker } \varphi. \quad (26)$$

4. **Theorem.** Thus $G/\text{Ker } \varphi$ is a group.

5. Two groups $G$ and $G'$ are said to be *isomorphic*

$$G \simeq G' \quad (27)$$

if their elements can be put into *one-to-one* correspondence which is preserved under multiplication.
6. An isomorphism \( \varphi : G \to G' \) is simply a one-to-one homomorphism, i.e. \( \text{Ker} \varphi = \{e\} \), so that the inverse map \( \varphi^{-1} : G' \to G \) is also a homomorphism.

7. An isomorphism \( \varphi : G \to G \) of a group with itself is called automorphism of the group.

8. Theorem. The set \( \text{Aut}(G) \) of all automorphisms of a group \( G \) is itself a group.

9. A homomorphism

\[ \varphi : G' \to G \]  

is called surjective mapping onto \( G \) if for any \( g \) in \( G \) there exists at least one element \( g' \) in \( G' \) such that \( \varphi(g') = g \).

10. Theorem. If homomorphism \( \varphi : \tilde{G} \to G \) is a surjective mapping onto \( G \), then the group \( G \) is isomorphic to \( \tilde{G}/\text{Ker} \varphi \)

\[ G \cong \tilde{G}/\text{Ker} \varphi. \]  

The group \( \tilde{G} \) is called the universal covering group of \( G \).

5 Direct and Semi-direct Products

1. The set of all ordered pairs

\[ (g, g') \]  

where \( g \) is an element of a group \( G \) and \( g' \) an element of another one \( G' \), with the product rule

\[ (g_1, g'_1)(g_2, g'_2) = (g_1g_2, g'_1g'_2) \]  

is called the direct product (or Cartesian product, or outer product, or simply product) \( G \times G' \) of the groups \( G \) and \( G' \). The unit element of \( G \times G' \) is \((e, e')\) and the inverse of \((g, g')\) is \((g', g'^{-1}) = (g^{-1}, g'^{-1})\).

2. Let \( G = \{g\} \) be a subgroup of the group of automorphisms \( \text{Aut}(H) \) of another group \( H = \{h\} \), i.e., the group \( G \) acts isomorphically on the group \( H \)

\[ g : h \in H \to g(h) \in H. \]  

The set of all ordered pairs \((g, h)\) with the product rule

\[ (g_1, h_1)(g_2, h_2) = (g_1g_2, h_1g_1(h_2)) \]  

defines the semi-direct product \( G \ltimes H \) of the groups \( G \) and \( H \). The unit element of \( G \ltimes H \) is \((e, 1)\) and the inverse of \((g, h)\) is

\[ (g, h)^{-1} = (g^{-1}, g^{-1}(h^{-1})). \]
3. If the group $T = \{ a \}$ is Abelian with the group multiplication denoted by $+$ and the group $L = \{ \Lambda \}$ acts linearly on $T$, then the semidirect product $L \ltimes T$ has the multiplication rule

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2).$$

(35)

4. \textbf{Theorem.} The semidirect product $G \ltimes H$ has the following properties:

(a) $H$ is a normal subgroup of $G \ltimes H$,

(b) $G \ltimes H/H$ is isomorphic to $G$.

6 \hspace{1em} \textbf{Group Representations}

1. If for each element $g$ of a group $G$ it is given an invertible operator $D(g)$ in a vector space $V$

$$D(g) : V \to V$$

(36)

and for all $g_1$ and $g_2$ in $G$

$$D(g_1 g_2) = D(g_1)D(g_2)$$

(37)

then the set of the operators $D(g)$ is said to form a \textit{linear representation} of the group $G$ on a vector space $V$.

2. Note that for any element $g$ in $G$

$$D(g^{-1}) = D^{-1}(g)$$

(38)

and

$$D(e) = I,$$

(39)

where $I$ is the identical operator in $V$.

3. In general, there are several elements in $G$ which are mapped on $I$.

4. An invertible operator in the vector space is called \textit{automorphism} of the vector space $V$.

5. \textbf{Theorem.} The set of all automorphisms of a vector space forms a group, denoted by $\text{Aut} (V)$.

6. Thus, a linear representation of a group $G$ on a vector space $V$ is a homomorphism

$$D : G \to \text{Aut} (V).$$

(40)

7. If $D$ is isomorphism, i.e. the correspondence (40) is one-to-one, the representation $D$ is said to be \textit{faithful} (or \textit{exact}).
8. If the dimension of the vector space \( p = \dim V \) is finite then the operators \( D(g) \) are described by \( p \times p \) matrices and we have a matrix representation of the group \( G \), \( I \) being the unit matrix.

9. If \( \text{Aut} (V) = G \) then the representation \( D : G \rightarrow \text{Aut} (V) \) is called the defining representation (or fundamental representation).

10. Every Lie group has the adjoint representation such that \( \dim V = \dim G \), which is determined by its structure constants (defined later).

11. For compact simple groups the adjoint representation is irreducible.

12. If the operators \( D(g) \) are unitary, i.e. \( D^\dagger (g)D(g) = I \), then the representation \( D \) is called unitary.

13. Not every Lie group has a faithful finite-dimensional (matrix) representation.

14. If two representations \( D_1 \) and \( D_2 \) of a group \( G \) on the vector space \( V \) are related by an invertible operator \( A \) on \( V \), i.e. an automorphism of the vector space \( V \),
\[
D_1(g) = A^{-1}D_2(g)A
\]
then the representations \( D_1 \) and \( D_2 \) are said to be equivalent.

15. With any matrix representation \( D \) of a group \( G \) it is associated a map
\[
\chi_D : G \rightarrow \mathbb{C},
\]
defined by the trace of the representation matrices
\[
\chi_D(g) = \text{tr} D(g),
\]
which is called the character of the representation \( D \). The value \( \chi_D(g) \) is called the character of the the element \( g \) in the representation \( D \).

16. The equivalent representations have the same characters.

17. A representation \( D \) of a Lie group \( G \) is called reducible if there is a proper invariant subspace \( V_1 \subset V \), i.e. \( D : V_1 \rightarrow V_1 \), so \( V_1 \) is closed under \( D \). Otherwise the representation is called irreducible.

18. Every reducible unitary representation \( D \) of a Lie group \( G \) is a direct sum of irreducible ones, i.e. \( D = D_1 \oplus \cdots \oplus D_n \).

19. For an Abelian Lie group all irreducible representations are one-dimensional.
7 Multiple-valued Representations. Universal Covering Group.

1. The matrix elements $D(g)$ of the representation $D$ of a Lie group $G$ are required to be continuous functions on the group $G$. Among continuous functions on the group $G$ there may be some functions which are multi-valued. Thus the representation can be, in principle, multiple-valued.

2. We say that a representation $D$ of $G$ is $m$-valued representation if with each element $g$ of the group $G$ there are associated $m$ diferent operators $D_1(g), \ldots, D_m(g)$.

3. Let us consider a continuous function $D(g)$ on the group $G$ and let us look at the values $D(g(\tau))$ along a continuous closed curve (loop) $g(\tau)$ on $G$, so that $g(0) = g(1) = g$. It could hapen, in principle, that $D(g(0)) \neq D(g(1))$. (44)

Let us fix an initial value $D_0 = D(g(0))$ and take all possible loops in $G$ starting at $g$.

If the maximal number of different values $D(g(1))$ is $m$, then the function $D(g)$ is $m$-valued. This number is a property of the group and reflects the connectedness of the group itself.

4. If the loop $g(\tau)$ on the group $G$ can be varied continuously so that it contracts to the initial point $g$, i.e., it is homotopic to the null loop, the continuous function $D(g)$ must return to its original value $D_0$. If this is the case for all loops on the group, i.e., if all loops are homotopic to the null loop, the group is called to be simply connected, and every continuous function on the group must be single-valued.

5. If there are $m$ different loops which cannot be deformed into each other, i.e., if there are $m$ homotopy classes, the group is said to be $m$-connected, and $m$-valued continuous functions can exist.

If the group is $m$-connected, we may expect that some of the representations will be $m$-valued. These multiple-valued representations cannot be simply ignored.

6. It can be shown that for any multiply-connected group $G$ there exists a simply connected group $\tilde{G}$, called the universal covering group of $G$, such that $\tilde{G}$ can be mapped homomorphically on $G$.

7. The group $\tilde{G}$ contains a discrete invariant subgroup $\Gamma$ such that $G$ is isomorphic to $\tilde{G}/\Gamma$

$$G \simeq \tilde{G}/\Gamma.$$ (45)
8. Every representation of the group \(G\) (whether single-valued or multiple-valued) is a single-valued representation of \(\tilde{G}\). Thus, one can study instead of the group \(G\) its universal covering group \(\tilde{G}\) which has only single-valued representations.

### 8 Matrix Lie Groups

1. The set \(M(n, \mathbb{R})\) of all real square \(n \times n\) matrices forms an Abelian Lie group under the law of matrix addition. It is not a group under the law of matrix multiplication since not all matrices have inverses. The dimension of the group \(M(n, \mathbb{R})\) is equal to the number of matrix elements, \(\dim M(n, \mathbb{R}) = n^2\).

2. The set of all invertible real \(n \times n\) matrices

\[
GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) : \det A \neq 0\} \tag{46}
\]

forms a general linear Lie group under the law of matrix multiplication. The set of all invertible real \(n \times n\) matrices with positive determinant forms a subgroup of \(GL(n, \mathbb{R})\) denoted by \(GL_+(n)\)

\[
GL_+(n) = \{A \in GL(n, \mathbb{R}) : \det A > 0\}. \tag{47}
\]

The dimension of both this groups is also equal to the number of the matrix elements: \(\dim GL_+(n) = \dim GL(n, \mathbb{R}) = n^2\).

3. The special linear group \(SL(n, \mathbb{R})\) is a subgroup of \(GL(n, \mathbb{R})\) which is formed by all invertible matrices of order \(n\) with unit determinant

\[
SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}. \tag{48}
\]

\(SL(n, \mathbb{R})\) is a Lie group of dimension \(n^2 - 1\).

4. The real orthogonal group \(O(n)\) is the subgroup of \(GL(n, \mathbb{R})\) of all real orthogonal matrices of order \(n\)

\[
O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = 1\}. \tag{49}
\]

\(O(n)\) is a Lie group of dimension \(n(n - 1)/2\).

5. The special orthogonal group \(SO(n)\) is a subgroup of \(O(n)\) of orthogonal matrices with unit determinant

\[
SO(n) = \{A \in O(n) : \det A = 1\}, \tag{50}
\]

\(\dim SO(n) = \dim O(n) = n(n - 1)/2\).
6. The pseudo-orthogonal group $O(p, q)$, $0 < p \leq q$, is a subgroup of $GL(n, \mathbb{R})$ of all pseudo-orthogonal matrices of type $(p, q)$

$$O(p, q) = \{ A \in GL(n, \mathbb{R}) : A^T \eta A = \eta \}, \quad (51)$$

where $\eta$ is the diagonal matrix of the form

$$\eta = \text{diag} \left( -1, \cdots, -1, +1, \cdots, +1 \right). \quad (52)$$

$$\dim O(p, q) = \dim O(n) = n(n - 1)/2.$$

7. The special pseudo-orthogonal group $SO(p, q)$, $0 < p \leq q$, is a subgroup of $O(p, q)$ of all pseudo-orthogonal matrices of type $(p, q)$ with unit determinant

$$SO(p, q) = \{ A \in O(p, q) : \det A = 1 \}, \quad (53)$$

$$\dim SO(p, q) = \dim SO(n) = n(n - 1)/2.$$

8. Similarly, one defines the groups of complex matrices $M(n, \mathbb{C})$, $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$. Obviously,

$$\dim M(n, \mathbb{C}) = 2 \cdot \dim M(n, \mathbb{R}) = \dim GL(n, \mathbb{C}) = 2 \cdot \dim GL(n, \mathbb{R}) = 2n^2$$

and

$$\dim SL(n, \mathbb{C}) = 2 \cdot \dim SL(n, \mathbb{R}) = 2(n^2 - 1). \quad (55)$$

9. Analogously to the real orthogonal group $O(n)$, the unitary group $U(n)$ is a subgroup of $GL(n, \mathbb{C})$ of unitary matrices

$$U(n) = \{ A \in GL(n, \mathbb{C}) : A^\dagger A = 1 \}. \quad (56)$$

where $\dagger$ means the Hermitian conjugate: $A^\dagger = A^{T*}$. $U(n)$ is a Lie group of dimension $n^2$.

10. The special unitary group $SU(n)$ is defined as a subgroup of $U(n)$ of unitary matrices with unit determinant

$$SU(n) = \{ A \in U(n) : \det A = 1 \}. \quad (57)$$

$SU(n)$ is a Lie group of dimension $n^2 - 1$.

**Theorem.**

i.) The groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $SO(n)$, $SU(n)$ and $U(n)$ are connected.

ii.) The groups $SL(n, \mathbb{C})$ and $SU(n)$ are simply connected.

iii.) The groups $GL(n, \mathbb{R})$ and $SO(p, q)$ ($0 < p \leq q$) have two connected components.

For convenience of further references we present the information about these matrix groups in form of a table
<table>
<thead>
<tr>
<th>group</th>
<th>dimension</th>
<th>connectedness</th>
<th>compactness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M(n, \mathbb{C})$</td>
<td>$2n^2$</td>
<td>simply connected</td>
<td>non-compact</td>
</tr>
<tr>
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<td>$n^2$</td>
<td>simply connected</td>
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</tr>
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<td>connected</td>
<td>non-compact</td>
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<td>$GL(n, \mathbb{R})$</td>
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<td>two connected components</td>
<td>non-compact</td>
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<td>connected</td>
<td>non-compact</td>
</tr>
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</tr>
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<td>compact</td>
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</tr>
</tbody>
</table>

Figure 4: Matrix Groups

9 The Structure Constants of a Lie Group.

Let us consider a finite-dimensional Lie group $G$ of dimension $p$, i.e. the group elements are parameterized by $p$ real parameters $(\lambda = (\lambda^a), \ a = 1, 2, \ldots, p)$. One can always choose the coordinates $\lambda^a$ so that the identity element $e$ is at the origin, i.e. $e = g(0)$. If it is not so we just multiply all the elements of the group by $g^{-1}(0)$.

Let $f^a(\lambda, \mu), (a = 1, 2, \ldots, p)$, be the coordinates of the product $g(\lambda)g(\mu)$ of two elements of a Lie group, $g(\lambda)$ and $g(\mu)$, and $\lambda^a(\lambda)$ be the coordinates of the inverse element $(g(\lambda))^{-1} = g(\bar{\lambda})$. By definition of the Lie group the functions $f^a(\lambda, \mu)$ and $\lambda^a(\lambda)$ are analytic functions. These functions are not arbitrary but satisfy very important functional identities:

\[
\begin{align*}
  f(\lambda, 0) &= f(0, \lambda) = \lambda, \\
  f(\lambda, \bar{\lambda}) &= 0, \\
  f(\lambda, f(\mu, \nu)) &= f(f(\lambda, \mu), \nu).
\end{align*}
\]

These are highly nontrivial identities that determine the functions $f(\lambda, \mu)$ and, hence, the group, up to a change of coordinates on the group. By differentiating these identities one can obtain a lot of other identities of higher order. Let us consider the neighbourhood of the identity element. The functions $f(\lambda, \mu)$
can be expanded then in the Taylor series in $\lambda$ and $\mu$:

$$f^a(\lambda, \mu) = \lambda^a + \mu^a + B^a_{\ b\ c}\lambda^b\mu^c + O_3(\lambda, \mu) \quad (61)$$

where

$$B^a_{\ b\ c} = \left. \frac{\partial^2 f^a(\lambda, \mu)}{\partial \lambda^b \partial \mu^c} \right|_{\lambda=\mu=0} \quad (62)$$

and $O_3(\lambda, \mu)$ denotes the terms of order higher than 3 in $\lambda$ and $\mu$. The numbers

$$C^a_{\ b\ c} = B^a_{\ b\ c} - B^a_{\ c\ b} \quad (63)$$

are called the structure constants of the Lie group. They can be also defined by

$$C^a_{\ b\ c} = \left. \frac{\partial^2}{\partial \lambda^b \partial \mu^c} [f^a(f(\lambda, \mu), f(\bar{\lambda}, \bar{\mu}))] \right|_{\lambda=\mu=0} \quad (64)$$

The structure constants are obviously antisymmetric in lower indices

$$C^a_{\ b\ c} = -C^a_{\ c\ b} \quad (65)$$

and satisfy the Jacobi identities

$$C^d_{\ e\ a}C^e_{\ b\ c} + C^d_{\ e\ b}C^e_{\ c\ a} + C^d_{\ e\ c}C^e_{\ a\ b} = 0 \quad (66)$$

or, in short,

$$C^d_{\ [e\ a\ b\ c]} = 0. \quad (67)$$

This identity, for example, can be obtained by differentiating the eq. (60) with respect to $\lambda^a$, $\mu^b$ and $\nu^c$, putting $\lambda = \mu = \nu = 0$ and antisymmetrizing over $a$, $b$ and $c$.

Let us consider a continuous curve $g(\tau) = g(\lambda(\tau))$ going through the unit element, so that $g(0) = e$. The components

$$X = \left. \frac{dg(\tau)}{d\tau} \right|_{\tau=0} \quad (68)$$

define a vector $X$, called the tangent vector to the curve $g(\tau)$ at $e$. The set of tangent vectors to all curves going through identity element forms a linear vector space $L$, called the tangent space, $T_eG$, at $e$.

Note that if $g(\lambda)$ is unitary, i.e. $g^\dagger = g^{-1}$, then $X$ is anti-Hermitian, i.e. $X^\dagger = -X$.

Let $X_a$ be the basis vectors, called generators, in the tangent space $L$. For example, one can always define the generators by

$$X_a = \left. \frac{\partial g}{\partial \lambda^a} \right|_{\lambda=0}, \quad (69)$$

so that

$$g(\lambda) = e + \lambda^a X_a + O(\lambda^2). \quad (70)$$
Then one can introduce the structure of a Lie algebra by defining for each ordered pair \((X_a, X_b)\) of tangent vectors \(X_a\) and \(X_b\) a composition rule, called the Lie multiplication (or Lie bracket, or simply commutator),

\[
(X_a, X_b) \in L \times L \rightarrow [X_a, X_b] \in L,
\]

so that

\[
[X_a, X_b] = C^c_{ab} X_c.
\]

This Lie algebra is called the Lie algebra of the Lie group \(G\). The Jacobi identity (66) can be rewritten in terms of double commutators in form

\[
[X_a[X_b, X_c]] + [X_b[X_c, X_a]] + [X_c[X_a, X_b]] = 0.
\]

For Abelian groups all structure constants vanish \(C^a_{bc} = 0\) and we have so called Abelian Lie algebra

\[
[X_a, X_b] = 0.
\]

If \(C^a_{bc}\) are the structure constants of a Lie group, then the matrices \(T_a\) defined by \((T_a)^b_c = C^b_{ac}\) form a representation of the Lie algebra called the adjoint representation under the standard matrix multiplication. The commutation relations

\[
[T_a, T_b] = C^c_{ab} T_c
\]

are then nothing but the Jacobi identities.

**Theorem.** In the class \(\Gamma\) of all connected Lie groups having isomorphic Lie algebras there exists one and only one simply connected group \(\tilde{G}\), called the universal covering group of the class \(\Gamma\). Any group of the class \(\Gamma\) is a factor group \(\tilde{G}/N\), where \(N\) is a discrete central invariant subgroup. Note that the members of the class \(\Gamma\), i.e. the groups having isomorphic Lie algebras, although locally isomorphic, may be totally different globally.

**Theorem.** A compact connected Lie group \(G\) is a direct product of its connected center \(G_0\) and of its simple compact connected Lie subgroups \(G_k, k = 1, 2, \ldots, n\),

\[
G = G_0 \times G_1 \times \cdots \times G_n
\]

**10 Exponential Mapping**

The exponential map \(\exp\) is a homomorphism of the Lie algebra \(L\) into the Lie group \(G\)

\[
\exp: X \in L \rightarrow \exp(X) \in G.
\]
**Theorem.** Let $G$ be a Lie group and $L$ its Lie algebra. Then for every tangent vector $X \in L$ there exists a one-parameter subgroup $\exp(\tau X)$ of $G$, i.e. a unique analytic homomorphism $g(\tau) = \exp(\tau X)$ of $\mathbb{R}$ into $G$, such that

$$g(\tau_1)g(\tau_2) = g(\tau_1 + \tau_2), \quad \text{(78)}$$

$$\left. \frac{dg(\tau)}{d\tau} \right|_{\tau=0} = X, \quad \text{(79)}$$

$$g(0) = e. \quad \text{(80)}$$

In some cases, (but not generically!), the exponential map $X \mapsto \exp(X)$, $X \in L$, covers the whole group $G$.

If $T_a$ are the generators in the adjoint representation then $g(\lambda) = \exp(\lambda^a T_a)$ forms the **adjoint representation of the Lie group**.

**References**


