Lecture Notes
Complex Analysis
MATH 435

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Textbook: J. E. Marsden and M. J. Hoffman, (Freeman, 1999)

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Chapter 1

Analytic Functions

1.1 Complex Numbers

1.1.1 Complex Number System

Definition 1.1.1 The system of complex numbers, \( \mathbb{C} \), is the set \( \mathbb{R} \) of ordered pairs of real numbers with two binary operations, addition and multiplication, defined by

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\
(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),
\]

and a scalar multiplication by a real number

\[ a(x, y) = (ax, ay). \]

- The numbers of the form \((x, 0)\) are called the real numbers and the numbers of the form \((0, y)\) are called pure imaginary numbers.

- The real part and the imaginary part of a complex number are real numbers defined by

\[
\text{Re} (x, y) = x, \quad \text{Im} (x, y) = y.
\]

- The number \(1_\mathbb{C} = (1, 0)\) is the complex identity and the number \(i = (0, 1)\) is called the imaginary unit.
An arbitrary complex number can be represented as
\[(x, y) = x + iy.\]

The property of the imaginary unit
\[i^2 = -1.\]

### 1.1.2 Algebraic Properties

- Addition and multiplication of complex numbers are: associative, commutative and distributive.
- Every non-zero complex number \(z = x + iy\) has the multiplicative inverse
  \[z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\]
  such that
  \[zz^{-1} = 1.\]
- The quotient of \(z\) by \(w\) is defined by
  \[\frac{z}{w} = zw^{-1}.\]
- All the usual algebraic rules of real numbers hold for complex numbers.
- The set of real numbers is a (complete ordered) field.
- The set of complex numbers forms a (complete but not ordered) field.

**Proposition 1.1.1** For any complex number \(z = x + iy\) there is a complex number \(w = a + ib\) with

\[a = \pm \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}, \quad b = \pm \text{sign}(y) \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}\]

such that \(w^2 = z.\)

**Proof:** Check directly.
1.1. COMPLEX NUMBERS

- The quadratic equation
  \[ az^2 + bz + c = 0 \]
  with complex numbers \( a, b, c \) has solutions
  \[ z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} . \]

- Examples.

1.1.3 Properties of Complex Numbers

- Complex numbers are represented as points on the real plane \( \mathbb{R}^2 \).
- Addition of complex numbers can be represented by the vector addition.
- Polar Representation of complex numbers
  \[ z = x + iy = r(\cos \theta + i \sin \theta) , \]
  where the real number \( r = |z| \) (called the absolute value (or the norm, or the modulus)) is defined by
  \[ |z| = \sqrt{x^2 + y^2} , \]
  and the angle \( \theta \) (called the argument) is defined by
  \[ \cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}. \]
- The argument \( \arg z \) can be thought as an infinite set of values
  \[ \arg z = \{ \theta + 2\pi n | n \in \mathbb{Z} \} \]
- Specific values of the argument are called branches of the argument.
- Multiplication of Complex Numbers in Polar Form.

<table>
<thead>
<tr>
<th>Proposition 1.1.2</th>
<th>For any complex numbers ( z_1 ) and ( z_2 ) there holds</th>
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<tbody>
<tr>
<td></td>
<td>[ r_1(\cos \theta_1 + i \sin \theta_1) r_2(\cos \theta_2 + i \sin \theta_2) = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] ]</td>
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<td>In other words, [</td>
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<td></td>
<td>[ \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi} . ]</td>
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CHAPTER 1. ANALYTIC FUNCTIONS

Proof: Follows from the trigonometric identities.

De Moivre Formula.

Proposition 1.1.3 For any complex numbers $z = r(\cos \theta + i \sin \theta)$ and a positive integer $n$ there holds

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

In other words,

$$|z^n| = |z|^n,$$

$$\arg(z^n) = n \arg z \pmod{2\pi}.$$  

Proof: By induction.

Roots of Complex Numbers.

Proposition 1.1.4 Let $w = r(\cos \theta + i \sin \theta)$ be a nonzero complex number and $n$ be a positive integer. Then there are $n$ nth roots of $w$, $z_0, z_1, \ldots, z_{n-1}$, given by

$$z_k = r^{1/n} \left[ \cos \left( \frac{\theta}{n} + \frac{2\pi k}{n} \right) + i \sin \left( \frac{\theta}{n} + \frac{2\pi k}{n} \right) \right]$$

with $k = 0, 1, \ldots, n - 1$.

Proof: Direct calculation.

Complex Conjugation. The complex conjugate of a complex number $z = x + iy$ is a complex number $\bar{z}$ defined by

$$\bar{z} = x - iy.$$
1.1. COMPLEX NUMBERS

Proposition 1.1.5

1. \( \overline{z + w} = \overline{z} + \overline{w} \),
2. \( \overline{zw} = \overline{z} \overline{w} \),
3. \( \frac{\overline{z}}{w} = \frac{\overline{z}}{\overline{w}} \) for \( w \neq 0 \),
4. \( z \overline{z} = |z|^2 \),
5. \( z^{-1} = \frac{\overline{z}}{|z|^2} \) for \( z \neq 0 \),
6. \( \text{Re } z = \frac{1}{2}(z + \overline{z}) \),
7. \( \text{Im } z = \frac{1}{2i}(z - \overline{z}) \),
8. \( \overline{\overline{z}} = z \),
9. \( z = \overline{z} \) if and only if \( z \) is real,
10. \( z = -\overline{z} \) if and only if \( z \) is purely imaginary.

Proof: Direct calculation.

• Properties of the Absolute Value.
Proposition 1.1.6

1. \(|zw| = |z||w|,\)
2. \(\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \text{ for } w \neq 0,\)
3. \(|\text{Re } z| \leq |z|, \quad |\text{Im } z| \leq |z|,\)
4. \(|\bar{z}| = |z|,\)
5. Triangle inequality
   \[|z + w| \leq |z| + |w|,\]
6. \[|z - w| \geq \left| |z| - |w| \right|,\]
7. Cauchy-Schwarz inequality
   \[\left| \sum_{k=1}^{n} z_k w_k \right| \leq \sqrt{\sum_{k=1}^{n} |z_k|^2} \sqrt{\sum_{j=1}^{n} |w_j|^2}.\]

Proof: Do Cauchy-Schwarz inequality.

1. Let \(a = \sum_{k=1}^{n} z_k w_k, \quad b = \sum_{k=1}^{n} |z_k|^2 \quad c = \sum_{j=1}^{n} |w_j|^2.\)
2. We have
   \[\sum_{k=1}^{n} |cz_k - \bar{a}w_k|^2 = c^2 b + |a|^2 c - 2c|a|^2 = bc^2 - c|a|^2 \geq 0\]
3. Therefore
   \[|a|^2 \leq bc.\]

• Examples.
• Exercises: 1.1[10,12,13], 1.2[9,11,25]
1.2 Elementary Functions

1.2.1 Exponential Function

- Define
  \[ e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} \]

- Then
  \[ e^{iy} = \cos y + i \sin y \]

- The exponential function exp : \( \mathbb{C} \rightarrow \mathbb{C} \) is defined now by
  \[ \exp(x + iy) = e^x(\cos y + i \sin y) \]

- A function \( f : \mathbb{C} \rightarrow \mathbb{C} \) is periodic if there is some \( w \in \mathbb{C} \) such that \( f(z + w) = f(z) \).

**Proposition 1.2.1** Let \( z, w \in \mathbb{C}, x, y \in \mathbb{R}, n \in \mathbb{Z} \). Then

1. \( e^{z+w} = e^z e^w \).
2. \( e^z \neq 0 \).
3. \( e^x > 1 \) if \( x > 0 \) and \( 0 < e^x < 1 \) if \( x < 0 \).
4. \( |e^{x+iy}| = e^x \).
5. \( e^{\pi i/2} = i, e^{\pi i} = -1, e^{3\pi i/2} = -i, e^{2\pi i} = 1 \).
6. \( e^{x+2\pi i} = e^x \).
7. \( e^z = 1 \) if and only if \( z = 2\pi ki \) for some \( k \in \mathbb{Z} \).

**Proof:** Easy.

1.2.2 Trigonometric Functions

The sine and cosine functions are defined by

\[ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \]
Proposition 1.2.2  Let $z, w \in \mathbb{C}$. Then

1. $\sin^2 z + \cos^2 z = 1$.
2. $\sin(z + w) = \sin z \cos w + \cos z \sin w$,
3. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Proof: Direct calculation.

1.2.3 Logarithm Function

Proposition 1.2.3  Let $y_0 \in \mathbb{R}$ and $A_{y_0} \subset \mathbb{C}$ be a set of complex numbers defined by

$$A_{y_0} = \{z = x + iy | y_0 \leq y < y_0 + 2\pi\}.$$ 

Let $f : A_{y_0} \to \mathbb{C} - \{0\}$ be a function defined by $f(z) = e^z$ for any $z \in A_{y_0}$. Then the function $f$ is an bijection (one-to-one onto map).

Proof: Show directly that it is one-to-one and onto.

Let $y_0 \in \mathbb{R}$. The branch of the logarithm function $\log : \mathbb{C} - \{0\} \to A_{y_0}$ lying in $A_{y_0}$ is defined by

$$\log z = \log |z| + i \arg z,$$

where $y_0 \leq \arg z < y_0 + 2\pi$.

Proposition 1.2.4  Any branch of the logarithm satisfies: for any $z \in \mathbb{C}$, $z \neq 0$,

$$\exp \log z = z.$$ 

Also, for a branch of the logarithm lying in $[y_0, y_0 + 2\pi)$ there holds: for any $z = x + iy$, $y \in [y_0, y_0 + 2\pi)$,

$$\log e^z = z.$$ 

Proof: Obvious.
1.2. ELEMENTARY FUNCTIONS

• Proposition 1.2.5 Let \( z, w \in \mathbb{C} \setminus \{0\} \). Then
\[
\log(zw) = \log z + \log w \mod (2\pi i)
\]

Proof: Obvious.

1.2.4 Complex Powers

• Let \( a, b \in \mathbb{C}, a \neq 0 \). Let \( y_0 \in \mathbb{R} \) and \( \log \) be the branch of the logarithm lying in \([y_0, y_0 + 2\pi]\). Then the complex power \( a^b \) is defined by
\[
a^b = \exp(b \log a).
\]

• Proposition 1.2.6 Let \( a, b \in \mathbb{C}, a \neq 0 \).

1. Then \( a^b \) is does not depend on the branch of the logarithm if and only if \( b \) is an integer.

2. Let \( b \) be a rational number such that \( b = p/q \) with \( p, q \in \mathbb{Z} \) relatively prime. Then \( a^b \) has exactly \( q \) distinct values, namely, the roots \((a^p)^{1/q}\).

3. If \( b \) is a real irrational number or \( \text{Im} \, b \neq 0 \), then \( a^b \) has infinitely many distinct values.

4. The distinct values of \( a^b \) differ by factors of the form \( e^{2\pi nbi} \).

Proof: Obvious.

1.2.5 Roots

• Let \( n \) be a positive integer. The a branch of the \( n \)th root function is defined by
\[
z^{1/n} = \exp \left( \frac{\log z}{n} \right)
\]
where a branch of the logarithm is chosen.
Proposition 1.2.7  Let \( z \in \mathbb{C} \) and \( n \) be a positive integer.

1. The \( n \)th root of \( z \) satisfies
\[
(z^{1/n})^n = z
\]

2. If \( z = re^{i\theta} \), then
\[
z^{1/n} = r^{1/n}e^{i\theta/n},
\]
where \( \theta \) lies within an interval of length \( 2\pi \) corresponding to the branch choice.

Proof: Obvious.

Exercises: 1.3[11,16,17,21,23]
1.3 Continuous Functions

1.3.1 Open Sets

**Definition 1.3.1**

1. A set \( A \subset \mathbb{C} \) is called **open** if for any point \( z_0 \in A \) there is \( \varepsilon > 0 \) such that for any \( z \in A \), if \( |z - z_0| < \varepsilon \), then \( z \in A \).

2. For a \( \varepsilon > 0 \) and \( z_0 \in \mathbb{C} \), the \( \varepsilon \)-**neighborhood**, (or **disk**, or **ball**) around \( z_0 \) is the set

\[
D(z_0, \varepsilon) = \{ z \in \mathbb{C} | |z - z_0| < \varepsilon \}
\]

3. A **deleted** \( \varepsilon \)-neighborhood is a \( \varepsilon \)-neighborhood whose center point has been removed, that is,

\[
D_{\text{del}}(z_0, \varepsilon) = \{ z \in \mathbb{C} | |z - z_0| < \varepsilon, \ z \neq z_0 \}.
\]

4. A neighborhood of a point \( z_0 \in \mathbb{C} \) is a set containing some \( \varepsilon \)-neighborhood around \( z_0 \).

• A set \( A \) is open if and only if each point \( z_0 \in A \) has a neighborhood wholly contained in \( A \).

**Definition 1.3.2**

1. The set \( \mathbb{C} \) is open.

2. The empty set \( \emptyset \) is open.

3. The union of any collection of open sets is open.

4. The intersection of any finite collection of open sets is open.

**Proof:** Exercise.
1.3.2 Mappings, Limits and Continuity

- Let $A \subset \mathbb{C}$. A mapping $f : A \to \mathbb{C}$ is a rule that assigns to every point $z \in A$ a point in $\mathbb{C}$ (called the value of $f$ at $z$ and denoted by $f(z)$). The set $A$ is called the domain of $f$. Such a mapping defines a complex function of a complex variable.

**Definition 1.3.3** Let $f : A \to \mathbb{C}$. Let $z_0 \in \mathbb{C}$ such that $A$ contains a deleted $r$-neighborhood $D^0(z_0, r)$ of $z_0$. The $f$ is said to have the limit $a$ as $z \to z_0$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in D^0(z_0, r)$, if $|z - z_0| < \delta$, then $|f(z) - a| < \varepsilon$.

Then we write
\[ \lim_{z \to z_0} f(z) = a. \]

- **Proposition 1.3.1** If a function has a limit, then it is unique.

**Proof:** Easy.

- **Proposition 1.3.2** Let $A \subset \mathbb{C}$ and $f, g : A \to \mathbb{C}$. Let $z_0 \in \mathbb{C}$ have a deleted neighborhood in $A$. Suppose that $f$ and $g$ have limits at $z_0$, $\lim_{z \to z_0} f(z) = a$ and $\lim_{z \to z_0} g(z) = b$. Then:
  1. $\lim_{z \to z_0} [f(z) + g(z)] = a + b$
  2. $\lim_{z \to z_0} [f(z)g(z)] = ab$
  3. $\lim_{z \to z_0} [f(z)/g(z)] = a/b$, if $b \neq 0$.

**Proof:** Easy.

- **Definition 1.3.4** Let $A \subset \mathbb{C}$ be open, $z_0 \in A$, and $f : A \to \mathbb{C}$ be a function.

  1. $f$ is continuous at $z_0$ if
  \[ \lim_{z \to z_0} f(z) = f(z_0). \]
  2. $f$ is continuous on $A$ if $f$ is continuous at each point in $A$. 
1.3. CONTINUOUS FUNCTIONS

Proposition 1.3.3

1. The sum, the product and the quotient of continuous functions is continuous.

2. The composition of continuous functions is continuous.

Proof: Easy.

1.3.3 Sequences

Definition 1.3.5 Let \((z_n)_{n=1}^{\infty}\) be a sequence in \(\mathbb{C}\) and \(z_0 \in \mathbb{C}\). The sequence \((z_n)_{n=1}^{\infty}\) converges to \(z_0\) if for every \(\varepsilon > 0\) there is a \(N \in \mathbb{Z}_+\) such that for any \(n \in \mathbb{Z}_+\) if \(n \geq N\), then \(|z_n - z_0| < \varepsilon\).

Then we write

\[
\lim_{n \to \infty} z_n = z_0 \text{ or } z_n \to z_0.
\]

Proposition 1.3.4 Let \((z_n)\) and \((w_n)\) be sequences in \(\mathbb{C}\) and \(z_0, w_0 \in \mathbb{C}\). Suppose that \(z_n \to z_0\) and \(w_n \to w_0\). Then:

1. \(z_n + w_n \to z_0 + w_0\)
2. \(z_n w_n \to z_0 w_0\)
3. \(z_n / w_n \to z_0 / w_0\), if \(w_0 \neq 0\).

Proof: Same as for functions.

Definition 1.3.6 Let \((z_n)_{n=1}^{\infty}\) be a sequence in \(\mathbb{C}\). The sequence \((z_n)\) is called a Cauchy sequence if for every \(\varepsilon > 0\) there is a \(N \in \mathbb{Z}_+\) such that for any \(n, m \in \mathbb{Z}_+\)

\[
if \ n, m \geq N, \ then \ |z_n - z_m| < \varepsilon.
\]
CHAPTER 1. ANALYTIC FUNCTIONS

Proposition 1.3.5  Completeness of $\mathbb{R}$. Every Cauchy sequence of real numbers is convergent. That is, for every Cauchy sequence $(x_n)$ in $\mathbb{R}$ there is $x_0 \in \mathbb{R}$ such that $x_n \to x_0$.

Proof: Real analysis.

Proposition 1.3.6  Completeness of $\mathbb{C}$. Every Cauchy sequence in $\mathbb{C}$ is convergent. That is, for every Cauchy sequence $(z_n)$ in $\mathbb{C}$ there is $z_0 \in \mathbb{C}$ such that $z_n \to z_0$.

Proof: Follows from completeness of $\mathbb{R}$.

Proposition 1.3.7  Let $A \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, and $f : A \to \mathbb{C}$. Then $f$ is continuous at $z_0$ iff for every sequence $(z_n)$ in $A$ such that $z_n \to z_0$ there holds $f(z_n) \to f(z_0)$.

Proof: Exercise.

1.3.4  Closed Sets

- The complement of a set $A \in \mathbb{C}$ is the set $\mathbb{C} \setminus A$.

Definition 1.3.7  A set $A \in \mathbb{C}$ is called closed if its complement is open.

Proposition 1.3.8

1. The empty set is closed.

2. $\mathbb{C}$ is closed.

3. The intersection of any collection of closed sets is closed.

4. The union of a finite collection of closed sets is closed.

Proof: Follows from properties of open sets.
Proposition 1.3.9  A set is closed iff it contains the limit of every convergent sequence in it.

Proof: Do.

Proposition 1.3.10  Let $f : \mathbb{C} \rightarrow \mathbb{C}$. The following are equivalent statements:

1. $f$ is continuous.
2. The inverse image of every closed set is closed.
3. The inverse image of every open set is open.

Proof: Do the cycle of implications.

Definition 1.3.8  Let $A \in \mathbb{C}$ and $B \subset A$.

1. $B$ is open relative to $A$ if $B = A \cap U$ for some open set $U$.
2. $B$ is closed relative to $A$ if $B = A \cap U$ for some closed set $U$.

Proposition 1.3.11  Let $f : A \rightarrow \mathbb{C}$. The following are equivalent statements:

1. $f$ is continuous.
2. The inverse image of every closed set is closed relative to $A$.
3. The inverse image of every open set is open relative to $A$.

Proof: Exercise.
1.3.5 Connected Sets

**Definition 1.3.9** Let $A \in \mathbb{C}$.

1. The set $A$ is **path-connected** if for every two points $z_1, z_2 \in A$ there is a continuous map $\gamma : [0, 1] \to A$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Then $\gamma$ is called a **path** joining the points $z_1$ and $z_2$.

2. The set $A$ is **disconnected** if there are open sets $U$ and $V$ such that
   
   (a) $A \subset U \cap V$,
   
   (b) $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$,
   
   (c) $(A \cap U) \cap (A \cap V) = \emptyset$.

3. A set is **connected** if it is not disconnected.

**Proposition 1.3.12** Let $A \subset \mathbb{C}$. The set $A$ is connected iff the only subsets of $A$ that are both open and closed relative to $A$ are $\emptyset$ and $A$.

**Proof:** Follows from the definition.

**Proposition 1.3.13** A path-connected set is connected.

**Proof:** Do.

**Example.** Let

$$A_1 = \left\{ z = x + iy \mid x \neq 0 \quad y = \sin \left( \frac{1}{x} \right) \right\}$$

$$A_2 = \left\{ z = x + iy \mid x = 0, \quad y \in [-1, 1] \right\}$$

and

$$A = A_1 \cup A_2.$$

Then $A$ is connected but not path-connected.

**Proposition 1.3.14** Every open connected set is path connected with a differentiable path. That is, if $A \subset \mathbb{C}$ is open connected and $z_1, z_2 \in A$. Then there is a differentiable map $\gamma : [0, 1] \to A$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. 
1.3. CONTINUOUS FUNCTIONS

Proof: See the book.

Definition 1.3.10  An open connected set in $\mathbb{C}$ is called a region (or a domain).

Proposition 1.3.15  A continuous image of a connected set is connected.

Proof: Easy.

1.3.6 Compact Sets

Definition 1.3.11

1. Let $A$ be a set and $O = \{U_\alpha\}_{\alpha \in A}$ be a collection of open sets. Let $K \subset \mathbb{C}$. Then the collection $O$ is called an open cover of $K$ if

$$K \subset \bigcup_{\alpha \in A} U_\alpha.$$  

2. A subcollection of an open cover is called a subcover if it is still an open cover.

3. A set is compact if every open cover of it has a finite subcover.

Proposition 1.3.16  Let $K \subset \mathbb{C}$. The following are equivalent:

1. $K$ is closed and bounded.

2. Every sequence in $K$ has a convergent subsequence whose limits is in $K$.

3. $K$ is compact.

Proof: In real analysis courses.

Proposition 1.3.17  A continuous image of a compact set is compact.

Proof: Easy.
Theorem 1.3.1  Extreme value Theorem. Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{R}$ be continuous. Then $f$ attains maximum and minimum values.

Proof: Easy.

Lemma 1.3.1  Distance Lemma. Let $A, K \subset \mathbb{C}$ be two disjoint sets. Suppose that $K$ is compact and $A$ is closed. Then the distance $d(K, A) > 0$. That is, there is $\varepsilon > 0$ such that for any $z \in K$ and $w \in A$, $|z - w| > \varepsilon$.

Proof: Do.

1.3.7 Uniform Continuity

Definition 1.3.12  Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$. Then $f$ is uniformly continuous on $A$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $z, w \in A$,

\[ |z - w| < \delta, \text{ then } |f(z) - f(w)| < \varepsilon. \]

Proposition 1.3.18  A continuous function on a compact set is uniformly continuous.

Proof:
1.3. CONTINUOUS FUNCTIONS

1.3.8 Path Covering Lemma

**Lemma 1.3.2** Path-Covering Lemma. Let \( a, b \in \mathbb{R} \) and \( G \subset \mathbb{C} \) be an open set. Let \( \gamma : [a, b] \rightarrow G \) be a continuous path. Then there is \( \rho > 0 \) and a partition of \([a, b]\), \( \{t_0 < t_1 < \cdots < t_n\} \) with \( t_0 = a \) and \( t_n = b \), such that

1. \( D(\gamma(t_k); \rho) \subset G \),
2. \( \gamma(t) \in D(\gamma(t_k); \rho) \) for \( t \in [t_k, t_{k+1}] \) and \( k = 1, \ldots, n - 1 \),
3. \( \gamma(t) \in D(\gamma(t_0); \rho) \) for \( t \in [t_0, t_1] \),
4. \( \gamma(t) \in D(\gamma(t_n); \rho) \) for \( t \in [t_{n-1}, t_n] \).

**Proof:** Do.

1.3.9 Riemann Sphere and Point at Infinity

- The extended complex plane, \( \tilde{\mathbb{C}} \), is obtained by adding the point at infinity, \( \infty \), to \( \mathbb{C} \), \( \tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \).
- **Limits at infinity.**
  - A point is close to infinity if it lies outside a sufficiently large circle.
  - The limit \( \lim_{z \to \infty} f(z) = a \) means that for any \( \varepsilon > 0 \) there is a \( R > 0 \) such that if \( |z| > R \) then \( |f(z) - a| < \varepsilon \).
  - The limit \( \lim_{z \to z_0} f(z) = \infty \) means that for any \( R > 0 \) there is a \( \delta > 0 \) such that if \( |z - z_0| < \delta \) then \( |f(z)| > R \).
  - The limit \( \lim_{n \to \infty} z_n = \infty \) means that for any \( R > 0 \) there is a \( N \in \mathbb{Z}_+ \) such that if \( n \geq N \) then \( |z_n| > R \).
- **Riemann sphere and stereographic projection.**
- \( \mathbb{S} \) and \( \tilde{\mathbb{C}} \) are compact.
- **Exercises:** 1.4[3, 5, 11, 12, 14, 16, 17, 20, 23, 24]
1.4 Analytic Functions

**Definition 1.4.1** Let \( A \subset \mathbb{C} \) be an open set and \( f : A \to \mathbb{C} \).

1. Let \( z_0 \in A \). Then \( f \) is **differentiable** at \( z_0 \) if the limit (called the derivative)
   \[
   f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
   \]
   exists.

2. The function \( f \) is **analytic** (or **holomorphic**) on \( A \) if it is differentiable at every point in \( A \).

3. The function \( f \) is analytic at \( z_0 \) if it is analytic in a neighborhood of \( z_0 \).

4. A function analytic on the whole complex plane is called **entire**.

**Remarks.** A differentiable function must be continuous. However, a continuous function does not have to be differentiable.

**Proposition 1.4.1** Let \( A \subset \mathbb{C} \) be an open set, \( f : A \to \mathbb{C} \), and \( z_0 \in A \). Suppose that \( f \) is differentiable at \( z_0 \). Then \( f \) is continuous at \( z_0 \).

**Proof:** Easy.

**A polynomial** of degree \( n \) is a function \( P : \mathbb{C} \to \mathbb{C} \) of the form

\[
P(z) = \sum_{k=0}^{n} a_k z^k,
\]

where \( a_k \in \mathbb{C} \).

Let \( P \) and \( Q \) be two polynomials. Let \( A = \{ z \in \mathbb{C} \mid Q(z) \neq 0 \} \) be an open set that is the complement of the set of roots of the polynomial \( Q \). A **rational function** is a function \( R : A \to \mathbb{C} \) of the form

\[
R(z) = \frac{P(z)}{Q(z)}.
\]
1.4. ANALYTIC FUNCTIONS

**Proposition 1.4.2** Let \( A \subset \mathbb{C} \) be an open set, \( f, g, h : A \to \mathbb{C} \) be analytic on \( A \), and \( a, b \in \mathbb{R} \). Suppose that \( h(z) \neq 0 \) for any \( z \in A \). Then the functions \((af + bg), fg \) and \( f/g \) are analytic on \( A \) and \( \forall z \in \mathbb{C} \)

\[
(af + bg)'(z) = af'(z) + bg'(z),
\]

\[
(fg)'(z) = f'(z)g(z) + f(z)g'(z),
\]

\[
(f/h)'(z) = \frac{f'(z)h(z) + f(z)h'(z)}{[h(z)]^2}.
\]

**Proof:** Easy.

---

**Proposition 1.4.3**

1. Every polynomial is an entire function.
2. Every rational function is analytic on its domain, that is, the complement of the set of roots of the denominator.

**Proof:** Easy.

---

**Proposition 1.4.4** Chain Rule. Let \( A, B \in \mathbb{C} \) be open sets, \( f : A \to \mathbb{C} \), \( g : B \to \mathbb{C} \). Suppose that \( f \) and \( g \) are analytic and \( f(A) \subset B \). Let \( h = g \circ f : A \to \mathbb{C} \) be the composition of \( f \) and \( g \). Then \( h \) is analytic on \( A \) and \( \forall z \in A \)

\[
h'(z) = g'(f(z))f'(z).
\]

**Proof:**
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Proposition 1.4.5 Let \(A, B \subset \mathbb{C}\) be open sets, and \(f : A \to \mathbb{C}\). Let \(a, b \in \mathbb{R}\) and \(\gamma : (a, b) \to B\), be a curve in \(B\). Let \(B \subset A\) and \(\sigma : (a, b) \to \mathbb{C}\) be the curve defined \(\forall t \in (a, b)\) by \(\sigma(t) = f(\gamma(t))\). Suppose that \(f\) is analytic and \(\gamma\) is differentiable. Then \(\sigma\) is also differentiable and

\[
\frac{d}{dt} \sigma(t) = f'(\gamma(t)) \frac{d}{dt} \gamma(t).
\]

Proof: Similar. 

Proposition 1.4.6 Let \(A \subset \mathbb{C}\) be a region (an open connected set), and \(f : A \to \mathbb{C}\). Suppose that \(f\) is analytic on \(A\) and \(f'(z) = 0\) for any \(z \in A\). Then \(f\) is constant on \(A\).

Proof: Let \(z_1, z_2 \in A\). Connect them by a differentiable path and use the chain rule.

1.4.1 Conformal Maps.

Definition 1.4.2 Let \(A \subset \mathbb{C}\) be an open and \(f : A \to \mathbb{C}\).

1. Let \(z_0 \in A\). Then \(f\) is a **conformal** at \(z_0\) if there exists \(\theta \in [0, 2\pi)\) and \(r > 0\) such that for any curve \(\gamma : (-\varepsilon, \varepsilon) \to A\) that is: i) differentiable at \(t = 0\), ii) \(\gamma(0) = z_0\), and iii) \(\gamma'(0) \neq 0\), the curve \(\sigma : (-\varepsilon, \varepsilon) \to \mathbb{C}\) defined by \(\sigma(t) = f(\gamma(t))\) is differentiable at \(t = 0\) and

\[
\sigma'(0) = re^{i\theta}\gamma'(0),
\]

that is,

\[
|\sigma'(0)| = r|\gamma'(0)|, \quad \arg \sigma'(0) = \arg \gamma'(0) + \theta \pmod{2\pi}.
\]

2. The function \(f : A\) is called a **conformal map** if it is conformal at every point of \(A\).

- A conformal map rotates and stretches tangent vectors to curves, that is, it preserves angles between intersecting curves.
1.4. ANALYTIC FUNCTIONS

- The numbers $r$ and $\theta$ are fixed at the point $z_0$. They should apply for any curve.

**Theorem 1.4.1 Conformal Mapping Theorem.** Let $A \subset \mathbb{C}$ be a open set, $z_0 \in A$, and $f : A \to \mathbb{C}$. Suppose that $f$ is analytic at $z_0$ and $f'(z_0) \neq 0$. Then $f$ is conformal at $z_0$.

**Proof:** Follows from the chain rule.

1.4.2 Cauchy-Riemann Equations

- **Multiple real variables.** A point in $\mathbb{R}^2$ is represented as a pair $(x, y)$. It can be also viewed as a column vector $\left( \begin{array}{c} x \\ y \end{array} \right)$, which can be added, multiplied by scalar etc. A linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by a matrix

$$J = \left( \begin{array}{cc} J_{11} & J_{12} \\ J_{21} & J_{22} \end{array} \right)$$

acting on column vectors $\left( \begin{array}{c} x \\ y \end{array} \right)$ by left multiplication, that is,

$$\langle J, (x, y) \rangle = \left( \begin{array}{cc} J_{11} & J_{12} \\ J_{21} & J_{22} \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right).$$

- Let $A \subset \mathbb{R}^2$ and $f : A \to \mathbb{R}^2$ with $f(x, y) = (u(x, y), v(x, y))$. The **Jacobian matrix** of $f$ is defined by

$$Df = \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right)$$

- The distance between the points $(x, y)$ and $(x_0, y_0)$ in $\mathbb{R}^2$ is given by $|| (x, y) - (x_0, y_0) || = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

- Let $(x_0, y_0) \in A$. The map $f$ is **differentiable** at $(x_0, y_0) \in A$ if there exists a matrix (called the derivative matrix) $Df(x_0, y_0)$ such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|| (x, y) - (x_0, y_0) || < \delta$ then

$$|| f(x, y) - f(x_0, y_0) - \langle Df(x_0, y_0), (x, y) - (x_0, y_0) \rangle || < \varepsilon ||(x, y) - (x_0, y_0)||.$$
• If \( f \) is differentiable, the partial derivatives of \( u \) and \( v \) exist and the derivative matrix is given by the Jacobian matrix.

• If the partial derivatives of \( u \) and \( v \) exist and are continuous then \( f \) is differentiable.

\[ \text{Theorem 1.4.2 Cauchy-Riemann Theorem.} \quad \text{Let} \quad A \in \mathbb{C} \text{ be an open set,} \quad f : A \to \mathbb{C}, \text{ and} \quad z_0 \in A. \text{ The} \quad f = u + iv \text{ is differentiable at} \quad z_0 = x_0 + iy_0 \quad \text{if and only if} \quad f \text{ is differentiable at} \quad (x_0, y_0) \quad \text{in the sense of real variables and the functions} \quad u \quad \text{and} \quad v \quad \text{satisfy the Cauchy-Riemann equations} \]

\[ \partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v. \]

\[ \text{Proof:} \]

• Cauchy-Riemann equations in polar form.

\[ \partial_r u = \frac{1}{r} \partial_\theta v, \quad \partial_r v = -\frac{1}{r} \partial_\theta u, \]

1.4.3 Inverse Functions

• **Real analysis.** A continuously differentiable function on an open set is bijective and has a differentiable inverse in a neighborhood of a point if the Jacobian matrix is not degenerate at that point.

\[ \text{Theorem 1.4.3 Real Variable Inverse Function Theorem.} \quad \text{Let} \quad A \subset \mathbb{R}^2 \text{ be an open set,} \quad f : A \to \mathbb{R}^2, \text{ and} \quad (x_0, y_0) \in A. \text{ Suppose that} \quad f \text{ is continuously differentiable and the Jacobian matrix} \quad Df(x_0, y_0) \text{ is non-degenerate. Then there is a neighborhood} \quad U \text{ of} \quad (x_0, y_0) \text{ and a neighborhood} \quad V \text{ of} \quad f(x_0, y_0) \text{ such that} \quad f : U \to V \text{ is a bijection,} \quad f^{-1} : V \to U \text{ is differentiable, and} \]

\[ Df^{-1} f(x, y) = [Df(x, y)]^{-1}. \]

\[ \text{Proof:} \quad \text{Advanced calculus.} \]

\[ \]

\[ \]

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1.4. ANALYTIC FUNCTIONS

Theorem 1.4.4  Inverse Function Theorem. Let $A \subset \mathbb{C}$ be an open set, $z_0 \in A$, and $f : A \to \mathbb{C}$. Suppose that $f$ is analytic, $f'$ is continuous and $f'(z_0) \neq 0$. Then there is a neighborhood $V$ of $z_0$ such that the function $f : U \to V$ is a bijection with analytic inverse $f^{-1} : V \to U$ with the derivative

$$\frac{d}{dz} f^{-1}(z) = \frac{1}{f'(f^{-1}(z))}.$$  

Proof: Compute $[Df]^{-1}$ by using CR equations.

The Laplacian is a partial differential operator acting on functions on $\mathbb{R}^2$ defined by

$$\Delta = \partial_x^2 + \partial_y^2.$$  

Let $A \in \mathbb{R}^2$ be an open set and $u : A \to \mathbb{R}$ be a twice-differentiable function. Then $f$ is harmonic if

$$\Delta u = 0.$$  

An analytic function is infinitely differentiable (show later).

Proposition 1.4.7  Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$. Suppose that $f$ is analytic on $A$. Then the real and the imaginary parts of $f$ are harmonic on $A$.

Proof: Use CR equations.

The real and the imaginary parts, $u$ and $v$, of an analytic function $f = u + iv$, are called harmonic conjugates of each other.

The level curve $u(x,y) = c$ is smooth if grad $u \neq 0$.

Let $f = u + iv$ be analytic. The level curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are smooth if $f'(z) \neq 0$.

The vector grad $u$ is orthogonal to the level curve $u(x,y) = c$.

Proposition 1.4.8  Let $A \subset \mathbb{C}$ be a region, $f = u + iv : A \to \mathbb{C}$ be analytic on $A$. Suppose that the level curves $u(x,y) = a$ and $v(x,y) = b$ defines smooth curves. Then they intersect orthogonally.

Proof: Use CR equations.
Proposition 1.4.9  Let $A \subset \mathbb{C}$ be an open set, $u : A \to \mathbb{R}$ be a real-valued function and $z_0 \in A$. Suppose that $u$ is twice differentiable and harmonic on $A$. Then there is a neighborhood $U$ of $z_0$ and an analytic function $f : U \to \mathbb{C}$ such that $u = \text{Re } f$.

Proof: Exercise.

Exercises: 1.5[7,8,10,11,13,14,15,16,17,25,30,31]
1.5 Differentiation

1.5.1 Exponential Function and Logarithm

- **Proposition 1.5.1** The exponential function \( f : \mathbb{C} \to \mathbb{C} \) defined by \( f(z) = e^z \) is entire and \( \frac{de^z}{dz} = e^z \).

**Proof:** Verify CR conditions.

- **Proposition 1.5.2** Let \( B \subset \mathbb{C} \) be a closed set defined by
  \[
  B = \{ z = x + iy \mid x \leq 0, y = 0 \}
  \]
and \( A = \mathbb{C} \setminus B \) be its complement (which is open). The principal branch of the logarithm is a branch on \( A \) defined by
  \[
  \log z = \log |z| + i \arg z,
  \]
where \(-\pi < \arg z < \pi\). Then \( \log z \) is analytic on \( A \) and
  \[
  \frac{d \log z}{dz} = \frac{1}{z}. \]

**Proof:** Verify CR equations.

1.5.2 Trigonometric Functions

- **Proposition 1.5.3** The sine and the cosine functions are entire and
  \[
  \frac{d \sin z}{dz} = \cos z, \quad \frac{d \cos z}{dz} = -\sin z. \]

**Proof:** Trivial.
1.5.3 Power Function

Proposition 1.5.4 Let \( a \in \mathbb{C} \). Then the power function \( z \mapsto z^a = e^{a \log z} \) (defined with the principal branch of the logarithm) is analytic on the domain of the logarithm and

\[
\frac{d z^a}{dz} = a z^{a-1}.
\]

Proof: Use chain rule.

1.5.4 The \( n \)-th Root Function

Proposition 1.5.5 Let \( n \in \mathbb{Z}, n \geq 1 \). Then the \( n \)-th root function \( z \mapsto z^{1/n} = e^{(\log z)/n} \) (defined with the principal branch of the logarithm) is analytic on the domain of the logarithm and

\[
\frac{d z^{1/n}}{dz} = \frac{1}{n} z^{(1/n)-1}.
\]

Proof: Follows from above.

Exercises: 1.6[6,8,13,14]
Chapter 2

Cauchy Theorem

2.1 Contour Integrals

• Let $h : [a, b] \to \mathbb{C}$. Let $h(t) = u(t) + iv(t)$. The integral of $h$ is defined by

$$\int_{a}^{b} h(t) dt = \int_{a}^{b} u(t) dt + i \int_{a}^{b} v(t) dt.$$ 

• A continuous curve in $\mathbb{C}$ is a continuous map $\gamma : [a, b] \to \mathbb{C}$. A contour is a continuous curve. A curve is continuously differentiable if the map $\gamma$ is continuously differentiable. A curve is piecewise continuously differentiable if it is continuous and consists of a finite number of continuously differentiable curves.

• Usually in analysis a function is called smooth if it is differentiable infinitely many times. However, in this course smooth will mean continuously differentiable. This applies to functions, curves, maps, etc.

• Let $A \subset \mathbb{C}$ be an open set, $\gamma : [a, b] \to A$ be a smooth curve in $A$ and $f : A \to \mathbb{C}$ be continuous. The integral of $f$ along $\gamma$ is defined by

$$\int_{\gamma} f(z) dz = \int_{a}^{b} f(\gamma(t)) \frac{d\gamma(t)}{dt} dt.$$ 

• If $\gamma$ is a closed curve the integral is denoted by $\oint_{\gamma} f(z) dz$. 
• Let $P, Q : A \to \mathbb{R}$ be real valued functions of two variables. Then the line integral along $\gamma$ is defined by

$$\int_{\gamma} P(x, y)dx + Q(x, y)dy = \int_{a}^{b} \left[ P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right] dt.$$ 

**Proposition 2.1.1**  Let $A \subset \mathbb{C}$ be an open set, $\gamma : [a, b] \to A$ be a smooth curve in $A$ and $f : A \to \mathbb{C}$ be continuous. Let $f(z) = u(x, y) + iv(x, y)$. Then

$$\int_{\gamma} f(z)dz = \int_{\gamma} [u(x, y)dx - v(x, y)dy] + i \int_{\gamma} [u(x, y)dy + v(x, y)dx].$$

**Proof:** Easy.

• Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. The **opposite curve** $-\gamma : [a, b] \to \mathbb{C}$ is defined by

$$(-\gamma)(t) = \gamma(a + b - t).$$

It is the same curve traversed in the opposite direction.

• The **zero curve** at the point $z_0$ is the curve $\gamma_0 : [a, b] \to \mathbb{C}$ defined by $\gamma_0(t) = z_0$ for any $t$. It is just the point $z_0$ itself.

• Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$ be two curves. The **sum of two curves** is the curve $(\gamma_1 + \gamma_2) : [a, c] \to \mathbb{C}$ defined by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} 
\gamma_1(t) & \text{if } a \leq t \leq b \\
\gamma_2(t) & \text{if } b \leq t \leq c
\end{cases}$$
2.1. CONTOUR INTEGRALS

Proposition 2.1.2 Let \( f, g : A \to \mathbb{C} \) be continuous, \( a, b \in \mathbb{C} \), and \( \gamma_1, \gamma_2 \) be two piecewise smooth curves. Then

1. \[ \int_{\gamma} (af + bg) = a \int_{\gamma} f + b \int_{\gamma} g, \]
2. \[ \int_{-\gamma} f = - \int_{\gamma} f, \]
3. \[ \int_{\gamma_1 \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f \]

Proof: Easy.

• Let \( \gamma : [a, b] \to \mathbb{C} \) be a piecewise smooth curve. A curve \( \tilde{\gamma} : [c, d] \to \mathbb{C} \) is a **reparametrization** of \( \gamma \) if there is a smooth function \( \alpha : [a, b] \to [c, d] \) such that \( \alpha'(t) > 0 \), \( \alpha(a) = c \), \( \alpha(b) = d \), and \( \tilde{\gamma}(t) = \gamma(\alpha(t)) \).

Proposition 2.1.3 Let \( A \subset \mathbb{C} \) be an open set, \( \gamma : [a, b] \to A \) be a smooth curve in \( A \) and \( f : A \to \mathbb{C} \) be continuous. Then

\[ \int_{\gamma} f = \int_{\tilde{\gamma}} f \]

Proof: Easy.

• The **arc length** of the curve \( \gamma : [a, b] \to \mathbb{C} \) is defined by

\[ l(\gamma) = \int_{a}^{b} \left| \frac{d\gamma}{dt} \right| dt. \]

• More generally, we define the integral

\[ \int_{\gamma} |f(\gamma(t))| \left| \frac{d\gamma}{dt} \right| dt. \]

Then

\[ l(\gamma) = \int_{\gamma} |dz| \]
CHAPTER 2. CAUCHY THEOREM

Proposition 2.1.4  Let $A \subset \mathbb{C}$ be an open set, $\gamma : [a, b] \to A$ be a smooth curve in $A$ and $f : A \to \mathbb{C}$ be continuous. Then

$$\left| \int_{\gamma} f \right| \leq \int_{\gamma} |f| |dz|$$

In particular, suppose that there is $M \geq 0$ such that for any $z$ on the curve $\gamma$ there holds $|f(z)| \leq M$. Then

$$\left| \int_{\gamma} f \right| \leq M l(\gamma).$$

Proof:

Theorem 2.1.1  Fundamental Theorem of Calculus for Contour Integrals. Let $A \subset \mathbb{C}$ be an open set, $\gamma : [a, b] \to A$ be a smooth curve in $A$ and $F : A \to \mathbb{C}$ be analytic in $A$. Then

$$\int_{\gamma} \frac{dF}{dz} dz = F(\gamma(b)) - F(\gamma(0))$$

In particular, if $\gamma$ is a closed curve then

$$\oint_{\gamma} \frac{dF}{dz} dz = 0$$

Proof:

Proposition 2.1.5  Let $A \subset \mathbb{C}$ be an open connected set, and $F : A \to \mathbb{C}$ be analytic in $A$ such that $F'(z) = 0$ for any $z \in A$. Then $F$ is constant on $A$.

Proof:
Theorem 2.1.2  Path Independence Theorem. Let \( A \subset \mathbb{C} \) be an open connected set, and \( f : A \rightarrow \mathbb{C} \) be continuous on \( A \). Then the following are equivalent:

1. Integrals of \( f \) are path independent.
2. Integrals of \( f \) around closed curves are equal to zero.
3. There is an antiderivative \( F \) for \( f \) on \( A \) such that \( F' = f \).

Proof:

Exercises: 2.1[3,5,7,8,12,13,14]
2.2 Cauchy Theorem

- A continuous curve is **simple** if it does not have self-intersection (except possibly the endpoints). A simple closed curve is called a **loop**.
- Let $A$ be an open set, $z_1$ and $z_2$ be two points in $A$ and $\gamma_1, \gamma_2 : [0, 1] \to A$ be two continuous curves in $A$ connecting the points $z_1$ and $z_2$ such that $\gamma_1(0) = \gamma_2(0) = z_1$ and $\gamma_1(1) = \gamma_2(1) = z_2$. Then $\gamma_1$ is **homotopic with fixed endpoints** to $\gamma_2$ in $A$ if there is a continuous functions $H : [0, 1] \times [0, 1] \to A$, such that: $H(0, t) = \gamma_1(t)$, $H(1, t) = \gamma_2(t)$, $H(s, 0) = z_1$ and $H(s, 1) = z_2$. For a fixed $s$ the $\gamma_s(t) = H(s, t)$ is a continuous one-parameter family of continuous curves.
- A homotopy $H : [0, 1] \times [0, 1] \to A$ is smooth if $H(s, t)$ is smooth function of both $s$ and $t$.
- Two homotopic closed curves $\gamma_1$ and $\gamma_2$ are just called **homotopic as closed curves** in $A$. That is, the loop $\gamma_1$ can be continuously deformed to the loop $\gamma_2$.
- A loop in $A$ is **contractible** if it is homotopic to a point (zero loop) in $A$, that is, it can be deformed to a point.
- Let $A \subset \mathbb{C}$ be an open set. Then $A$ is **simply connected** if $A$ is connected and every loop in $A$ is contractible. A simply connected region does not have any holes.

**Theorem 2.2.1 Green’s Theorem.** Let $A$ be a simply connected region with a boundary $\gamma = \partial A$ that is a loop oriented counterclockwise. Let $P, Q$ be smooth functions defined on an open set that contains $A$. Then

$$\oint_{\gamma} [P(x, y)dx + Q(x, y)dy] = \int_{A} \left[ \partial_x Q(x, y) - \partial_y P(x, y) \right] dx \ dy.$$

**Proof:** Vector analysis.
2.2. **CAUCHY THEOREM**

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**Theorem 2.2.2**  
**Cauchy’s Theorem.** The integral of an analytic function in a simply connected domain along any loop is equal to zero. More precisely, let \( A \subset \mathbb{C} \) be a simply connected region, \( \gamma \) be a loop in \( A \) and \( f : A \to \mathbb{C} \) be an analytic function. Then

\[
\oint_{\gamma} f = 0.
\]

**Proof:** Use CR equations.

---

**Theorem 2.2.3**  
**Deformation Theorem.** The integrals of an analytic function along homotopic loops are equal to each other. More precisely, let \( A \subset \mathbb{C} \) be a region, \( f : A \to \mathbb{C} \) be an analytic function and \( \gamma_1 \) and \( \gamma_2 \) be two homotopic loops in \( A \). Then

\[
\oint_{\gamma_1} f = \oint_{\gamma_2} f.
\]

**Proof:** Use Cauchy theorem.

---

**Theorem 2.2.4**  
**Path Independence Theorem.** The integral of an analytic function in a simply connected domain along any contours connecting two points are equal to each other. More precisely, let \( A \subset \mathbb{C} \) be a simply connected region, \( f : A \to \mathbb{C} \) be an analytic function, \( \gamma_1 \) and \( \gamma_2 \) be two contours in \( A \) connecting the points \( z_1 \) and \( z_2 \) in \( A \). Then

\[
\int_{\gamma_1} f = \int_{\gamma_2} f.
\]

**Proof:** Use Cauchy theorem.

---

**Theorem 2.2.5**  
**Antiderivative Theorem.** An analytic function on a simply connected domain has an antiderivative. More precisely, let \( A \subset \mathbb{C} \) be a simply connected region, \( f : A \to \mathbb{C} \) be an analytic function. Then there is a function \( F : A \to \mathbb{C} \) such that \( F'(z) = f(z) \) for any \( z \in A \).

**Proof:** Use path independence theorem.
Proposition 2.2.1  Let $A \subset \mathbb{C}$ be a simply connected region not containing 0. Then there is an analytic function $F : A \to \mathbb{C}$ (called a branch of the logarithm) such that $\exp[F(z)] = z$. This function is unique modulo $2\pi i$.

Proof: In the book.

Exercises: 2.2[2,4,5,9,11]
2.3 Cauchy Integral Formula

- Example. Let \( z_0 \in \mathbb{C} \) and \( \gamma \) be a loop (without selfcrossings) not passing through \( z_0 \). Then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = \begin{cases} 
1 & \text{if } z_0 \text{ is inside } \gamma \text{ and } \gamma \text{ is traversed counterclockwise around } z_0 \\
-1 & \text{if } z_0 \text{ is inside } \gamma \text{ and } \gamma \text{ is traversed clockwise around } z_0 \\
0 & \text{if } z_0 \text{ is outside } \gamma
\end{cases}
\]

- Let \( z_0 \in \mathbb{C} \) and \( \gamma \) be a closed curve not passing through \( z_0 \). Then the index (or winding number) of \( \gamma \) with respect to \( z_0 \) is defined by

\[
I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}.
\]

**Proposition 2.3.1**

1. The index of a closed curve counts the number of times the curve winds around the point. More precisely, let \( z_0 \in \mathbb{C} \), \( \gamma : [0, 2\pi n] \rightarrow \mathbb{C} \) be the circle of radius \( r \) around \( z_0 \), \( \gamma(t) = z_0 + re^{it} \) and \( \text{gamma} : [0, 2\pi n] \rightarrow \mathbb{C} \), \( (-\gamma)(t) = z_0 + re^{-it} \), be the opposite curve. Then

\[
I(\gamma; z_0) = -I(-\gamma; z_0) = n.
\]

2. Indices of homotopic curves are equal. More precisely, let \( z_0 \in \mathbb{C} \) and \( \gamma_1 \) and \( \gamma_2 \) be two homotopic curves in \( \mathbb{C} \setminus \{z_0\} \) not passing through \( z_0 \). Then

\[
I(\gamma_1; z_0) = I(\gamma_2; z_0).
\]

**Proof:** Easy.

**Proposition 2.3.2** The index of a curve is an integer. More precisely, let \( z_0 \in \mathbb{C} \) and \( \gamma : [a, b] \rightarrow \mathbb{C} \) be a smooth closed curve not passing through \( z_0 \). Then \( I(\gamma; z_0) \) is an integer.

**Proof:** Use change of variables.
Proposition 2.3.3  Cauchy’s Integral Formula. Let \( A \) be a region, \( z_0 \in A \), \( \gamma \) be a closed contractible curve in \( A \) not passing through \( z_0 \), and \( f : A \to \mathbb{C} \) be analytic on \( A \). Then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} \, dz = I(\gamma; z_0) f(z_0) .
\]

In particular, if \( \gamma \) is a loop and \( z_0 \) is inside \( \gamma \) then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} \, dz = f(z_0) .
\]

Proof:

1. Let

\[
g(z) = \frac{f(z) - f(z_0)}{z-z_0}
\]

if \( z \neq z_0 \) and \( g(z_0) = f'(z_0) \).

2. Then \( g \) is analytic except at \( z_0 \) and is continuous at \( z_0 \).

3. Therefore \( \int_{\gamma} g = 0 \).

4. The statement follows.

Theorem 2.3.1  Differentiability of Cauchy Integrals. Let \( A \) be a region, \( \gamma \) be a curve in \( A \), \( z \) be a point not on \( \gamma \), and \( f : A \to \mathbb{C} \) be continuous on \( A \). Let \( F : \mathbb{C} \setminus \gamma([a, b]) \) be a function defined by

\[
F(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta-z} \, d\zeta .
\]

Then \( F \) is analytic on \( \mathbb{C} \setminus \gamma([a, b]) \).
Moreover, \( F \) is infinitely differentiable with the \( k \)th derivative given by

\[
F^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta-z)^{k+1}} \, d\zeta,
\]

for \( k = 1, 2, \ldots \).

Proof:
2.3. CAUCHY INTEGRAL FORMULA

**Theorem 2.3.2 Cauchy’s Integral Formula for Derivatives.** Let $A$ be a region, $z_0 \in A$, $\gamma$ be a closed contractible curve in $A$ not passing through $z_0$, and $f : A \to \mathbb{C}$ be analytic on $A$. Then $f$ is infinitely differentiable and

$$
\frac{k!}{2\pi i} \int_\gamma dz \frac{f(z)}{(z-z_0)^{k+1}} = I(\gamma; z_0) f^{(k)}(z_0).
$$

In particular, if $\gamma$ is a loop and $z_0$ is inside $\gamma$ then

$$
\frac{k!}{2\pi i} \int_\gamma dz \frac{f(z)}{(z-z_0)^{k+1}} = f^{(k)}(z_0).
$$

**Proof:** Follows from above.

**Theorem 2.3.3 Cauchy’s Inequalities.** Let $A$ be a region, $z_0 \in A$, $\gamma$ be a circle of radius $R$ centered at $z_0$ such that the whole disk $|z - z_0| < R$ lies in $A$, and $f : A \to \mathbb{C}$ be analytic on $A$ such that for all $z \in \gamma$,

- $|f(z)| \leq M$ for some $M > 0$. Then for any $k = 0, 1, 2, \ldots$

$$
|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M.
$$

**Proof:** Follows from Cauchy integral formula.

**Theorem 2.3.4 Liouville Theorem.** An entire bounded function is constant. More precisely, let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that for any $z \in \mathbb{C}$, $|f(z)| \leq M$ for some $M$.

**Proof:** Follows from above.

**Theorem 2.3.5 Fundamental Theorem of Algebra.** Every non-constant polynomial has at least one root. More precisely, let $n \geq 1$, $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$ and $P(z) = \sum_{k=0}^n a_k z^k$. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

**Proof:** Use Liouville theorem.
Theorem 2.3.6  Morera’s Theorem. Let $f : A \to \mathbb{C}$ be a continuous function such that $\int_{\gamma} f = 0$ for every closed curve in $A$. Then $f$ is analytic on $A$ and has an analytic antiderivative on $A$, that is, $f = F'$ for some analytic function $F$.

Proof:

Corollary 2.3.1  Let $A$ be a region, $z_0 \in A$ and $f : A \to \mathbb{C}$ be continuous on $A$ and analytic on $A \setminus \{z_0\}$. Then $f$ is analytic on $A$.

Proof:

Exercises: 2.4[5,6,8,13,16,17,21]
2.4 Maximum Modulus Theorem and Harmonmic Functions

2.4.1 Maximum Modulus Theorem

**Proposition 2.4.1 Mean value Property.** Let $A$ be a simply connected region, $z_0 \in A$, and $\gamma$ be a circle (in $A$) of radius $r$ centered at $z_0$. Let $f : A \rightarrow \mathbb{C}$ be analytic. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, f(z_0 + re^{i\theta}).$$

**Proof:** Use Cauchy formula. 

**Theorem 2.4.1 Local Maximum Modulus Theorem.** Let $A$ be a region, $z_0 \in A$, and $f : A \rightarrow \mathbb{C}$ be analytic on $A$. Suppose that the modulus $|f|$ has a relative maximum at $z_0$. Then $f$ is constant in some neighborhood of $z_0$.

**Proof:**

1. Assume $|f(z)| \leq |f(z_0)|$ for any $z$ in a disk around $z_0$.
2. Then there is no point $z_1$ such that $|f(z_1)| < |f(z_0)|$.
3. So, $|f(z)| = |f(z_0)|$ for any $z$.
4. Thus $|f|$ is constant.
5. Then by using CR equations show that $f$ is constant.

- Let $A$ be a set. A point $z \in A$ is an **interior** point of $A$ if $A$ contains a neighborhood of $z$. A point $z$ is an **exterior** point of $A$ if the complement $\mathbb{C} \setminus A$ contains a neighborhood of $A$ (that is, if $z$ is an interior point of the complement). A point $z$ is a **boundary point** of $A$ if it is neither interior nor exterior. The set of all interior points of $A$ is the **interior** of $A$, denoted by $\text{int}(A)$. The set of all exterior points of $A$ is the **exterior** of $A$, denoted by $\text{ext}(A)$. The set of all boundary points of $A$ is the **boundary** of $A$, denoted by $\partial A$. The closure of the set $A$ is the set $\text{cl}(A) = A \cup \partial A$. 
Definition 2.4.1

1. Let $A$ be a set. The **closure** of the set $A$ is the set $\bar{A}$ (or $\text{cl}(A)$) consisting of $A$ together with all limit points of $A$ (that is, the limits of all converging sequences in $A$).

2. The **boundary** of $A$ is the set $\partial A$ defined by
   \[
   \partial A = \text{cl}(A) \cap \text{cl}(\mathbb{C} \setminus A),
   \]
   so that $\text{cl}(A) = A \cup \partial A$.

Proposition 2.4.2  Let $A, B \subset \mathbb{C}$.

1. Every set is a subset of its closure, that is, $A \subset \text{cl}(A)$.

2. The closure of any set is closed, that is, $\text{cl}(A)$ is closed.

3. The interior of any set is open, that is, $\text{int}(A)$ is open.

4. A set is closed if and only if it coincides with its closure. That is, the set $A$ is closed if and only if $A = \text{cl}(A)$.

5. A set is open if and only if it coincides with its interior. That is, the set $A$ is open if and only if $A = \text{int}(A)$.

6. A closed set contains the closure of all its subsets. That is, if $A \subset B$ and $B$ is closed, then $\text{cl}(A) \subset B$.

Proof:

Theorem 2.4.2  **Maximum Modulus Principle.** Let $B$ and $f : B \to \mathbb{C}$ be analytic. Let $A \subset B$ be a closed bounded connected region in $B$.

- Then the modulus $|f|$ has a maximum value on $A$, which is attained on the boundary of $A$. If it is also attained in the interior of $A$, then $f$ is constant on $A$.

Proof:
2.4. MAXIMUM MODULUS THEOREM AND HARMONMIC FUNCTIONS

**Lemma 2.4.1 Schwarz Lemma.** Let \( A = D(0; 1) \) be the open unit disk centered at the origin, \( f : A \to \mathbb{C} \) be analytic. Suppose that \( f(0) = 0 \) and \( |f(z)| \leq 1 \) for any \( z \in A \). Then:

1. \( |f'(0)| \leq 1 \) and \( |f(z)| \leq |z| \) for any \( z \in A \).
2. If \( |f'(0)| = 1 \) or if there is a point \( z_0 \neq 0 \) such that \( |f(z_0)| = |z_0| \), then \( f(z) = cz \) for all \( z \in A \) with some constant \( c \) such that \( |c| = 1 \).

**Lemma 2.4.2 Lindelöf Principle.** Let \( A = D(0; 1) \) be the open unit disk centered at the origin, \( f : A \to f(A) \), \( g : A \to g(A) \) be analytic. Suppose that \( g \) is a bijection, \( f(A) \subset g(A) \), \( f(0) = g(0) \). Then:

1. \( |f'(0)| \leq |g'(0)| \) and \( |f(z)| \leq |z| \) for any \( z \in A \).
2. The image of an open disk \( D(0, r) \) of radius \( r < 1 \) centered at 0 under \( f \) is contained in its image under \( g \), that is, \( f(D(0, r)) \subset g(D(0, r)) \).

**Proof:**

---

### 2.4.2 Harmonic Functions

**Proposition 2.4.3** Let \( A \) be a region, \( u \in C^2(A) \) be a harmonic function in \( A \). Then:

1. \( u \in C^\infty(A) \).
2. For any \( z_0 \in A \) there is a neighborhood \( V \) of \( z_0 \) and an analytic function \( f : V \to \mathbb{C} \) such that \( u = \text{Re } f \).
3. If \( A \) is simply connected then there is an analytic function \( g : A \to \mathbb{C} \) such that \( u = \text{Re } g \).

**Proof:** By construction.

---

If \( f = u + iv \) is analytic, then \( u \) and \( v \) are **harmonic conjugates**.
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Theorem 2.4.3 Mean Value Property for Harmonic Functions. Let A be a simply connected region, \( z_0 = x_0 + iy_0 \in A \) such that A contains the circle of radius \( r \) centered at \( z_0 \), and \( u \) be a harmonic function on A. Then

\[
u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ u(x_0 + r \cos \theta, y_0 + r \sin \theta).
\]

Proof:

Theorem 2.4.4 Local Maximum Principle for Harmonic Functions. Let A be a region, \( u : A \to \mathbb{R} \) be harmonic on A. Suppose that \( u \) has a relative maximum at \( z_0 \in A \). Then \( u \) is constant in a neighborhood of \( z_0 \).

Proof:

Theorem 2.4.5 Global Maximum Principle for Harmonic Functions. Let B and \( u : B \to \mathbb{R} \) be continuous harmonic function. Let \( A \subset B \) be a closed bounded connected region in B and \( M \) be the maximum of \( u \) on the boundary of \( A \) and \( m \) be the minimum of \( u \) on the boundary of \( A \). Then:

1. \( m \leq u(x, y) \leq M \) for all \( (x, y) \in A \).
2. If \( u(x, y) = M \) for some interior point \( (x, y) \in A \) or if \( u(x, y) = m \) for some interior point \( (x, y) \in A \), then \( u \) is constant on \( A \).

Proof:

2.4.3 Dirichlet Boundary Value Problem

Theorem 2.4.6 Uniqueness of Solution of Dirichlet Problem. Let A be a closed bounded region and \( u_0 : \partial A \to \mathbb{R} \) be a continuous function on the boundary of A. If there is a continuous function \( u : A \to \mathbb{R} \) that is harmonic in the interior of A and equals \( u_0 \) on the boundary of A, then it is unique.

Proof:
Theorem 2.4.7 Poisson’s Formula. Let $D(0; r)$ be an open disk of radius $r$ centered at the origin and $A = \text{cl}(D(0; r))$ be its closure. Let $u : A \to \mathbb{R}$ be a continuous function that is harmonic in the interior of $A$. Then for $\rho < r$ and any $\varphi$

$$u(\rho e^{i\varphi}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} d\theta \frac{u(re^{i\theta})}{r^2 - 2\rho r \cos(\varphi - \theta) + \rho^2}.$$ 

Alternatively, let $\gamma = \partial A$ be the circle of radius $r$ centered at the origin. Then for any $z \in A$

$$u(z) = \frac{1}{2\pi} \oint_{\gamma} d\zeta \frac{u(\zeta)}{r^2 - |z|^2} \frac{r^2 - |\zeta|^2}{|\zeta - z|^2}$$

Proof:

Exercises: 2.5[1,6,7,10,12,18]
Chapter 3

Series Representation of Analytic Functions

3.1 Series of Analytic Functions

- **Definition 3.1.1**

1. Let \( (z_n) \) be a sequence in \( \mathbb{C} \) and \( z_0 \in \mathbb{C} \). The sequence \( z_n \) **converges** to \( z_0 \) (denoted by \( z_n \rightarrow z_0 \), or \( \lim_{n \rightarrow \infty} z_n = z_0 \)) if for any \( \varepsilon > 0 \) there is \( N \in \mathbb{Z}_+ \) such that for any \( n \in \mathbb{Z}_+ \) if \( n \geq N \), then \( |z_n - z_0| < \varepsilon \).

2. Let \( \sum_{k=1}^{\infty} a_k \) be a series and \( (s_n) \) be the corresponding sequence of partial sums defined by \( s_n = \sum_{k=1}^{n} a_k \), and \( S \in \mathbb{C} \). The series \( \sum_{k=1}^{\infty} a_k \) **converges** to \( S \) (denoted by \( \sum_{k=1}^{\infty} a_k = S \)) if \( s_n \rightarrow S \).

- The limit of a convergent sequence is unique.

- A sequence \( (z_n) \) is **Cauchy** if for any \( \varepsilon > 0 \) there is \( N \in \mathbb{Z}_+ \) such that for any \( n, m \in \mathbb{Z}_+ \)

  if \( n, m \geq N \), then \( |z_n - z_m| < \varepsilon \).

- A sequence converges if and only if it is Cauchy.
• **Cauchy Criterion for Series.** A series \( \sum_{k=1}^{\infty} a_k \) converges if and only if for any \( \varepsilon > 0 \) there is \( N \in \mathbb{Z}_+ \) such that: for any \( n, m \in \mathbb{Z}_+ \)

\[
\text{if } n, m \geq N, \text{ then } |s_n - s_m| < \varepsilon ,
\]

or

\[
\text{if } n \geq N, \text{ then } \left| \sum_{k=n+1}^{n+m} a_k \right| < \varepsilon ,
\]

• In particular, if the series \( \sum_{k=1}^{\infty} a_k \) converges, then the sequence \((a_k)\) converges to zero, \( a_k \to 0 \).

• A series \( \sum_{k=1}^{\infty} a_k \) converges absolutely if the series \( \sum_{k=1}^{\infty} |a_k| \) converges.

**Proposition 3.1.1**  If a series converges absolutely, then it converges.

*Proof:*

**Proposition 3.1.2**  Tests for Convergence.

1. **Geometric series.** If \(|r| < 1\), then

\[
\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} .
\]

If \(|r| \geq 1\), then \( \sum_{k=0}^{\infty} r^k \) diverges.

2. **Comparison test.** If the series \( \sum_{k=1}^{\infty} b_k \) converges and \( 0 \leq a_k \leq b_k \), then the series \( \sum_{k=1}^{\infty} a_k \) converges. If the series \( \sum_{k=1}^{\infty} c_k \) diverges and \( 0 \leq c_k \leq d_k \), then the series \( \sum_{k=1}^{\infty} d_k \) diverges.

3. **\( p \)-series.** If \( p > 1 \), then the series \( \sum_{k=1}^{\infty} k^{-p} \) converges. If \( p \leq 1 \), then \( \sum_{k=1}^{\infty} k^{-p} \) diverges to \( \infty \).

4. **Ratio test.** Suppose that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \) exist. If \( L < 1 \), then \( \sum_{k=1}^{\infty} a_k \) converges absolutely. If \( L > 1 \), then \( \sum_{k=1}^{\infty} c_k \) diverges. if \( L = 1 \), then the test is inconclusive.

5. **Root test.** Suppose that \( \lim_{n \to \infty} |a_n|^{1/n} = L \) exist. If \( L < 1 \), then \( \sum_{k=1}^{\infty} a_k \) converges absolutely. If \( L > 1 \), then \( \sum_{k=1}^{\infty} c_k \) diverges. if \( L = 1 \), then the test is inconclusive.
3.1. SERIES OF ANALYTIC FUNCTIONS

**Proof:** In calculus.

- Let $A$ be a set and $f_n : A \to \mathbb{C}$ be a sequence of functions on $A$. The sequence $(f_n)$ converges **pointwise** if it converges for any point $z \in A$.

**Definition 3.1.2 Uniform Convergence.** Let $A$ be a set, $f : A \to \mathbb{C}$, and $f_n : A \to \mathbb{C}$ be a sequence of functions on $A$. The sequence $(f_n)$ converges uniformly to $f$ (denoted by “$f_n \to f$ uniformly on $A$”) if for any $\varepsilon > 0$ there is an $N \in \mathbb{Z}_+$ such that for any $n \in \mathbb{Z}_+$ and any $z \in A$

$$|f_n(z) - f(z)| < \varepsilon .$$

- Similarly a series $\sum_{k=1}^{\infty} g_n$ converges uniformly on $A$ if the sequence of partial sums converges uniformly.

- Uniform convergence implies pointwise convergence.

**Theorem 3.1.1 Cauchy Criterion.** Let $A \subset \mathbb{C}$, and $g_n, f_n : A \to \mathbb{C}$ be sequences of functions on $A$.

1. Then $f_n$ converges uniformly if and only if for any $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for any $n, m \in \mathbb{Z}_+$, and any $z \in A$

$$|f_n(z) - f_{n+m}(z)| < \varepsilon .$$

2. The series $\sum_{k=1}^{\infty} g_n$ converges uniformly on $A$ if and only if for any $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for any $n, m \in \mathbb{Z}_+$, and any $z \in A$

$$\left| \sum_{k=n+1}^{n+m} g_k(z) \right| < \varepsilon .$$

**Proof:**
**Proposition 3.1.3**

1. The limit of a uniformly convergent sequence of continuous functions is continuous. Let \( A \subset \mathbb{C} \), \( f : A \rightarrow \mathbb{C} \) and \( f_n : A \rightarrow \mathbb{C} \) be a sequence of functions on \( A \). Suppose that all functions \( f_n \) are continuous on \( A \) and \( f_n \rightarrow f \) uniformly on \( A \). Then the function \( f \) is continuous on \( A \).

2. The sum of a uniformly convergent series of continuous functions is continuous. That is, if \( g = \sum_{k=1}^{\infty} g_k \) uniformly on \( A \) and \( g_k \) are continuous on \( A \) then \( g \) is continuous on \( A \).

**Proof:**

---

**Theorem 3.1.2** **Weierstrass M Test.** Let \( A \subset \mathbb{C} \) and \( g_n : A \rightarrow \mathbb{C} \). Let \( M_n \) be a sequence in \( \mathbb{R} \) such that:

1. \( |g_n(z)| \leq M \) for any \( z \in A \),

2. \( \sum_{k=1}^{\infty} M_k \) converges.

Then the series \( \sum_{k=1}^{\infty} g_k \) converges absolutely and uniformly on \( A \).

**Proof:** By Cauchy criterion.

---

**Theorem 3.1.3** **Analytic Convergence Theorem.**

1. Let \( A \) be an open set, \( f : A \rightarrow \mathbb{C} \), \( f_n : A \rightarrow \mathbb{C} \) be a sequence of functions on \( A \). Suppose that \( f_n \) are analytic on \( A \) and \( f_n \rightarrow f \) uniformly on every closed disk \( D \) contained in \( A \). Then \( f \) is analytic on \( A \). Moreover, \( f_n' \rightarrow f' \) uniformly on every closed disk in \( A \).

2. Let \( A \) be an open set, \( g : A \rightarrow \mathbb{C} \), \( g_n : A \rightarrow \mathbb{C} \) be a sequence of functions on \( A \). Suppose that \( g_n \) are analytic on \( A \) and \( g = \sum_{k=1}^{\infty} g_k \) uniformly on every closed disk \( D \) contained in \( A \). Then \( g \) is analytic on \( A \). Moreover, \( g' = \sum_{k=1}^{\infty} g_k' \) uniformly on every closed disk in \( A \).

**Proof:**
Proposition 3.1.4 Let $A$ be a region, $\gamma : [a, b] \to A$ be a curve in $A$, $f : A \to \mathbb{C}$, and $f_n : A \to \mathbb{C}$. Suppose that $f_n$ are continuous on the curve $\gamma([a, b])$ and $f_n \to f$ uniformly on $\gamma([a, b])$. Then

$$\int_{\gamma} f_n \to \int_{\gamma} f.$$ 

Similarly, if the series $\sum_{k=1}^{\infty} g_k$ converges uniformly on $\gamma$, then

$$\int_{\gamma} \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \int_{\gamma} g_k.$$ 

Proof:

Exercises: 3.1[2,4,5,12,19]
3.2 Power Series and Taylor Theorem

- A power series is a series of the form
  \[ \sum_{k=0}^{\infty} a_k (z - z_0)^k. \]

**Lemma 3.2.1 Abel-Weierstrass Lemma.** Let \((a_n)_{n=1}^{\infty} \in \mathbb{C}\) and \(r_0, M \in \mathbb{R}\) such that \(|a_n| r_0^n \leq M\) for any \(n \in \mathbb{Z}_+\). Then for any \(r\) such that \(0 \leq r < r_0\) the series \(\sum_{k=0}^{\infty} a_k (z - z_0)^k\) converges uniformly and absolutely on the closed disk \(\text{cl}(D(z_0, r)) = \{ z \in \mathbb{C} \mid |z - z_0| \leq r \}\).

**Proof:**

**Theorem 3.2.1 Power Series Convergence Theorem.**
Let \(\sum_{k=0}^{\infty} a_k (z - z_0)^k\) be a power series.

1. There is a unique real number \(R \geq 0\), (which can also be \(\infty\)), called the **radius of convergence**, such that the series converges on the open disk \(D(z_0, R) = \{ z \in \mathbb{C} \mid |z - z_0| < R \}\) of radius \(R\) centered at \(z_0\) and diverges in the exterior of this disk \(\text{ext}(D(z_0, R)) = \{ z \in \mathbb{C} \mid |z - z_0| > R \}\).

2. The convergence is uniform and absolute on any closed disk \(\text{cl}(D(z_0, r)) = \{ z \in \mathbb{C} \mid |z - z_0| \leq r \}\) of radius \(r\) smaller than \(R\), \(r < R\).

3. The series may diverge or converge on the boundary \(\partial D(z_0, R)\) of the disk, i.e. the circle \(|z - z_0| = R\), called the **circle of convergence**. There is a least one point on this circle such that the series diverges.

**Proof:**

**Theorem 3.2.2 Analyticity of Power Series.** Every power series is analytic inside its circle of convergence.

**Proof:**
Theorem 3.2.3 Differentiation of Power Series. Let $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a power series and $A$ be the inside of its circle of convergence. Let $f : A \rightarrow \mathbb{C}$ be the analytic function defined by

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$ 

1. Then for any $z \in A$

$$f'(z) = \sum_{k=1}^{\infty} ka_k(z - z_0)^{k-1} = \sum_{k=0}^{\infty} (k + 1)a_{k+1}(z - z_0)^k.$$

2. Moreover, for any $k = 0, 1, \ldots$,

$$a_k = \frac{f^{(k)}(z_0)}{k!},$$

and, therefore,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

This series is called the Taylor series of $f$ around $z_0$.

**Proof:**

---

Theorem 3.2.4 Uniqueness of Power Series. Power series expansions around the same point are unique.

**Proof:**

---

Proposition 3.2.1 Let $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a power series.

1. Ratio test.

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

2. Root test.

$$R = \lim_{n \to \infty} |a_n|^{1/n}.$$
Proof:

\textbf{Theorem 3.2.5 \ Taylor's Theorem.} Let $A$ be an open set and $f : A \to \mathbb{C}$ be analytic on $A$. Let $z_0 \in A$ and $A_\rho = D(z_0, \rho)$ be the open disk of radius $\rho$ around $z_0$. Let $r$ be the largest value of the radius $\rho$ such that $A_r \subset A$. Then for every $z \in A_r$ the series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

converges.

Proof:

\textbf{Corollary 3.2.1} Let $A$ be an open set and $f : A \to \mathbb{C}$ be analytic on $A$. Then $f$ is analytic on $A$ if and only if for each point $z_0 \in A$ the Taylor series of $f$ around $z_0$ has a non-zero radius of convergence and $f$ is equal to the sum of that series.

Proof:

- Let $A$ be an open set, $z_0 \in A$ and $f : A \to \mathbb{C}$ be analytic on $A$. We say that $f$ has a \textbf{zero of order} $n$ at $z_0$ if

$$f^{(k)}(z_0) = 0 \quad \text{for } k = 0, 1, \ldots, (n - 1), \quad \text{and } f^{(n)}(z_0) \neq 0.$$

- If $f$ has a zero of order $n$ at $z_0$ then

$$f(z) = (z - z_0)^n g(z),$$

for some analytic function $g$ such that $g$ does not have zero at $z_0$, $g(z_0) \neq 0$.

- If a function $f$ has a zero at $z_0$ and does not have zeroes in a neighborhood of $z_0$ then the zero of $f$ is called \textbf{isolated}.

\textbf{Proposition 3.2.2} Let $A$ be an open set and $f : A \to \mathbb{C}$ be analytic on $A$. Suppose that $f$ has a zero at a point $z_0 \in A$. Then either $f$ is identically zero or $f$ has an isolated zero of some order $n$ at $z_0$. 
Proof:

Proposition 3.2.3  Local Isolation of Zeroes. Let $A$ be an open set, $z_0 \in A$, and $f : A \to \mathbb{C}$ be analytic on $A$. Let $z_k$ be a sequence of zeroes of $f$ in $A$ converging to $z_0$, $z_k \to z_0$. Then $f$ is identically zero in $A$.

Proof:

Exercises: 3.2[1c,2b,2d,12,17,22]
3.3  Laurent Series and Classification of Singularities

Theorem 3.3.1  Laurent Expansion. Let \( 0 \leq r_1 < r_2, \ z_0 \in \mathbb{C} \), and \( A = \{ z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2 \} \). (It is possible that \( r_1 = 0 \) and \( r_2 = \infty \)). Let \( f : A \to \mathbb{C} \) be an analytic function. Then:

1. There is an expansion (called Laurent series around \( z_0 \) in \( A \))
   \[
   f(z) = \sum_{n=\infty}^{\infty} \sum_{n=0} \sum_{-\infty}^{\infty} a_n (z-z_0)^n.
   \]

2. The series converges absolutely on \( A \) and uniformly on every set \( B = \{ z \in \mathbb{C} \mid \rho_1 \leq |z - z_0| \leq \rho_2 \} \) with \( r_1 < \rho_1 < \rho_2 < r_2 \).

3. The coefficients are given by
   \[
   a_n = \frac{1}{2\pi i} \oint_{\gamma} d\zeta \frac{f(z)}{(z-z_0)^{n+1}}, \quad n \in \mathbb{Z},
   \]
   where \( \gamma \) is a circle centered at \( z_0 \) of radius \( r \) with \( r_1 < r < r_2 \).

4. The Laurent series is unique.

Proof:

- The Laurent series can also be written in the form
  \[
  f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{k=1}^{\infty} a_{-k} (z-z_0)^{-k}.
  \]

- Example.
### Definition 3.3.1

Let $A$ be a region and $z_0 \in A$. Let $f$ be a function and $a_n$ be the coefficients of the Laurent series around $z_0$.

1. If $f$ is analytic on a deleted neighborhood of $z_0$, then $z_0$ is called an **isolated singularity**.

2. If there are only a finite number of coefficients $a_n$ with negative $n$, then $z_0$ is a **pole** of $f$. The **order** of the pole is the largest integer $k$ such that $b_k \neq 0$. A pole of order one is called a **simple pole**.

3. If there are infinitely many coefficients $a_n$ with negative $n$, then $z_0$ is called an **essential singularity**.

4. The coefficient $a_{-1}$ is called the residue of $f$ at $z_0$.

5. If $a_n = 0$ for any negative $n$, then $z_0$ is called a **removable singularity**.

6. If the only singularities of a function $f$ are poles, then $f$ is called **meromorphic**.

7. If $z_0$ is a pole of order $k$, then the principal part of $f$ at $z_0$ is

$$
\sum_{n=0}^{k} a_{-n}(z-z_0)^{-n} = \frac{a_k}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{(z-z_0)}. 
$$

- If $z_0$ is a removable singularity, then by defining $f(z_0) = a_0$, $f$ becomes analytic at $z_0$.

### Proposition 3.3.1

Let $f$ be an analytic function with an isolated singularity at $z_0$. Let $\gamma$ be a circle centered at $z_0$ of a sufficiently small radius so that $f$ is analytic on an annulus containing $\gamma$. Then

$$
a_{-1} = \frac{1}{2\pi i} \oint_{\gamma} df(z).
$$

**Proof:**

\[ \blacksquare \]
Proposition 3.3.2 \ Let \( z_0 \in \mathbb{C} \) Let \( f \) and \( g \) be analytic in a neighborhood of \( z_0 \) with zeroes of order \( n \) and \( k \) respectively. Let \( h = f / g \). Then

1. If \( k > n \), then \( h \) has a pole of order \( (k - n) \) at \( z_0 \).
2. If \( k = n \), then \( h \) has a removable singularity at \( z_0 \).
3. If \( k < n \), then \( h \) has a removable singularity at \( z_0 \), and by defining \( h(z_0) = 0 \) we get an analytic function \( h \) with zero of order \( (n - k) \) at \( z_0 \).

Proof:

Proposition 3.3.3 \ Let \( z_0 \in \mathbb{C} \). Let \( f \) be an analytic function with an isolated singularity at \( z_0 \).

1. \( z_0 \) is a removable singularity if and only if one of the following holds:
   
   (a) \( f \) is bounded in a deleted neighborhood of \( z_0 \).
   (b) \( f \) has a limit at \( z_0 \).
   (c) \( \lim_{z \to z_0} (z - z_0)f(z) = 0 \).

2. \( z_0 \) is a simple pole if and only if \( \lim_{z \to z_0} (z - z_0)f(z) \) exists and is not equal to 0.

3. \( z_0 \) is a pole of order \( m \) (or a removable singularity) if and only if one of the following holds
   
   (a) There are \( M > 0 \) and \( k \in \mathbb{Z}_+ \) such that for any \( z \) in a deleted neighborhood of \( z_0 \) \( |f(z)| \leq \frac{M}{|z - z_0|^k} \).
   (b) for any \( k \geq m \), \( \lim_{z \to z_0} (z - z_0)^{k+1}f(z) = 0 \).
   (c) for any \( k \geq m \), \( \lim_{z \to z_0} (z - z_0)^k f(z) \) exists.

4. \( z_0 \) is a pole of order \( m \geq 1 \) if and only if there is an analytic function \( g \) on a neighborhood of \( z_0 \) such that \( g(z_0) \neq 0 \) and for any \( z \neq z_0 \)
   
   \[
   f(z) = \frac{g(z)}{(z - z_0)^m}.
   \]
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Proof:

• For a simple pole the residue is equal to

\[ a_{-1} = \lim_{z \to z_0} (z - z_0)f(z). \]

**Theorem 3.3.2 Picard Theorem.** Let \( z_0 \in \mathbb{C} \) and \( U \) be a deleted neighborhood of \( z_0 \). Let \( f \) be an analytic function with an essential singularity at \( z_0 \). Then for all \( w \in \mathbb{C} \) (except possibly one value), the equation \( f(z) = w \) has infinitely many solutions in \( U \).

Proof:

• **Theorem 3.3.3 Casorati-Weierstrass Theorem** Let \( z_0, w \in \mathbb{C} \). Let \( f \) be an analytic function with an essential singularity at \( z_0 \). Then there is a sequence \( (z_n) \) in \( \mathbb{C} \) such that \( z_n \to z_0 \) and \( f(z_0) \to w \).

Proof:

• **Exercises:** 3.3[4,7,8,10,18,19]
Chapter 4

Calculus of Residues

4.1 Calculus of Residues

4.1.1 Removable Singularities

- **Proposition 4.1.1** Let $f$ and $g$ be analytic in a deleted neighborhood of $z_0 \in \mathbb{C}$. Suppose that $f$ and $g$ have zeros of the same order at $z_0$. Then the function $h = f/g$ has a removable singularity at $z_0$.

**Proof:**

- Examples.

4.1.2 Simple Poles.

- **Proposition 4.1.2** Let $f$ and $g$ be analytic in a deleted neighborhood of $z_0 \in \mathbb{C}$. Suppose that $f(z_0) \neq 0$ and $g'(z_0) \neq 0$. Then the function $h = f/g$ has a simple pole at $z_0$ and

\[
\text{Res}(h, z_0) = \frac{f(z_0)}{g'(z_0)}.
\]

**Proof:**
CHAPTER 4. CALCULUS OF RESIDUES

Proposition 4.1.3 \textit{Let }f\textit{ and }g\textit{ be analytic in a deleted neighborhood of }z_0 \in \mathbb{C}. \textit{Suppose that }f\textit{ has a zero of order }k\textit{ at }z_0\textit{ and }g\textit{ has a zero of order } (k + 1)\textit{ at }z_0\textit{. Then the function }h = f/g\textit{ has a simple pole at }z_0\textit{ and}

$$\text{Res}(h, z_0) = (k + 1) \frac{f^{(k)}(z_0)}{g^{(k+1)}(z_0)}.$$ 

\textit{Proof:} 

Examples.

4.1.3 Double Poles.

Proposition 4.1.4 \textit{Let }f\textit{ and }g\textit{ be analytic in a deleted neighborhood of }z_0 \in \mathbb{C}. \textit{Suppose that }f(z_0) \neq 0\textit{ and }g(z_0) = g'(z_0) = 0\textit{ and }g''(z_0) \neq 0\textit{. Then the function }h = f/g\textit{ has a double pole at }z_0\textit{ and}

$$\text{Res}(h, z_0) = 2 \frac{f'(z_0)}{g''(z_0)} - 2 \frac{f(z_0)g'''(z_0)}{3 [g''(z_0)]^2}.$$ 

\textit{Proof:} 

Examples.

Proposition 4.1.5 \textit{Let }f\textit{ and }g\textit{ be analytic in a deleted neighborhood of }z_0 \in \mathbb{C}. \textit{Suppose that }f(z_0) \neq 0\textit{ and }g(z_0) = g'(z_0) = g''(z_0) = 0\textit{ and }g'''(z_0) \neq 0\textit{. Then the function }h = f/g\textit{ has a double pole at }z_0\textit{ and}

$$\text{Res}(h, z_0) = 3 \frac{f''(z_0)}{g'''(z_0)} - 3 \frac{f'(z_0)g^{(4)}(z_0)}{2 [g'''(z_0)]^2}.$$ 

\textit{Proof:} 

Examples.
4.1.4 Higher-Order Poles.

**Proposition 4.1.6** Let \( f \) be analytic in a deleted neighborhood of \( z_0 \in \mathbb{C} \). Let \( k \) be the smallest non-negative integer such that the limit \( \lim_{z \to z_0} (z - z_0)^k f(z) \) exists. Let \( \varphi \) be a function in the deleted neighborhood of \( z_0 \) defined by \( \varphi(z) = (z - z_0)^k f(z) \). Then \( f \) has a pole of order \( k \) at \( z_0 \) and \( \varphi \) has a removable singularity at \( z_0 \) and

\[
\text{Res}(f, z_0) = \frac{\varphi^{(k-1)}(z_0)}{(k-1)!}.
\]

**Proof:**

\[
\begin{vmatrix}
\frac{g^{(k)}}{k!} & 0 & \ldots & 0 & \frac{f(z_0)}{0!} \\
\frac{g^{(k+1)}}{(k+1)!} & \frac{g^{(k)}}{k!} & \ldots & 0 & \frac{f'(z_0)}{1!} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{g^{(2k-1)}}{(2k-1)!} & \frac{g^{(2k-2)}}{(2k-2)!} & \ldots & \frac{g^{(k+1)}}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!}
\end{vmatrix}
\]

**Proposition 4.1.7** Let \( f \) and \( g \) be analytic in a deleted neighborhood of \( z_0 \in \mathbb{C} \). Suppose that \( f(z_0) \neq 0 \) and \( g(z_0) = g'(z_0) = \cdots = g^{(k-1)}(z_0) = 0 \) and \( g^{(k)}(z_0) \neq 0 \). Then the function \( h = f/g \) has a pole of order \( k \) at \( z_0 \) and

\[
\text{Res}(h, z_0) = \left( \frac{k!}{g^{(k)}(z_0)} \right)^k \det
\]

**Proof:**

- Summary.
- Examples.

4.1.5 Essential Singularities.

- Exercises: 4.1[2,5,8,11,14]
4.2 Residue Theorem

**Theorem 4.2.1** Residue Theorem. Let $A$ be a region and $z_1, \ldots, z_n \in A$ be $n$ distinct points in $A$. Let $B = A - \{z_1, \ldots, z_n\}$ and $f : B \rightarrow \mathbb{C}$ be analytic with isolated singularities at $z_1, \ldots, z_n$. Let $\gamma$ be a closed contractible curve in $A$ that does not pass through any points $z_1, \ldots, z_n$. Then

$$\oint_{\gamma} dz \, f(z) = 2\pi i \sum_{k=1}^{n} I(\gamma, z_k) \text{Res}(f, z_k),$$

where $I(\gamma, z_k)$ is the winding number of $\gamma$ with respect to $z_k$.

In particular, if $\gamma$ is a loop traversed counterclockwise and $z_1, \ldots, z_p$ are the points inside $\gamma$, then

$$\oint_{\gamma} dz \, f(z) = 2\pi i \sum_{k=1}^{p} \text{Res}(f, z_k).$$

**Proof:**

**Definition 4.2.1** Let $F(z) = f(1/z)$. Then:

1. $f$ has a pole of order $k$ at $\infty$ if $F$ has a pole of order $k$ at 0;
2. $f$ has a zero of order $k$ at $\infty$ if $F$ has a zero of order $k$ at 0;
3. The residue of $f$ at $\infty$ is defined by

$$\text{Res}(f, \infty) = -\text{Res}[F(z)/z^2; 0].$$

The $R$-neighborhood of $\infty$ is the set $\{z \mid |z| > R\}$ for some $R > 0$.

**Proposition 4.2.1** Let $f$ be analytic in a $R_0$-neighborhood of infinity. Let $\gamma$ be a sufficiently large circle centered at 0 with radius $R$, $R > R_0$, traversed once counterclockwise. Then

$$\oint_{\gamma} dz f(z) = -2\pi i \text{Res}(f, \infty).$$

**Proof:**
Proposition 4.2.2  Let $\gamma$ be a simple closed curve in $\mathbb{C}$ traversed once counterclockwise. Let $f$ be analytic along $\gamma$ and have only finitely many isolated singularities, $z_1, \ldots, z_n$, outside $\gamma$. Then

$$\oint_{\gamma} dz \, f(z) = -2\pi i \left\{ \sum_{k=1}^{n} \text{Res}(f, z_k) + \text{Res}(f, \infty) \right\}.$$

Proof:

Exercises: 4.2[4,5,7,12,15]
4.3 Evaluation of Definite Integrals

4.3.1 Rational Functions of Sine and Cosine

**Proposition 4.3.1** Let \( R(\cos \theta, \sin \theta) \) be a rational function of \( \cos \theta \) and \( \sin \theta \) for any \( \theta \in [0, 2\pi] \). Then

\[
\int_{0}^{2\pi} d\theta \, R(\cos \theta, \sin \theta) = 2\pi \sum_{|z_k| < 1} \text{Res}(f, z_k),
\]

where

\[
f(z) = \frac{1}{z} R \left[ \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2} \left( z - \frac{1}{z} \right) \right].
\]

**Proof:**

\[\blacksquare\]

**Example.**

4.3.2 Improper Integrals

**Proposition 4.3.2** Let \( a \in \mathbb{R} \) and \( f, g : [a, \infty) \rightarrow \mathbb{C} \) are two complex-valued functions of a real variable. Suppose that \( |f(x)| \leq |g(x)| \) for any \( x \in [a, \infty) \) and the integral \( \int_{a}^{\infty} dx \, |g(x)| \) converges. Then the integral \( \int_{a}^{\infty} dx \, f(x) \) also converges and

\[
\left| \int_{a}^{\infty} dx \, f(x) \right| \leq \int_{a}^{\infty} dx \, |g(x)|.
\]

This also holds if, formally, \( a = -\infty \).

**Proof:**

\[\blacksquare\]

**Remark.** The convergence of the integral \( \int_{-\infty}^{\infty} dx \, f(x) \) means that both integrals \( \int_{a}^{\infty} dx \, f(x) \) and \( \int_{-\infty}^{a} dx \, f(x) \) exist for any \( a \), that is,

\[
\int_{-\infty}^{\infty} dx \, f(x) = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} dx \, f(x).
\]
4.3. EVALUATION OF DEFINITE INTEGRALS

- If the integral $\int_{-\infty}^{\infty} dx \, f(x)$ exists then it is equal to the symmetric limit

$$\int_{-\infty}^{\infty} dx \, f(x) = \lim_{R \to \infty} \int_{-R}^{R} dx \, f(x).$$

- However, the existence of this symmetric limit does not imply the convergence of the integral. If this limit exists, it is called the (Cauchy) principal value integral and denoted by $P \int_{-\infty}^{\infty} f(x)$, that is, by definition,

$$P \int_{-\infty}^{\infty} dx \, f(x) = \lim_{R \to \infty} \int_{-R}^{R} dx \, f(x).$$

**Proposition 4.3.3** Let $f : \mathbb{R} \to \mathbb{C}$. Suppose that the limit $\lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} dx \, f(x)$ exists. Then the integral $\int_{-\infty}^{\infty} dx \, f(x)$ exists and

$$\int_{-\infty}^{\infty} dx \, f(x) = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} dx \, f(x).$$

**Proof:**

**Proposition 4.3.4** Let $f$ be an function that is analytic on an open set containing the closed upper half plane except for a finite number of isolated singularities which do not lie on the real axis. Suppose that there are $M, R_0 > 0$ and $p > 1$ such that for any $z$ in the upper half plane, if $|z| > R_0$, then

$$|f(z)| \leq \frac{M}{|z|^p}.$$ 

Then

$$\int_{-\infty}^{\infty} dx \, f(x) = 2\pi i \sum_{\lim z_k > 0} \text{Res}(f, z_k).$$

**Proof:**
Proposition 4.3.5  Let $f$ be an function that is analytic on an open set containing the closed lower half plane except for a finite number of isolated singularities which do not lie on the real axis. Suppose that there are $M, R_0 > 0$ and $p > 1$ such that for any $z$ in the lower half plane, if $|z| > R_0$, then

$$|f(z)| \leq \frac{M}{|z|^p}.$$  

Then

$$\int_{-\infty}^{\infty} dx \ f(x) = -2\pi i \sum_{\text{Im} \ z_k < 0} \text{Res}(f, z_k).$$

Proof:

\[ \blacksquare \]

- Let $P_n$ and $Q_m$ be polynomials such that $m \geq n + 2$. Suppose that $Q_m$ does not have any zeroes on the real line. Let $f = P/Q$. Then both of the above formulas hold.

- Example.

4.3.3 Fourier Transform.

- Let $f : \mathbb{R} \to \mathbb{C}$. The Fourier transform of the function $f$ is a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ e^{-i\omega x} f(x).$$

The Fourier cosine transform and Fourier sine transform are defined by

$$\hat{f}_c(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ \cos(\omega x) f(x).$$

$$\hat{f}_s(\omega) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ \sin(\omega x) f(x).$$

Obviously,

$$\hat{f} = \hat{f}_c + i\hat{f}_s$$
• The inverse Fourier transform is defined by

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \ e^{i\omega x} \hat{f}(\omega). \]

**Proposition 4.3.6** Let \( f \) be analytic on an open set containing the closed upper half plane except for a finite number of isolated singularities not lying on the real axis. Suppose that \( f(z) \to 0 \) as \( z \to \infty \) in the upper half plane, that is, for any \( \varepsilon > 0 \) there is \( R > 0 \) such that if \( |z| > R \) in the upper half plane, then \( |f(z)| < \varepsilon \). Then for \( \omega < 0 \)

\[ \int_{-\infty}^{\infty} dx \ e^{-i\omega x} f(x) = 2\pi i \sum_{\text{Re} \ z_k > 0} \text{Res}(e^{-i\omega z} f(z), z_k). \]

**Proof:**

• **Proposition 4.3.7** Let \( f \) be analytic on an open set containing the closed lower half plane except for a finite number of isolated singularities not lying on the real axis. Suppose that \( f(z) \to 0 \) as \( z \to \infty \) in the lower half plane, that is, for any \( \varepsilon > 0 \) there is \( R > 0 \) such that if \( |z| > R \) in the lower half plane, then \( |f(z)| < \varepsilon \). Then for \( \omega > 0 \)

\[ \int_{-\infty}^{\infty} dx \ e^{-i\omega x} f(x) = -2\pi i \sum_{\text{Im} \ z_k < 0} \text{Res}(e^{-i\omega z} f(z), z_k). \]

**Proof:**

• **Remark.** Let \( P_n \) and \( Q_m \) be polynomials and \( m > n \). Suppose that \( Q \) has no zeros on the real axis. Let \( f = P/Q \). Then both of the above formulas are valid.

**Lemma 4.3.1** Jordan’s Lemma. Let \( R > 0 \) and \( f \) be analytic outside a semicircle \( |z| > R \) in the upper half plane. Suppose that \( f \to 0 \) as \( z \to \infty \) uniformly in the upper half plane. Let \( \omega < 0 \) and \( \gamma_r \) be a semicircle of radius \( r > R \) centered at the origin in the upper half plane. Then as \( r \to \infty \)

\[ \int_{\gamma_r} dz \ e^{-i\omega z} f(z) \to 0. \]
**Proof:** Advanced books.

---

- **Example.**

### 4.3.4 Cauchy Principal Value

- Let \( c \in (a, b) \). Let \( f : [a, b] \to \mathbb{C} \) be continuous on \([a, c) \) and \((c, b] \) but not necessarily at \( c \). The integral integral \( \int_{a}^{b} dx \, f(x) \) converges if both integrals \( \int_{a}^{c} dx \, f(x) \) and \( \int_{c}^{b} dx \, f(x) \) exist, that is,

\[
\int_{a}^{b} dx \, f(x) = \lim_{\varepsilon_{1}, \varepsilon_{2} \to 0} \left( \int_{a}^{c-\varepsilon_{1}} dx \, f(x) + \int_{c+\varepsilon_{2}}^{b} dx \, f(x) \right).
\]

- If the integral \( \int_{a}^{b} dx \, f(x) \) exists then it is equal to the symmetric limit

\[
\int_{a}^{b} dx \, f(x) = \lim_{\varepsilon \to 0} \left( \int_{a}^{c-\varepsilon} dx \, f(x) + \int_{c+\varepsilon}^{b} dx \, f(x) \right).
\]

- However, the existence of this symmetric limit does not imply the convergence of the integral. If this limit exists, it is called the **(Cauchy) principal value** integral and denoted by \( P \int_{-\infty}^{\infty} p.v. \int \), or \( P \int \), that is, by definition,

\[
P \int_{a}^{b} dx \, f(x) = \lim_{\varepsilon \to 0} \left( \int_{a}^{c-\varepsilon} dx \, f(x) + \int_{c+\varepsilon}^{b} dx \, f(x) \right).
\]

---

**Proposition 4.3.8** Let \( f \) be analytic in a deleted neighborhood of \( z_{0} \). Suppose that \( f \) has a simple pole at \( z_{0} \). Let \( \gamma_{r} \) be an arc of a circle of radius \( r \) and angle \( \varphi \) centered at \( z_{0} \). Then as \( r \to 0 \)

\[
\lim_{r \to 0} \int_{\gamma_{r}} dz \, f(z) = \varphi i \text{ Res} \left( f, z_{0} \right).
\]

**Proof:**
Proposition 4.3.9  Let $f$ be analytic on an open set containing the closed upper (lower) half-plane except for a finite number of isolated singularities, with the isolated singularities on the real axis being simple poles. Suppose that:

1. there are $M, R_0 > 0$ and $p > 1$ such that for any $z$ in the upper (lower) half-plane if $|z| > R_0$ then
   \[ |f(z)| \leq \frac{M}{|z|^p}, \]

or,

2. there is $\omega < 0$ ($\omega > 0$) and a function $g$ such that $f(z) = e^{-i\omega z}g(z)$, where $g$ is analytic on an open set containing the closed upper (lower) half-plane except for a finite number of isolated singularities and $g(z) \to 0$ as $z \to \infty$ in the upper (lower) half-plane.

Then the principal value integral exists and

\[ P \int_{-\infty}^{\infty} dz \ f(z) = \pm 2\pi i \sum_{\text{upper (lower) half-plane}} \text{Res} \ (f, z_k) \pm \pi i \sum_{\text{real axis}} \text{Res} \ (f, z_k). \]

Proof:


Example.

4.3.5 Integrals with Branch Cuts

- Mellin transform of a function $f : [0, \infty) \to \mathbb{R}$ is a function $\hat{f}$ of a complex variable $s$ defined by

\[ \hat{f}(s) = \int_0^{\infty} dt \ t^{s-1} f(t). \]
CHAPTER 4. CALCULUS OF RESIDUES

Proposition 4.3.10 Let $f$ be analytic on $\mathbb{C}$ except for a finite number of isolated singularities none of which lie on the positive real axis. Let $s \in \mathbb{R}$ such that $s > 0$ and $s$ is not a positive integer. Suppose that:

1. there are $M > 0, R > 0$ and $p > 1$ such that for any $z \in \mathbb{C}$, if $|z| > R$, then
   \[ |z^{-1}f(z)| \leq \frac{M}{|z|^p}. \]

2. there are $m > 0, r > 0$ and $q < 1$ such that for any $z \in \mathbb{C}$, if $|z| < r$, then
   \[ |z^{-1}f(z)| \leq \frac{m}{|z|^q}. \]

Let $z^{-1} = \exp[(s-1)\log z]$ be the branch with $0 < \arg < 2\pi$. Then the following integral converges absolutely and is equal to

\[ \int_{0}^{\infty} dt \, t^{s-1}f(t) = -\pi \left[ \cot(\pi s) - i \right] \sum_{\text{all singularities except 0}} \text{Res} \left( z^{s-1}f(z), z_k \right). \]

Proof:

• Example. Rational functions.

4.3.6 Logarithms

• Example.

• Exercises: 4.3[3,5,6,8,9,12,13,16,19,20b]
4.4 Evaluation of Infinite Series

- **Proposition 4.4.1 Summation Theorem.** Let \( f \) be a function that is analytic in a region containing the real axis. Let \( a_k = f(k), \ k \in \mathbb{Z} \). Let \( G \) be a function that is analytic in a region containing the real axis with simple poles at the real integers. Let \( g_k = \text{Res}(G, k), \ k \in \mathbb{Z} \). Suppose that the series \( \sum_{k=-\infty}^{\infty} g_k a_k \) converges absolutely. Let \( \gamma = \gamma_1 + \gamma_2 \), where \( \gamma_1 \) is a line above the real axis and \( \gamma_2 \) is a line below the real axis, such that \( f \) is analytic in the strip between \( \gamma_1 \) and \( \gamma_2 \). Suppose that \( \gamma_1 \) goes from \( \infty + i \varepsilon \) to \( -\infty + i \varepsilon \) and \( \gamma_2 \) goes from \( -\infty - i \varepsilon \) to \( \infty - i \varepsilon \), where \( \varepsilon > 0 \). Then there holds
\[
\sum_{k=-\infty}^{\infty} g_k a_k = \frac{1}{2\pi i} \int_{\gamma} dz \ G(z) f(z).
\]

**Proof:** Let \( \tilde{\gamma} \) be a closed contour oriented counterclockwise enclosing the points \(-N_1, \ldots, -1, 0, 1, \ldots, N_2\). Then
\[
\frac{1}{2\pi i} \oint_{\tilde{\gamma}} dz \ G(z) f(z) = \sum_{k=-N_1}^{N_2} g_k a_k.
\]
As \( N_1, N_2 \to \infty \) we have \( \tilde{\gamma} \to \gamma \) and we obtain the result.

- Similarly we can obtain a summation formula for the series \( \sum_{k=0}^{\infty} a_k \).

- **Proposition 4.4.2** Let \( f \) be a function that is analytic in a region containing the positive real axis. Let \( a_k = f(k), \ k = 0, 1, 2, \ldots \). Let \( G \) be a function that is analytic in a region containing the positive real axis with simple poles at the nonnegative integers. Let \( g_k = \text{Res}(G, k), \ k = 0, 1, 2, \ldots \). Suppose that the series \( \sum_{k=0}^{\infty} g_k a_k \) converges absolutely. Let \( \gamma \) be counter encircling the positive real axis in the counterclockwise direction, that is, \( \gamma \) goes from \( \infty + i \varepsilon \) around 0 to \( \infty - i \varepsilon \), where \( \varepsilon > 0 \). Then there holds
\[
\sum_{k=0}^{\infty} g_k a_k = \frac{1}{2\pi i} \int_{\gamma} dz \ G(z) f(z).
\]
Proof: Let \( \tilde{\gamma} \) be a closed contour oriented counterclockwise enclosing the points \( 0, 1, \ldots, N \). Then

\[
\frac{1}{2\pi i} \oint_{\tilde{\gamma}} dz \, G(z)f(z) = \sum_{k=0}^{N} g_k a_k.
\]

As \( N \to \infty \) we have \( \tilde{\gamma} \to \gamma \) and we obtain the result.

- If the functions \( f \) and \( g \) are meromorphic and decrease sufficiently fast at infinity, then these integrals can be computed by residue calculus.

- Examples.

- Exercises: 4.4[1,3,4,5]
Chapter 5

Analytic Continuation

5.1 Analytic Continuation and Riemann Surfaces

- A function analytic in a domain is uniquely determined by the values in some set with at least one limit point, such as a curve, a subdomain etc. It does not matter how small that set is.

**Lemma 5.1.1**  
Let $A$ be a region in $\mathbb{C}$ and $h : A \to \mathbb{C}$. Let $B \subset A$ be a subset of $A$ with a limit point. Suppose that $h$ is analytic on $A$ and $h = 0$ on $B$. Then $h = 0$ everywhere on $A$.

**Proof:** By isolation of zeros theorem or by using Taylor series.

- In particular, $B$ can be a region, a curve, a neighborhood of a point etc.

**Corollary 5.1.1 Principle of Analytic Continuation.** Let $f, g : A \to \mathbb{C}$ be two analytic functions and $B \subset A$ be a subset of $A$ with a limit point. Suppose that $f = g$ on $B$. Then $f = g$ everywhere in $A$.

**Proof:** By using Taylor series.
Proposition 5.1.1  Let \( A_1 \) and \( A_2 \) be two disjoint simply connected domains whose boundaries share a common contour \( \gamma \). Let \( A = A_1 \cap \gamma \cap A_2 \). Let \( f \) be analytic in \( A_1 \) and continuous in \( A_1 \cap \gamma \) and \( g \) be analytic in \( A_2 \) and continuous in \( A_2 \cap \gamma \). Suppose that \( f \) and \( g \) coincide on \( \gamma \). Let \( h \) be a function on \( A \) defined by
\[
h(z) = \begin{cases} 
  f(z), & z \in A_1 \\
  f(z) = g(z), & z \in \gamma \\
  g(z), & z \in A_2
\end{cases}
\]
Then \( h \) is analytic on \( A \).

The function \( g \) is called the analytic continuation of the function \( f \) from \( A_1 \) to \( A_2 \) through \( \gamma \).

**Proof:** Show that for any contour \( C \) in \( A \), \( \int_C h = 0 \).

Proposition 5.1.2  Let \( A_1 \) and \( A_2 \) be two overlapping regions so that \( A_1 \cap A_2 \neq \emptyset \). Let \( f : A_1 \to \mathbb{C} \) and \( g : A_2 \to \mathbb{C} \). Suppose \( f = g \) on \( A_1 \cup A_2 \). Then there is a unique analytic function \( h : A_1 \cup A_2 \to \mathbb{C} \) such that \( h = f \) on \( A_1 \) and \( h = g \) on \( A_2 \) defined by
\[
h(z) = \begin{cases} 
  f(z), & z \in A_1 \\
  g(z), & z \in A_2
\end{cases}
\]

The function \( h \) is called the analytic continuation of the function \( f \) from \( A_1 \) and of the function \( g \) from \( A_2 \).

**Proof:** Show that for any contour \( C \) in \( A \), \( \int_C h = 0 \).

Examples.

Analytic continuation can be done along curves by using Taylor series.
Proposition 5.1.3 Monodromy Theorem. Let $A$ be a simply connected domain and $D$ be a disk in $A$. Let $f$ be analytic in $D$. Suppose that the function $f$ can be analytically continued from $D$ along two distinct contours $\gamma_1$ and $\gamma_2$ to functions $f_1$ and $f_2$ at a point $z$ in $A$. Suppose that in the region enclosed by the contours $\gamma_1$ and $\gamma_2$ the function $f$ contains at most isolated singularities. Then the result of each analytic continuation is the same, that is, $f_1(z) = f_2(z)$.

Proof:

- Example.
- There are singularities of analytic functions that prevent analytic continuation to a larger domain. Such singularities are called natural boundary, or natural barrier.

- Example. Let $f$ be defined by

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}.$$  

This series converges in the unit disk $|z| < 1$ and defines an analytic function there. However, it cannot be analytically continued to a larger domain.

- One can show that $f$ satisfies the equation

$$f(z^2) = f(z) - z.$$  

Therefore

$$f(z^4) = f(z) - z - z^2,$$

and, by induction,

$$f(z^{2^m}) = f(z) - \sum_{k=0}^{m-1} z^{2^k}.$$  

- Let $z_m$ be defined by $z_m^{2^m} = 1$. These are roots of unity that are uniformly spread on the unit circle. As $m \to \infty$ they form a dense set on the unit circle.

- On another hand, for any $z_m$, $f(z_m) = \infty$. That is, all these points are singular points of $f$. Since they dense on the unit circle, $f$ cannot be analytically continued to any $z$ such that $|z| \geq 1$. 

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• **Riemann Surfaces.** Square root and logarithm.

• **Exercises:** 6.1[1,2,3,4,8,11]
5.2 Roché Theorem and Princile of the Argument

5.2.1 Root and Pole Counting Formulas

**Proposition 5.2.1 Root-Pole Counting Theorem.** Let \( f \) be an analytic function on a region \( A \) except for a finite number of poles at \( b_1, \ldots, b_m \) of orders \( p_1, \ldots, p_m \). Suppose that \( f \) has zeros at \( a_1, \ldots, a_n \) of orders \( q_1, \ldots, q_n \). Let \( \gamma \) be a closed contractible curve in \( A \) not passing through the points \( A_k \) and \( b_j \). Then

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f} = \sum_{k=1}^{n} q_k I(\gamma, a_k) - \sum_{j=1}^{m} p_j I(\gamma, b_k).
\]

**Proof:**

**Corollary 5.2.1** Let \( \gamma \) be a simple closed contractible curve. Let \( f \) be an analytic function on a region \( A \) containing \( \gamma \) and its interior except for a finite number of poles. Suppose that the poles and zeros of \( f \) do not lie on \( \gamma \). Then

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f} = Z - P,
\]

where \( Z \) and \( P \) are the number of zeros and poles counted with multiplicities inside \( \gamma \).

**Proof:**

**Corollary 5.2.2 Root Counting Formula.** Let \( \gamma \) be a simple closed contractible curve. Let \( f \) be an analytic function on a region \( A \) containing \( \gamma \) and its interior. Let \( w \in \mathbb{C} \). Suppose that \( f \neq w \) on \( \gamma \). Then

\[
N = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f - w}
\]

is the number of roots of the equation \( f(z) = w \) inside \( \gamma \) counted with their multiplicities.

**Proof:**
5.2.2 Principle of the Argument

- Let $z_0 \in \mathbb{C}$ and $\gamma$ a closed contour. Recall the definition of the \textbf{winding number} (or \textbf{index})
  \[ I(\gamma, z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z - z_0}. \]

- The change of the argument of $z$ as $z$ traverses $\gamma$ is equal to
  \[ \Delta_\gamma \arg z = 2\pi I(\gamma, 0). \]

- Now let $f$ be an analytic function and $\gamma : [a, b] \to \mathbb{C}$ be a closed curve. Let us choose a branch of $\arg f(\gamma(t))$ that varies continuously with $t$. Then the change in the argument of $w = f(z)$ as $z = \gamma(t)$ traverses $\gamma$ is equal to
  \[ \Delta_\gamma \arg f = \arg[f(\gamma(b))] - \arg[f(\gamma(a))]. \]
  Note that $f(\gamma(b)) = f(\gamma(a))$.

- Equivalently, $\Delta_\gamma f$ is equal to the change in the argument of $z$ along the image curve $f \circ \gamma$, that is,
  \[ \Delta_\gamma \arg f = 2\pi I(f \circ \gamma, 0). \]

**Proposition 5.2.2 Principle of the Argument.** Let $f$ be an analytic function on a region $A$ except for a finite number of poles at $b_1, \ldots, b_m$ of orders $p_1, \ldots, p_m$. Suppose that $f$ has zeros at $a_1, \ldots, a_n$ of orders $q_1, \ldots, q_n$. Let $\gamma$ be a closed contractible curve in $A$ not passing through the points $A_k$ and $b_j$. Then

\[ \Delta_\gamma \arg f = 2\pi \left\{ \sum_{k=1}^{n} q_k I(\gamma, a_k) - \sum_{j=1}^{m} p_j I(\gamma, b_k) \right\}. \]

In particular, if $\gamma$ is a simple closed curve, then
\[ \Delta_\gamma \arg f = 2\pi (Z - P), \]
where $Z$ and $P$ are the number of zeros and poles counted with multiplicities inside $\gamma$.

**Proof:**

\[ \boxed{\text{\(\square\)}} \]
5.2. ROCHÉ THEOREM AND PRINCIPLE OF THE ARGUMENT

**Proposition 5.2.3 Rouché’s Theorem.** Let $f$ and $g$ be analytic on a region $A$ except for a finite number of poles. Suppose that $f$ and $g$ have a finite number of zeros in $A$. Let $\gamma$ be a closed contractible curve in $A$ not passing through zeros and poles of $f$ and $g$. Let $a_k$ be the zeros of $f$ of order $q_k$ and

$$Z_f = \sum_{k=1}^{n} q_k I(\gamma, a_k),$$

and let $Z_g$, $P_f$, $P_g$ be defined correspondingly for zeros and poles of functions $f$ and $g$. Suppose that on $\gamma$

$$|f(z) - g(z)| < |f(z)|.$$

Then:

1. $\Delta_{\gamma} \arg f = \Delta_{\gamma} \arg g$,

2. $Z_f - P_f = Z_g - P_g$.

**Proof:**

---

**Corollary 5.2.3** Let $\gamma$ be a simple closed contractible curve. Let $f$ and $g$ be analytic on a region $A$ containing $\gamma$. Suppose that on $\gamma$

$$|g(z)| < |f(z)|.$$

Then $f$ and $f + g$ have the same number of zeros inside $\gamma$.

**Proof:**

---

5.2.3 Injective Functions

**Proposition 5.2.4** Let $f : A \to \mathbb{C}$ be an analytic function. If $f$ is locally one-to-one, then $f' \neq 0$ in $A$. If $f$ is globally one-to-one, then $f^{-1} : f(A) \to A$ is analytic function.
Proposition 5.2.5  Injection Theorem. Let $f$ be analytic on a region $A$. Let $\gamma$ be a closed contractible contour in $A$. Suppose that for any $z \in A$, $I(\gamma, z) = 0$ or $1$. Let $B \subset A$ be a subset of $A$ (the inside of $\gamma$) defined by

$$B = \{z \in A \mid I(\gamma, z) \neq 0\}.$$ 

Suppose that for any point $w \in f(B)$

$$I(f \circ \gamma, w) = 1.$$ 

Then $f$ is one-to-one on $B$.

Proof: Let $z_0 \in B$ and $w_0 = f(z_0) \in f(B)$. Then

$$N = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z) - w_0} \, dz$$

is the number of solutions of the equation $f(z) = w_0$ on $B$. Further,

$$N = \frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{dw}{w - w_0} = 1.$$ 

Thus $f$ is injective in $B$. 

To show that an analytic function is injective on a region it is sufficient to show that it is injective on the boundary.

Exercises: 6.2[2,5,13,17]
5.3 Mapping Properties of Analytic Functions
Bibliography

Answers to Exercises
Notation