1 Objectives and Significance of the Proposed Project

The PI proposes to study new heat invariants of second-order and first-order elliptic partial differential operators acting on sections of vector bundles over Riemannian manifolds with and without boundary. The long term goal of this project is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants. The PI will:

- develop new methods for calculation of the heat kernel for second-order partial differential operators on vector bundles (both Laplace type and non-Laplace type) over Riemannian manifolds (both without boundary and with boundary),
- define and study new heat invariants of differential operators; compute explicitly some leading terms of the asymptotic expansion of new heat invariants,
- study the asymptotics of the heat invariants for singular time-dependent operators,
- study the heat invariants of a one-parameter family of operators and their relation to the integrable systems.

The heat kernel is one of the most important tools of global analysis, spectral geometry, differential geometry and mathematical physics, in particular, quantum field theory [76, 44, 69, 77], even financial mathematics (see, eg. PI’s book [28]). In quantum field theory the main objects of interest are described by the Green functions of self-adjoint elliptic partial differential operators on manifolds and their spectral invariants such as the functional determinants. In spectral geometry one is interested in the relation of the spectrum of natural elliptic partial differential operators to the geometry of the manifold, more precisely, one studies the question: “To what extent does the spectrum of a differential operator determine the geometry of the underlying manifold?”

There are also non-trivial links between the spectral invariants and the non-linear completely integrable evolution systems, such as Korteweg-de Vries hierarchy (see, e.g. [77] and the PI’s papers [41, 16]). In many interesting cases such systems are, in fact, infinite-dimensional Hamiltonian systems, and the spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of this system.
In financial mathematics one uses stochastic differential equations with some Wiener processes built in to model the random behavior of financial assets like equities (stocks). Then the behavior of the corresponding derivative securities (options) is determined by deterministic parabolic partial differential equations of diffusion type with an elliptic partial differential operator of second order. These equations are the generalization of the classical Fokker-Planck equation, and forward and backward Kolmogorov equations. The conditional probability density is then nothing but the fundamental solution of this equation, in other words, the heat kernel. For more details see PI's book [28].

Instead of studying the spectrum of a differential operator directly one usually studies its spectral functions, that is, spectral traces of some functions of the operator, such as the zeta function, and the heat trace. Usually one does not know the spectrum exactly; that is why, it becomes very important to study various asymptotic regimes. It is well known, for example, that one can get information about the asymptotic properties of the spectrum by studying the short time asymptotic expansion of the heat trace.

The coefficients of this expansion, called the heat trace coefficients (or global heat kernel coefficients), play very important role in spectral geometry and mathematical physics [77, 69].

The simplest case of a Laplace operator on a compact manifold without boundary is well understood and there is a vast literature on this subject, see [69] and the references therein. Many ideas and techniques do not apply directly in more general cases:

- For a Laplace type operator on a compact manifold without boundary there is a well defined local asymptotic expansion of the heat kernel, which enables one to compute its diagonal and then the heat trace by directly integrating the heat kernel diagonal; this gives all heat trace coefficients. For manifolds with boundary, the situation is much more complicated. One has to derive the local asymptotic expansion of the heat kernel separately in the interior of the manifold and in a narrow strip along the boundary (see PI’s paper [20]).

- For oblique boundary conditions containing tangential derivatives the heat kernel near the boundary depends non-polynomially on the symbol of the tangential boundary operator; this complicates the calculations significantly (see PI’s papers [35, 36] (with G. Esposito)).

- For discontinuous boundary conditions (called Zaremba boundary conditions) that jump from Dirichlet to Neumann boundary conditions along a submanifold of codimension 2 one has to study the heat kernel separately near that singular interface (see [85] and the PI’s paper [20]); in this case one looses universality and, therefore, the powerful functorial methods no longer work [60].

- For non-Laplace type operator the situation is further complicated by the fact that the leading symbol of the operator is not scalar and, as a result, the heat kernel is non-polynomial in the principal symbol, which makes the calculations much more complicated, in general (see PI’s papers [29, 17] (with T. Branson)).

- The case of non-Laplace type operators on manifolds with boundary is even more difficult; there are just some results for the next to leading asymptotics (see PI’s paper [23]).

In the situation when it is impossible to compute the trace of the heat kernel exactly, it becomes very important to study various asymptotic regimes. That is why it is vital to develop adequate approximation schemes for the heat kernel by studying deformations of the operators under consideration. This is very important and extremely complicated problem, which, in general setting, remains open.
The standard short-time asymptotic expansion of the heat trace has a limited applicability. From a practical point of view one needs a good approximation for the heat trace. Of course, the short-time asymptotic expansion gives a good approximation if the curvatures and their derivatives are rather small. In the situation when either the curvature or its derivatives are large this asymptotic expansion gives a good approximation only in the limit when the time goes to zero. Thus, it makes more sense to consider one-parameter family of metrics and connections which defines a one-parameter family of operators. Now, the heat trace is a function of two variables, time and a new deformation parameter. Therefore, one can study an alternative asymptotic expansion in the deformation parameter. By choosing different types of deformation one can probe various aspects of the geometry (see PI’s papers [10, 13, 26]). For example, one could deform symmetric spaces, Lie groups, and some other homogeneous spaces that allow an exact solution. Another extreme would be a deformation of a flat metric. Such techniques have been used to derive the leading derivatives terms in all heat trace invariants and to study the non-local asymptotic form of the heat trace in the PI’s papers [2, 3]. It would be interesting to study the heat kernel asymptotics under some deformation flow, in particular, conformal flow, Ricci flow etc.

Also, one can get new spectral information by equating the deformation parameter with the time variable, that is, by considering time-dependent metric and connection and study the short-time asymptotic expansion of the heat trace. In particular, it is possible to consider singular deformations of the connection which provide well defined asymptotics (see PI’s paper [25] (with G. Fucci)).

Another point; the existence of non-isometric isospectral manifolds demonstrates that the spectrum alone does not determine the geometry (see, e.g. [43]). That is why, we propose to study more general invariants of partial differential operators that are not spectral invariants, that is, they depend not only on the eigenvalues but also on the eigenfunctions, and, therefore, contain much more information about the geometry of the manifold.

In the past three decades the heat kernel asymptotics have been extensively studied in the literature, and many important results have been discovered. The early developments are summarized in the books [69, 44] with extensive bibliography, see also PI’s book [16] and PI’s expository articles [15, 17, 26]. The heat kernel asymptotics for the Laplace type operator with classical boundary conditions were studied in [50, 53, 78] and in the PI’s paper [4]. The leading terms in heat trace invariants were studied in [54] and in the PI’s papers [2, 3]. The non-Laplace type operators on manifolds without boundary were studied in [70, 1, 48, 55, 56, 75] and the PI’s papers [29, 17]. The heat kernel asymptotics of the oblique boundary value problem for Laplace type operators were studied in [81, 61, 62, 57] and in the PI’s papers [35, 36]. The discontinuous Zaremba boundary value problem for laplace type operator was studied in [59, 60, 85] and in the PI’s paper [20].

The proposed activities are further development of the prior work of the PI and constitute a natural part of his long-term research plans. In the proposed project the PI intends to continue the research outlined above and investigate the new heat invariants he is suggesting. The proposed research will further extend the heat kernel methods to new non-standard problems described above. The progress in this area will be a significant contribution to the spectral theory of natural geometric differential operators, conformal geometry, spectral geometry, quantum field theory, quantum gravity, mathematical physics as well as applied mathematics. It will discover new non-perturbative phenomena that were inaccessible by the previous methods. The project will advance one’s understanding of the spectral asymptotics of partial differential operators and enhance one’s ability to use these methods in spectral geometry and mathematical physics. This is important from both the theoretical and the practical point of view. It is expected that the proposed project will further broaden potential applicability of heat kernel methods to
a variety of practical problems in quantum field theory and quantum gravity.

The detailed description of the proposed research is organized in Section 2. Section 3 is devoted to the broader impacts of the proposed activity and Section 4 summarizes the PI’s research experience related to the project.

2 Proposed Research

2.1 Partial Differential Operators

Let \( M \) be a smooth compact (possibly with smooth boundary \( \partial M \)) of dimension \( n \) and let \( \mathcal{V} \) be a smooth vector bundle over the manifold \( M \). Let \( L : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \) be a second-order formally self-adjoint elliptic partial differential operator with a positive definite leading symbol. If the leading symbol of the operator \( L \) is scalar then the operator \( L \) is called of Laplace type; in this case one can define a Riemannian metric by the leading symbol of the operator \( L \), a connection \( \nabla \) on the vector bundle \( \mathcal{V} \) and an endomorphism \( Q \) of the vector bundle \( \mathcal{V} \) so that

\[
L = \nabla^\ast \nabla + Q.
\] (1)

In general, if the leading symbol of the operator \( L \) is not scalar then the operator \( L \) is called of non-Laplace type; in this case there is no naturally defined Riemannian metric, instead the leading symbol of the operator \( L \) defines a “non-commutative” Riemannian metric which is much closer rather to Finsler geometry (see PI’s papers [18, 20, 21, 22, 23]). Let \( D : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \) be a first-order formally self-adjoint elliptic partial differential operator. Obviously, the operator \( D^2 \) is a second-order formally self-adjoint elliptic partial differential operator with a positive definite leading symbol. If the operator \( D^2 \) is of Laplace type then the operator \( D \) is said to be of Dirac type; otherwise, we call it a non-Dirac type operator.

In the case when there is a boundary we assume that some suitable boundary conditions are imposed both on the operator \( L \) and \( D \) to make them elliptic and essentially self-adjoint in \( L^2(\mathcal{V}) \) [74, 45, 69]. Apart from the classical Dirichlet or Neumann (or, more generally, Robin) boundary conditions one can consider more general mixed boundary conditions that involve Dirichlet conditions on a subbundle and Neumann conditions on another subbundle. More generally, one could study the oblique boundary conditions [73, 74, 71, 72], which include tangential derivatives along the boundary on a subbundle. For the oblique boundary value problem to be elliptic the tangential operator has to be small compared to the normal derivative, more precisely, it has to satisfy the so-called Shapiro-Lopatinsky condition (see [74] and the PI’s papers [35, 36]). One can generalize the setup even further to include discontinuous (or Zaremba) boundary conditions as well ([85] and PI’s paper [19]). In this case the boundary \( \partial M \) is decomposed as a disjoint union \( \partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are smooth compact submanifolds of codimension 1 sharing the same boundary \( \Sigma_0 = \partial \Sigma_1 = \partial \Sigma_2 \), that is a smooth compact submanifold without boundary of codimension 2. Then one can impose different boundary conditions on \( \Sigma_1 \), \( \Sigma_2 \) and \( \Sigma_0 \).

It is well known [69] that if the manifold \( M \) is compact then both operators \( L \) and \( D \) have only point spectrum consisting of real discrete eigenvalues with finite multiplicities. The spectrum of the operator \( L \) is bounded from below and the spectrum of the operator \( D \) is not. Let \( (\lambda_n, \varphi_n) \) be the eigenvalues and the eigensections of the operator \( L \) and \( (\mu_n, \psi_n) \) be the eigenvalues and the eigensections of the operator
\( D \); in both cases counted with multiplicity. Without loss of generality we can assume that the set of eigenfunctions is orthonormal. Then for \( t > 0 \) the heat semigroups \( \exp(-tL) \) and \( \exp(-tD^2) \) are bounded operators with the integral kernels

\[
U_L(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k(x) \otimes \varphi_k^*(x'),
\]

\[
U_{D^2}(t; x, x') = \sum_{k=1}^{\infty} e^{-t\mu_k^2} \psi_k(x) \otimes \psi_k^*(x')
\]
called the heat kernels.

### 2.2 Spectral Functions

Thus, the operator \( D^2 \) is non-negative and the operator \( L \) can only have finitely many negative eigenvalues. To study the spectrum of these operators one defines the zeta and the eta functions [52, 51, 69]. To avoid the complications with the possible zero eigenvalues we introduce an extra regularization. Let \( \lambda \ll 0 \) be a sufficiently large negative parameter such that the operators \((L - \lambda)\) and \((D^2 - \lambda)\) are positive, that is, \( \lambda_n > \lambda \) and \( \mu_n^2 > \lambda \) for all \( n \), so that, \( \text{Ker}(L - \lambda) = \text{Ker}(D^2 - \lambda) = 0 \). Then we define

\[
\zeta(s; L - \lambda) = \text{Tr} (L - \lambda)^{-s} = \sum_{n=1}^{\infty} (\lambda_n - \lambda)^{-s},
\]

\[
\zeta(s; D^2 - \lambda) = \text{Tr} (D^2 - \lambda)^{-s} = \sum_{n=1}^{\infty} (\mu_n^2 - \lambda)^{-s},
\]

\[
\eta(s, \lambda; D) = \text{Tr} D(D^2 - \lambda)^{-s-\frac{1}{2}} = \sum_{n=1}^{\infty} \mu_n (\mu_n^2 - \lambda)^{-s-\frac{1}{2}}.
\]

These functions have meromorphic extensions to the whole complex plane and enable one to define important spectral invariants of these operators. In doing so we let \( \lambda \) go to zero at the very end, after all other limits have been taken. Then the invariants \( \zeta(0, 0, L) \) and \( \zeta(0, 0, D^2) \) describe the regularized number of all eigenvalues of these operators, including the zero eigenvalues. The derivatives of the zeta functions at \( s = 0 \) as \( \lambda \to 0 \) reveal the number of zero eigenvalues, \( l = \dim \text{Ker} L \) and \( m = \dim \text{Ker} D \), and define the regularized determinants via

\[
\zeta'(0; L - \lambda) \sim -l \log (-\lambda) - \log \det L + O(\lambda),
\]

\[
\zeta'(0; D^2 - \lambda) \sim -m \log (-\lambda) - \log \det D^2 + O(\lambda).
\]

The eta invariant \( \eta(D) = \eta(0, 0; D) \) describes the regularized difference between the number of positive and negative eigenvalues. The derivative of the eta function at \( s = 0 \) as \( \lambda \to 0 \) defines the regularized super determinant via

\[
\eta'(0, 0; D) = -2 \log \text{sdet} D,
\]

which is equal to the regularized product of all positive eigenvalues divided by the product of the absolute values of all negative eigenvalues. Both invariants \( \eta(D) \) and \( \text{sdet} D \) measure the spectral asymmetry of the operator \( D \). Obviously, if the spectrum of the operator \( D \) is symmetric then \( \eta(D) = 0 \) and \( \text{sdet} D = 1 \).
All these invariants can be described by the corresponding heat traces,
\[
\zeta(s; L - \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} e^{t\lambda} \Theta(t; L), \tag{10}
\]
\[
\zeta(s; D^2 - \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} e^{t\lambda} \Theta(t; D^2), \tag{11}
\]
\[
\eta(s, \lambda; D) = \frac{1}{\Gamma(s + \frac{1}{2})} \int_0^\infty dt \ t^{s-\frac{1}{2}} e^{t\lambda} \Phi(t; D), \tag{12}
\]
where
\[
\Theta(t; L) = \text{Tr} \exp(-tL) = \sum_{n=1}^\infty e^{-t\lambda_n} = \int_M U_L^\text{diag}(t), \tag{13}
\]
\[
\Theta(t; D^2) = \text{Tr} \exp(-tD^2) = \sum_{n=1}^\infty e^{-t\mu_n^2} = \int_M U_{D^2}^\text{diag}(t), \tag{14}
\]
\[
\Phi(t; D) = \text{Tr} \ D \exp(-tD^2) = \sum_{n=1}^\infty \mu_n e^{-t\mu_n^2} = \int_M [DU_{D^2}(t)]^\text{diag}, \tag{15}
\]
where \(F^\text{diag}(x) = F(x, x)\) denotes the diagonal value of a two-point function \(F\) and \(\text{tr}\) is the fiber trace. That is why it is so important to investigate the heat traces of the form \(\text{Tr} \exp(-tL)\) and \(\text{Tr} \ D \exp(-tD^2)\).

Suppose that there is an operator \(\Gamma\) that anti-commutes with the operator \(D\), \(\Gamma D = -D \Gamma\). Then one can show that
\[
\text{Tr} \ D^2 \exp(-tD^2) = 0, \tag{16}
\]
and, therefore, the invariant
\[
\text{Index}(\Gamma, D) = \text{Tr} \Gamma \exp(-tD^2) \tag{17}
\]
does not depend on \(t\) and is equal to
\[
\text{Index}(\Gamma, D) = \text{Tr} \ \Gamma P, \tag{18}
\]
where \(P\) is the projection onto the kernel of the operator \(D\); it can be called the index of the operator \(D\) with respect to the operator \(\Gamma\).

The PI intends to study the asymptotics of all these spectral functions as \(t \to 0\) and the related residues of the zeta and the eta functions. This can be done for manifolds without boundary as well as for manifolds with boundary for various boundary conditions. It would be interesting to develop the heat kernel methods for the calculation of the eta function \(\eta(s, \lambda; D)\) and the eta invariant \(\eta(D)\) for the Dirac operator as well.

### 2.3 Deformation of Spectral Invariants

The PI proposes to study the heat traces
\[
\Theta(t; L(\tau)) = \text{Tr} \exp(-tL(\tau)), \quad \Theta(t; D(\tau)^2) = \text{Tr} \exp(-tD^2(\tau)), \tag{19}
\]
for a one-parameter family of operators $L(\tau)$ and $D(\tau)$ (we call this a deformation). The PI is interested in studying the deformation of the heat trace induced by some flow of the metric and the connection, in particular, conformal flow, Ricci flow and others. He will try to rewrite these flow equations, if possible, in form of an infinite-dimensional Hamiltonian system. Notice that there are important relations

$$\partial_\tau \Theta(t; L(\tau)) = -t \text{Tr} \left[ \partial_\tau L \exp(-tL) \right],$$

$$\partial_\tau \Theta(t; D^2(\tau)) = -2t \text{Tr} \left[ \partial_\tau D D \exp(-tD^2) \right].$$

(20)

(21)

The PI intends to study the asymptotic expansion of this combined trace for a fixed $t$ and for $\tau \to 0$ (or $\tau \to \infty$).

In particular, the PI plans to study more general zeta functions of the form

$$\zeta(s; D^2 + \tau D - \lambda) = \text{Tr} (D^2 + \tau D - \lambda)^{-s},$$

(22)

and the heat traces of the form

$$\Theta(t; D^2 + \tau D) = \text{Tr} \exp \left[ -t \left( D^2 + \tau D \right) \right].$$

(23)

This would give, in particular, all the traces considered above, for example,

$$\eta(s, \lambda; D) = -\frac{1}{s} \frac{\partial}{\partial \tau} \zeta \left( s - \frac{1}{2}; D^2 + \tau D - \lambda \right) \bigg|_{\tau=0},$$

(24)

$$\Phi(t; D) = -\frac{1}{t} \frac{\partial}{\partial \tau} \Theta(t; D^2 + \tau D) \bigg|_{\tau=0}.$$  

(25)

One could go even further and study the invariants of the form

$$\text{Tr} \exp [-t(L + \tau D)],$$

(26)

where $L$ is a second-order operator and $D$ is a first order operator (not necessarily related to $L$). More generally, given a collection of $N$ first-order operators $G_i$, $i = 1, \ldots, N$, let

$$L = \sum_{i=1}^N G_i^* G_i.$$  

(27)

Obviously, this operator is self-adjoint and non-negative. Suppose that it is also elliptic. Then we can study the invariants

$$\zeta(s; L + \langle p, G \rangle - \lambda), \quad \text{Tr} \exp \left[ -t \left( L + \langle p, G \rangle \right) \right],$$

(28)

where $\langle p, G \rangle = \sum_{i=1}^N p_i G_i$ and $p_i$ are some real parameters. The derivatives of these functions with respect to $p_i$ will give the invariants of the form

$$\text{Tr} G_i (L - \lambda)^{-s}, \quad \text{Tr} G_i \exp(-tL)$$

(29)

An interesting case if when the operators $G_i$ form a representation of a Lie algebra, for example, the Lie algebra of the isometry group. This could be important in the case of Lie groups or symmetric spaces.
The PI made some progress in this area in the papers [5, 6, 7, 9, 24, 25]. One could also study the invariants of the form
$$\text{Tr} \exp(\tau D) \exp(-tL) \text{ and } \text{Tr} \exp\langle p, G \rangle \exp(-tL). \quad (30)$$

Now, let $D$ be a Dirac type operator and $Q$ be an anti-self-adjoint endomorphism. The PI intends to study the heat trace of the self-adjoint operator
$$L(\tau) = (D - \tau M)(D + \tau M) = D^2 + \tau[D, M] - \tau^2 M^2. \quad (31)$$
It is obvious that it is non-negative $L \geq 0$. In particular, the PI plans to study the kernel $\text{Ker} L(\tau)$ of this operator and its dependence on the deformation parameter $\tau$. One can show that for large $\tau$ there are no zero eigenvalues and the kernel is empty. It would be very interesting to study the critical value of the deformation parameter $\tau_{\text{crit}}$ at which the kernel appears. Of course, this depends on the endomorphism $M$ as well as on the geometry.

Next, let $\Omega$ be another operator and let the deformation of the operator $L(\tau)$ be defined by the equation
$$\partial_\tau L = a \Omega L + b L \Omega, \quad (32)$$
where $a$ and $b$ are some real constants. Then it follows
$$\partial_\tau \text{Tr} \exp(-tL) = (a + b) t \partial_t \text{Tr} \Omega \exp(-tL). \quad (33)$$
A very important particular case is $a = -b$, i.e.
$$\partial_\tau L = a[\Omega, L]. \quad (34)$$
Such deformation is a particular case of isospectral deformations, that is,
$$\partial_\tau \Theta(t; L(\tau)) = 0. \quad (35)$$

More generally, one can construct isospectral deformation as follows. Suppose that there is a self-adjoint operator $N$ commuting with $L$ and an anti-self-adjoint operator $P$ such that the commutator $[P, N]$ is a self-adjoint differential operator. Then it is easy to see that the deformation
$$\partial_\tau L = [P, N] \quad (36)$$
is isospectral. Now, assume that $L = -\sum_{i=1}^N G_i^2$ and $G_i$ commute with $L$. Then the deformation of the form
$$\partial_\tau G_i = \sum_{j=1}^N [A_{ij}, G_j] \quad (37)$$
with symmetric operator-valued matrix $A_{ij}$, is isospectral.

One could also consider the combined heat kernel asymptotics
$$\Theta(t; L(t)) = \text{Tr} \exp[-tL(t)] \quad (38)$$
of a time-dependent operator when $\tau = t$ and $t \to 0$. In this case
$$\partial_t \Theta(t; L(t)) = -\text{Tr} \left[ (t\partial_t L + L) \exp(-tL) \right]. \quad (39)$$
The most interesting (and the most challenging case) is when the deformation is singular as $t \to 0$, for example,

$$L(t) \sim \frac{1}{t^2} L_0 + \frac{1}{t} L_1 + \tilde{L}(t),$$

(40)

where $L_0$ is a zero-order operator (simply an endomorphism), $L_1$ is a first-order operator and $\tilde{L}(t)$ has a Taylor series expansion at $t = 0$.

Isospectral deformations play an important role in the theory of integrable systems [77, 80]. For example, it is well known [77] (see also the PI’s papers [41, 16] (with R. Schimming)) that the Korteweg-de Vries hierarchy is an infinite dimensional Hamiltonian system whose flows are exactly the isospectral deformations of the potential of a one-dimensional Schrödinger operator $L(\tau) = -\partial_x^2 + Q(x, \tau)$ acting on smooth functions on a circle $S^1$.

It would be very interesting to generalize the KdV hierarchy to higher dimensions, that is, to find an isospectral deformation of a general elliptic partial differential (or pseudo-differential) operator $L(\tau)$ and the corresponding infinite-dimensional (completely integrable) Hamiltonian system. If this is impossible we will try to understand why. We will also study the asymptotic expansions of the trace of the heat kernel with respect to the deformation parameter $\tau$.

Another very important application of the above ideas is the study the extremal properties of the heat invariants, in particular, the zeta-regularized determinant, of natural conformally covariant geometric differential operators, such as the Yamabe operator and the Dirac operator, on smooth compact manifolds without boundary under smooth perturbations of the Riemannian metric $g$. Of special interest are conformal perturbations $\partial_\tau g = 2\omega g$, where $\omega$ is a smooth function, with fixed volume $\text{Vol}_g(M)$. The conformal deformation of a conformally covariant operator $L$ is described by the equation (32) with $\Omega$ being the operator of multiplication by $\omega$, i.e. a zero-order differential operator. This is a very active and exciting area of research [49, 47].

### 2.4 Deformation of Riemann Zeta Function

The PI proposes to study a generalized zeta function defined as follows. Let $q$ be a real valued smooth ($2\pi$-periodic) function defined on the unit circle $S^1$ with zero average, that is, $\int_{2\pi} q(x)dx = 0$. Let $D(\tau) = i\partial_\tau + i\tau q$ be a first-order differential operator on $S^1$ and $L$ be a second-order self-adjoint non-negative operator defined by

$$L(\tau) = D^*(\tau)D(\tau) = -\partial_x^2 + \tau q' + \tau^2 q^2.$$

(41)

The kernel of the operator $L$ coincides with the kernel of the operator $D$. It is not difficult to see that the operator $D$ has one zero eigenvalue, so that $\dim \ker L = \dim \ker D = 1$. Then one can define the invariant

$$Z(s, \tau) = \pi^{s/2} \Gamma\left(\frac{s}{2}\right) \left\{ \frac{1}{s(s-1)} + \frac{1}{2} \int_1^\infty \frac{dt}{t} \left( t^{s/2} + t^{1-s/2} \right) \left[ \Theta(\pi t; L(\tau)) - 1 \right] \right\},$$

(42)

Notice that this formula exhibits the simple pole of the zeta function at $s = 1$ and satisfies the functional equation for the Riemann zeta function

$$Z(s, \tau) = 2 \sin \left( \frac{\pi}{2} s \right) (2\pi)^{s-1} \Gamma(1-s) Z(1-s, \tau).$$

(43)
As \( \tau \to 0 \) the operator \( D \) becomes \( D(0) = i\partial_x \) with the simple eigenvalues \( \mu_n = n, n \in \mathbb{Z} \), and one gets nothing else but an integral representation of the Riemann zeta function \([58]\). The reason why it is possible to rewrite the Riemann zeta in this form is the Poisson summation formula which describes the fundamental duality between the spectrum and the closed geodesics. Therefore, \( Z(s,0) = \zeta_R(s) \) and the invariant defined above can be called a deformation of the Riemann zeta function which preserves the analytical structure and the functional equation. Recall that the functional equation is one of the reasons for the Riemann hypothesis about the zeros of the zeta function. It would be interesting to study this deformation of the Riemann zeta function, in particular, the location of its zeros as functions of the deformation parameter \( \tau \).

### 2.5 New Heat Invariants

Since the spectral invariants alone do not fix the geometry (they allow some isospectral deformations) The PI is interested in introducing and studying more general invariants that depend not only on the eigenvalues but also on the eigensections.

#### 2.5.1 Heat Traces

Let \( \psi \) be a unit section of the vector bundle \( V \) and \( P_\psi \) be the projection operator on \( \psi \). The heat content is defined by

\[
\Pi(t; \psi, L) = \text{Tr} P_\psi \exp(-tL) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \psi_k^* \psi_k
\]

where \( \psi_k = (\varphi_k, \varphi) = \int_M \langle \varphi_k, \psi \rangle \) with \( \langle , \rangle \) denoting the fiber inner product. Let \( Y \) be another first-order differential operator (or just a endomorphism). Then one can define an invariant by

\[
\Gamma(t; Y, L) = \text{Tr} Y \exp(-tL) Y \exp(-tL) = \sum_{k,l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} Y_{kl}^* Y_{kl},
\]

where \( Y_{kl} = (\varphi_k, Y \varphi_l) = \int_M \langle \varphi_k, Y \varphi_l \rangle \).

#### 2.5.2 Heat Determinants

The PI is proposing to study new invariants of second-order elliptic self-adjoint operators defined as follows. For a scalar operator \( L \) it can be defined by

\[
\tilde{K}(t; L) = \int_{M \times M} dx \, dx' \det \left\{ \nabla_{x'} \nabla_{x'} U(t; x, x') \right\} = \sum_{1 \leq k_1 < k_2 < \cdots < k_n} \exp \left\{ -t(\lambda_{k_1} + \cdots + \lambda_{k_n}) \right\} \tilde{E}_{k_1\cdots k_n}^2,
\]

where

\[
\tilde{E}_{k_1\cdots k_n} = \int_M d\varphi_{k_1} \wedge \cdots \wedge d\varphi_{k_n} = \int dM \frac{\partial}{\partial M} \varphi_{k_1} \wedge \cdots \wedge \varphi_{k_n}.
\]
Recall that \( n \) is the dimension of the manifold and all eigenvalues and eigenfunctions are counted with multiplicity. Also, the eigenfunctions are supposed to be orthonormal. Obviously, the coefficients \( \tilde{E}_{k_1...k_n} \) are anti-symmetric in all their indices. Also, we see that for manifolds without boundary or with Dirichlet boundary conditions all the invariants \( E_{k_1...k_n} \) vanish and, therefore, the invariant \( \tilde{K}(t; L) \) vanishes as well.

For non-scalar operators \( L \) acting on sections of a vector bundle the above construction obviously does not work. In this case we proceed as follows. Let \( \mathcal{A} \) be the connection one-form; then the covariant exterior derivative has the form

\[
D\varphi = d\varphi + \mathcal{A} \wedge \varphi .
\]

This defines the one-forms \( B_{kl} = \langle \varphi_k, D\varphi_l \rangle \) and the invariants

\[
E_{k_1...k_n}^{l_1...l_n} = \int_M B_{k_1l_1} \wedge ... \wedge B_{k_nl_n} .
\]

Then a similar invariant can be defined by

\[
K(t; L) = \frac{1}{n!} \sum_{k_1, l_1, ..., k_n, l_n = 1}^\infty \exp \left\{ -t(\lambda_{k_1} + \lambda_{l_1} + ... + \lambda_{k_n} + \lambda_{l_n}) \right\} E_{k_1...k_n}^{l_1...l_n} E_{k_1...k_n}^{l_1...l_n} ,
\]

This invariant can be also defined directly in terms of the heat kernel

\[
K(t; L) = \int_{M \times M} dx \, dx' \det \left\{ \mathrm{tr} \left[ U(t; x', x) \nabla_{\mu} \nabla_{\nu} U(t; x, x') \right] \right\} .
\]

Notice that both the integrands in the above equations are densities and, therefore, there is no need for the usual Riemannian volume element.

Both these invariants measure the correlations between the eigensections and the eigenvalues. Contrary to the heat traces they can be called heat determinants.

### 2.6 Heat Trace Asymptotics

The PI proposes to study the short-time asymptotics of the new heat invariants defined above for second-order and first order elliptic partial differential operators. In the smooth category there is an asymptotic expansion of the trace of the heat kernel as \( t \to 0^+ \) [69, 44, 74]

\[
\mathrm{Tr} \exp(-tL) \sim \sum_{k=0}^\infty \frac{t^{(k-n)/2}}{k!} A_k ,
\]

where \( A_k \) are the heat invariants of the operator \( L \) (see [83, 69, 44] and the PI’s papers [16, 3, 15, 12]). The coefficients \( A_k \) are given by integrals over the manifold and its boundary of some local invariants constructed from the jets of the symbol of the operator \( L \) and the boundary operator [68, 69, 54, 78, 53] and the PI’s papers [3, 16, 4] (for reviews, see PI’s papers [15, 17, 26]). In the non-smooth category the asymptotic expansion contains, in general, logarithmic terms [67]. The coefficients of the asymptotic expansion exhibit more complicated structure.

The computation of the heat trace asymptotics in the non-classical situations (non-Laplace type operators, oblique boundary conditions, non-smooth (Zaremba type) boundary conditions) is a much more
complicated problem. Their study is still quite new and the available methodology is underdeveloped in comparison with the theory of classical problems.

First of all, non-classical problems are not automatically elliptic. For oblique boundary value problem one needs an additional condition on the leading symbol of the boundary operator [35]. For Zaremba boundary value problem one has to specify the boundary operator on the singular co-dimension two submanifold $\Sigma_0$ yet to be understood in physical terms. Second, in most of non-classical problems the heat kernel coefficients are non-polynomial in the jets of the operator $L$ and the boundary operator. This complicates their computation significantly. Third, sometimes (e.g. in Zaremba problem) they are not necessarily “locally computable”, which means that there is a need for some (two-dimensional) global analysis.

For oblique boundary conditions only some special cases have been studied in the literature [81, 35, 62, 61, 57]. In the case of non-Laplace type operators only antisymmetric forms have been studied extensively [70, 55, 75, 56, 1]. The study of non-smooth boundary value problems began only recently [42, 60, 59]. It has been shown [85] that at least in the Zaremba type problems the logarithmic terms do not appear.

Despite the importance of non-Laplace type operators in quantum field theory and quantum gravity, their study is still quite new, and the available methodology is still underdeveloped in comparison with the Laplace type theory. The PI hopes to lay the groundwork for a systematic attack on the spectral asymptotics of this larger class of operators (the PI has made some progress in this area in the papers [29, 30] (with T. Branson)).

It is well known that the Einstein functional (with the cosmological constant) is a linear combination of the first two heat trace coefficients for the scalar Laplacian, that is, $S = 6A_1 - 2\Lambda A_0$ (see PI’s papers [18, 20, 21, 22]). The study of Einstein spaces with the Riemannian metrics that are the extremals of the Einstein functional is a very important subject in Riemannian geometry. In these papers, the PI proposed to define a “non-commutative” (matrix-valued) Riemannian metric by the leading symbol of a non-Laplace type operator acting on a $N$-dimensional vector bundle and to study the noncommutative Einstein spaces defined as the extremals of the non-commutative Einstein functional. This would also have deep connections to noncommutative spectral geometry. It would be interesting to study the limit as $N \to \infty$ that may also have some relation to the random matrix theory.

These cases present a significant difficulty in comparison with standard smooth category. It is a new intriguing area and we hope to understand the general structure of the asymptotic expansion as well as to compute explicitly some low-order coefficients.

### 2.7 Non-perturbative Heat Trace Asymptotics

The deformation ideas described above were exploited in the PI’s paper [25] (with G. Fucci). Let $\mathcal{V}$ be a complex vector bundle with the structure group $G \times U(1)$ over a Riemannian manifold $M$ without boundary. The total connection on the vector bundle naturally splits into a $G$-connection and a $U(1)$-connection, which is assumed to have a parallel curvature $F$. We deformed a Laplace type operator $L$ by assuming that the curvature $F$ is of order $t^{-1}$ so that the operator $L$ has exactly an expansion of the form (40) and found a new asymptotic expansion of the off-diagonal heat kernel and the heat trace as $t \to 0$

$$\text{Tr} \exp(-tL) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} B_k(t),$$

(53)
where $B_k(t)$ are new time-dependent spectral invariants of the operator $L$. They are integrals of differential polynomials in the Riemann curvature tensor and the curvature of the $G$-connection and their derivatives with universal coefficients depending in a non-polynomial but analytic way on the curvature $F$, more precisely, on $tF$.

These ideas can find useful applications in various fields of mathematical physics. For instance, our results can be applied to the study of the heat kernel asymptotic expansion on Kähler manifolds. The complex structure on Kähler manifolds is a parallel antisymmetric two-tensor which plays the role of the curvature $F$.

In [3, 2] the PI studied the leading derivatives in heat kernel coefficients of a Laplace type operator $L = \nabla^* \nabla + Q$ and obtained an non-perturbative heat trace asymptotic expansion of the form

$$
\Theta(t; L) \sim (4\pi t)^{-n/2} \int_M \text{tr} \left\{ I - t \left( Q - \frac{1}{6} R \right) + t^2 \left[ Q \gamma_1(t \Delta) Q + (\nabla_\alpha R_\gamma) \gamma_2(t \Delta) \nabla_\beta R_\gamma \right. 
+ Q \gamma_3(t \Delta) R + R_{\alpha\beta} \gamma_4(t \Delta) R_{\alpha\beta} + R \gamma_5(t \Delta) R \right] + O(\mathcal{R}^3) \right\},
$$

(54)

where $Q$ is a smooth endomorphism, $R$ is the scalar curvature, $R_{\mu\nu}$ is the Ricci tensor, $\mathcal{R}_{\mu\nu}$ is the curvature of the connection $\nabla$ and $\gamma_i(\tau)$ are some entire functions computed explicitly in the papers [3, 2]. This expansion can also be obtained by considering a deformed operator $L(t)$ such that the derivatives $\nabla \sim t^{-1/2}$ are of order $t^{-1/2}$ as $t \to 0$.

## 2.8 Heat Invariants on Symmetric Spaces

The PI proposes to study the new heat invariants defined above in some specific cases such as line bundles with parallel curvature, symmetric spaces, Lie groups, Kähler manifolds, and general homogeneous spaces. An interesting question is the relationship between the heat invariants for the dual symmetric spaces. The PI studied and made progress in some of these problems before (see PI’s papers [5, 6, 7, 9, 24, 25, 27, 31, 32]). The high symmetry of these spaces enables one to express the heat semigroup of a Laplace type operator in terms of the heat semigroup of some first-order differential operators such as Killing vectors and Lie derivatives, which allows one to compute the heat traces without solving the differential equation for the heat kernel. He also applied his methods and the results to various problems in quantum field theory and quantum gravity.

## 3 Broader Impacts of the Proposed Activity

### Integration of Research and Education

An important objective of the project is to integrate the proposed research with the training of students in related fields. The proposed activity will provide multidisciplinary education and training opportunities to undergraduate and graduate students. Graduate students from mathematics or physics will be involved in this project. Working at an appropriate level, a student will focus upon one or more of the above mentioned (sub)-problems. In addition to the research benefits, the project will also be used to develop and strengthen the mathematical physics community at New Mexico Tech. The participating students will benefit from studying a range of topics, including functional analysis, partial differential equations, differential geometry, perturbation techniques, asymptotic expansions, applications of theory of complex variables, integral transforms, special functions etc,
almost all of which is accessible to upper level undergraduates. Whilst, at graduate level, the students might also be motivated to pursue advanced studies in these topics. The results of the research in the proposed area will definitely have a positive impact on the content and the attraction of the upper level (beginning graduate) courses as well as on the development of new courses, such as a new graduate course in global analysis. The results from the research will benefit teaching at all levels: undergraduate, graduate as well as thesis supervision. The PI supervises two graduate students, one specializing in mathematical physics and another in analysis, and will offer research oriented projects for both graduate and undergraduate students. A concrete direct benefit from the proposal is the preparation of new course materials and a new book. This project will certainly have a concrete impact on the graduate program in mathematics at New Mexico Tech.

Broading the Participation of Underrepresented Groups. New Mexico Tech is a public school that is identified as an EPSCoR institution, has a large underrepresented minority student body and is dedicated to increasing minority participation in science and engineering. In the state of New Mexico the minorities (Hispanics and Native Americans) represent a major part of the population. This is also reflected in the student body of New Mexico Tech. New Mexico Tech is designated as an eligible institution by the US Department of Education as a Hispanic-Serving Institution. The PI is committed to recruit and train minority students. Many of these students are interested in mathematics and physics. Currently, the PI supervises one Hispanic graduate student whose research is directly related to the proposed activity. The current project will attract students at both undergraduate and graduate levels to the research in the area of geometric analysis and mathematical physics. The current award will help in the research training and education of these students, increase the reputation of New Mexico Tech and attract even more students to the graduate programs at New Mexico Tech. This would also lead to establishing solid collaborative partnerships between New Mexico Tech and leading research schools in the country.

Enhancement of the Infrastructure for Research and Education The project will enhance active collaborations with other groups and individuals active in this area. The collaboration on this project will also be particularly useful for motivating students from other departments to interact and study topics of common interest in both mathematics and physics departments.

Dissemination of the Results of the Project The results obtained in this proposal will be posted in the electronic online journals and archives, presented at the national and international conferences, published in the professional scientific internationally recognized journals as well as used in the partnerships with scientists around the world.

4 Prior Research of the PI Related to the Project

The PI has been working in the area of heat kernel asymptotics for more than twenty years. He published one book on the heat kernel and quantum gravity [16]. Recently another book [28] on the heat kernel and its application to finance has been accepted for publication by Springer. The results of his research are published in many research papers in refereed scientific journals. They have been presented at many international scientific conferences and are widely known among the experts working in this area.

Specifically, the PI studied and made progress in the following areas:
1. Developed a new method for calculating the local off-diagonal heat kernel coefficients for Laplace type operators on vector bundles over Riemannian manifolds [3]. Applied this method to compute the effective action in quantum field theory. This method has been implemented on computers and is widely used [46, 88, 89, 86, 82].

2. Studied non-perturbative heat kernel expansion for Laplace type operators on vector bundles with partially parallel connection over a Riemannian manifold without boundary [40, 38] (with G. Fucci). Applied these results to the study of non-perturbative Quantum Electrodynamics in curved spacetime.

3. Studied the general structure of higher-order global heat kernel coefficients and the asymptotic form of the heat trace [2, 3]. Applied these results to the study of the asymptotic non-local structure of the effective action in quantum field theory.

4. Studied the heat kernel asymptotics for boundary value problems for Laplace type operators: Dirichlet problem [4], oblique boundary value problem [36, 35] (with G. Esposito) and (discontinuous) Zaremba boundary value problem [19]. Applied these results to the study of quantum gravity and gauge theories on manifolds with boundary [33, 34, 35, 37] (with G. Esposito).

5. Studied the Hadamard expansion of the resolvent and the heat kernel of higher order differential operators [11, 12]. Applied these results to the study of Huygens principle of these operators.

6. Studied the heat kernel coefficients for Schrödinger operators as well as Schrödinger operators with matrix-valued potentials and their relation to the Korteweg-de Vries Hierarchy [41, 42] (with R. Schimming).

7. Studied the heat kernel for non-Laplace type operators on spin-tensor bundles over Riemannian manifolds without boundary [29, 30] (with T. Branson) as well as with boundary [23].

8. Developed a non-commutative Riemannian geometry from the heat kernel asymptotics of non-Laplace type operators [18, 20, 21, 22]. Proposed and studied a non-commutative General Relativity, [20, 21, 22] and [64, 65, 39] (with G. Fucci).

9. Studied the heat kernel for Laplace type operators with parallel connection and parallel curvature: (a) in flat Euclidean space with parallel connection and parallel or quadratic potential [5, 7], (b) scalar Laplacian on symmetric spaces [6, 9]. (c) Laplace type operator on homogeneous vector bundles with parallel connection over symmetric space [24, 25]. Applied these results to the calculation of the effective action in quantum gravity and gauge theories [8, 14, 27].

10. Studied the heat kernel for Laplace type operators acting on sections of spin-tensor bundles valued in the Lie algebra of a gauge group with a parallel bundle curvature over $S^2$ and $S^3$ [31, 32] (with S. Collopy). Applied these results to the study of the stability of Yang-Mills theory.

This research experience will be very helpful for the PI to carry out the proposed project.
References


