Problem 3. Let \( A = \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 1 & 2 & 3 & 0 & 3 \\ 2 & 3 & 3 & -1 & 5 \end{pmatrix} \).

1. Find a basis for the column space of \( A \).

\[ A \rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 & -2 & 1 \\ 0 & 1 & 3 & 1 & 1 \end{pmatrix} \]

Columns 1 and 2 are pivot => 
\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \text{Col}(A). \]

2. Find a basis for the row space of \( A \).

\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \\ -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \text{Row}(A). \]

3. Find a basis for the null space of \( A \).

\( \text{RREF}(A) \): Variables \( x_3, x_4, x_5 \) are free

\[ x_5 = t \\
 x_4 = s \\
 x_3 = u \\
 x_2 = -3u - s + t \\
 x_1 = 3u + 2s + t \]

\[ x = t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \]

\[ \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \text{Null}(A). \]

4. Determine the rank and the nullity of \( A \).

\[ \text{rank}(A) = \# \text{pivot columns} = 2, \quad \text{Nullity}(A) = n - \text{rank}(A) = 5 - 2 = 3. \]

5. What is the rank and the nullity of \( A^T \)?

\[ \text{rank}(A) = \text{rank}(A^T) = 2, \quad \text{Nullity}(A^T) = m - \text{rank}(A^T) = 3 - 2 = 1. \]