Test 2

<table>
<thead>
<tr>
<th>Problem</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>Score</th>
</tr>
</thead>
</table>

NAME: __________

Solution Key

Show all your work for full credit. Electronic devices, the textbook, and lecture notes are not allowed. You may use a single page with your notes.

Problem 1. Use the method of washers to find the volume of the solid of revolution obtained by revolving about the x-axis a planar region in the xy-plane (see the following figure) bounded by the curves and lines given by

\[ y = 2 - (x - 1)^2 \text{ and } y = 1. \]

\[
V = \pi \int_0^2 \left( (2 - (1-x)^2) - 1 \right)^2 \, dx
\]

\[
= \pi \int_0^2 \left( 4 - 4(x-1)^2 + (x-1)^4 - 1 \right) \, dx
\]

\[
= \left. \pi \int \left[ 4 - 4u^2 + u^4 - 1 \right] \, du \right|_{-1}^{1}
\]

\[
= \pi \left( 3 \cdot 2 - 4 \cdot \frac{1^3}{3} \right) + \pi \left( \frac{1^5}{5} \right)
\]

\[
= \pi \left( 6 - \left( \frac{4}{3} + \frac{4}{3} \right) + \left( \frac{1}{5} + \frac{1}{5} \right) \right) = \pi \left( 6 - \frac{8}{3} + \frac{2}{5} \right)
\]

\[
= \frac{56}{15} \pi
\]
Problem 2. Use the method of shells to find the volume of the solid of revolution obtained by revolving the planar region described in Problem 1 about the line $x = -1$. 

$$r(x) = x + 1$$

$$h(x) = f(x) - 1 = 2 - (x - 1)^2 - 1 = 1 - (x - 1)^2.$$ 

$$V = 2\pi \int_0^2 r(x) h(x) \, dx = 2\pi \int_0^2 (x + 1) \left( 1 - (x - 1)^2 \right) \, dx$$

$$= \left. 2\pi \int_0^2 \left( u + 1 \right) \left( 1 - u^2 \right) \, du \right|_1^1 = 2\pi \int_{-1}^1 \left( u + 2 - u^3 - 2u^2 \right) \, du$$

$$= 2\pi \left( \frac{u^2}{2} + 2u - \frac{u^4}{4} - \frac{2}{3} u^3 \right) \Bigg|_1^1 = 2\pi \left( 2u - \frac{2}{3} u^3 \right) \Bigg|_{-1}^1$$

$$= 2\pi \cdot 2 \left( 2 - \frac{2}{3} \right) = \frac{16\pi}{3}.$$
Problem 3. Find the length of the curve given by the function

\[ y = \frac{2}{3}x^{3/2} + 1, \quad 0 \leq x \leq 3. \]

\[ L = \int_a^b \sqrt{1 + (y')^2} \, dx \]

\[ y' = \frac{2}{3} \cdot \frac{3}{2} \cdot x^{1/2} = x^{1/2}. \]

\[ \sqrt{1 + (y')^2} = \sqrt{1 + x}. \]

\[ L = \int_0^3 \sqrt{1 + x} \, dx = \left. \frac{u^{3/2}}{3} \right|_0^4 = \frac{2}{3} \left. u^{3/2} \right|_1^4 \]

\[ = \frac{2}{3} \left( 4^{3/2} - 1 \right) = \frac{2}{3} \left( 8 - 1 \right) = \frac{14}{3}. \]
Problem 4. Find the surface area of the solid of revolution obtained by revolving about the \( x \)-axis the curve given by the function

\[
y = \sqrt{x}, \quad 0 \leq x \leq 2.
\]

\[
A = 2\pi \int_{a}^{b} y(x) \sqrt{1 + (y'(x))^2} \, dx
\]

\[
y(x) \sqrt{1 + (y'(x))^2} = \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} = \sqrt{x \left(1 + \frac{1}{4x}\right)}
\]

\[
= \sqrt{x + \frac{1}{4}}.
\]

\[
A = 2\pi \int_{0}^{2} \sqrt{x + \frac{1}{4}} \, dx = \left| \begin{array}{c}
u = x + \frac{1}{4} \\
\frac{du}{dx} = \frac{1}{4}
\end{array} \right| = 2\pi \int_{\frac{2}{4}}^{\frac{2+1/4}{4}} u^{1/2} \, du
\]

\[
= 2\pi \int_{1/4}^{9/4} u^{1/2} \, du = 2\pi \cdot \frac{2}{3} u^{3/2} \bigg|_{1/4}^{9/4}
\]

\[
= \frac{4\pi}{3} \left( \left(\frac{9}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2} \right) = \frac{4\pi}{3} \left( \frac{27}{8} - \frac{1}{8} \right)
\]

\[
= \frac{13\pi}{6}.
\]
Problem 5. A tank is filled with water. The end of the tank is a planar region in the $xy$-plane bounded by the curve of $y = x^2 - 1$ and the $x$-axis (see the following figure). Set up the integral that gives the value of the hydrostatic force against the end of the tank. Do not evaluate the integral. The weight density of water is $\omega = 62.5 \text{ lb/ft}^3$.

\[ F = \omega \int_{-1}^{1} d(y) W(y) \, dy \]

where

- $d(y) = -y - \text{ depth}$
- $W(y) = 2 \cdot x(y) - \text{ width}$.
- $x(x) = x-1$ $\Rightarrow$ $x(y) = \sqrt{y+1}$

\[ F = \omega \int_{-1}^{0} (-y) \cdot 2 \cdot \sqrt{y+1} \, dy \]

\[ F = -\omega \int_{-1}^{0} y \sqrt{y+1} \, dy \] (1b)

**Integral evaluation (optional)**:

\[ F = -2 \omega \int_{-1}^{0} y \sqrt{y+1} \, dy = \left. \frac{t^{3/2}}{1/2} \right|_{0}^{1} \]

\[ = -2 \omega \int_{0}^{1} (t^3 - t^{1/2}) \, dt = -2 \omega \left( \frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right) \bigg|_{0}^{1} \]

\[ = -2 \omega \left( \frac{2}{5} - 0 \right) = 2 \omega \frac{10 - 6}{15} = \frac{8}{15} \omega. \] (1b)

\[ F \approx 33 \frac{1}{3} \text{ (lb)} \]
Problem 6. Find the work required to empty a tank full of water in the shape of an upright cylinder with height 10 feet and the base radius 5 feet by pumping water out the outlet on the top of the tank.

\[ R = 5, \quad H = 10. \]

\[
W = \omega \int_a^b d(x) A(x) \, dx
\]

\[
A(x) = \pi R^2 = \pi \cdot 5^2 = 25\pi.
\]

\[
d(x) = 10 - x.
\]

\[
W = \omega \int_0^{10} (10-x) \cdot 25\pi \, dx
\]

\[
W = \omega \left[ 25\pi \left( 100 - \frac{x^2}{2} \right) \right]_0^{10} = 25\pi \omega \left( 100 - 50 \right)
\]

\[
W = 1250\pi \omega \quad (\text{lb} \cdot \text{ft})
\]

\[
W \approx 245,430 \quad (\text{lb} \cdot \text{ft})
\]
Problem 7. Determine if the sequence is convergent or divergent and explain why. If convergent, find the limit of the sequence.

Let \( f(x) = \frac{x}{e^x} \).

\( f(x) \) is a differentiable function.

\[
f'(x) = \frac{e^x - x e^x}{e^{2x}} = \frac{1-x}{e^x} \leq 0 \text{ for } x \geq 1.
\]

\( \Rightarrow \) function \( f(x) \) is decreasing

\( \Rightarrow \) the sequence \( \left\{ \frac{k}{e^k} \right\} \) is decreasing.

Since \( f(k) = \frac{k}{e^k} \geq 0 \) for \( k = 1, 2, \ldots \),
the sequence is bounded below.

Since \( \left\{ \frac{k}{e^k} \right\} \) is a monotonic bounded sequence,
it is convergent.

\[
\lim_{k \to \infty} \frac{k}{e^k} = \lim_{k \to \infty} \frac{k'}{(e^k)'} = \lim_{k \to \infty} \frac{1}{(e^k)'} = \lim_{k \to \infty} \frac{1}{e^k} = 0.
\]
Problem 8. Determine if the series is convergent or divergent. If the series is convergent, then find its limit.

1. \( S = \sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k} = 2 \cdot 3 \sum_{k=0}^{\infty} \left( \frac{2}{3} \right)^k \), Geometric Series

\( r = \frac{2}{3} \Rightarrow |r| = \frac{2}{3} < \infty \Rightarrow \text{Convergent} \)

\( \Rightarrow S = 6 \cdot \frac{1}{1-r} = 6 \cdot \frac{1}{1-\frac{2}{3}} = 6 \cdot 3 = 18 \)

2. \( S = \sum_{k=1}^{\infty} \frac{2}{k(k+1)} \)

Note \( \frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1} \)

\( \Rightarrow S = \sum_{k=1}^{\infty} \left( \frac{2}{k} - \frac{2}{k+1} \right) = \text{Telescoping Series} \)

\( S_n = \sum_{k=1}^{n} \left( \frac{2}{k} + \frac{2}{k+1} \right) = \left( \frac{2}{1} - \frac{2}{2} \right) + \left( \frac{2}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{2}{4} \right) + \ldots + \left( \frac{2}{n} - \frac{2}{n+1} \right) = 2 - \frac{2}{n+1} \)

\( S_n \to 2 \text{ as } n \to \infty \Rightarrow \text{Convergent} \)

\( \Rightarrow \boxed{S = 2} \)