Stability Analysis

- Von Neumann (harmonic) Analysis – apply Fourier series to analyze stability of the numerical scheme

- Substitute for all dependent variables in original difference equation and find criterion as to whether it remains bounded as t$\rightarrow$ large.

$$f(x,t) = \psi(t)e^{j\beta x}$$

time

space

- Where β is a positive constant and $j = \sqrt{-1}$

Stability Analysis

- For $\theta = 0$, **Explicit Method**

$$\left[D \frac{C_{i+1}^{n} - 2C_{i}^{n} + C_{i-1}^{n}}{\Delta x^{2}} \right] = \frac{C_{i}^{n+1} - C_{i}^{n}}{\Delta t}$$

- Substitute, rearrange and simplify,

$$A = \frac{\psi(t+\Delta t)}{\psi(\Delta t)} = 1 - 4\lambda \sin^{2}\left(\frac{\beta \Delta x}{2}\right)$$

- Where A is defined as the Amplification Factor and $\lambda = \frac{D\Delta t}{\Delta x^{2}}$

- If $|A| \geq 1$, means the error term will be amplified as time increases…**numerical scheme will be unstable**.

Stability Analysis

- Requirement,

$$\left|1 - 4\lambda \sin^2\left(\frac{\beta \Delta x}{2}\right)\right| \leq 1$$

- Furthermore, must guard against unbounded amplification for all β

$$\left|\sin^2\left(\frac{\beta \Delta x}{2}\right)\right|_{\max} = 1$$

- Thus,

$$|1 - 4\lambda| \leq 1$$

or

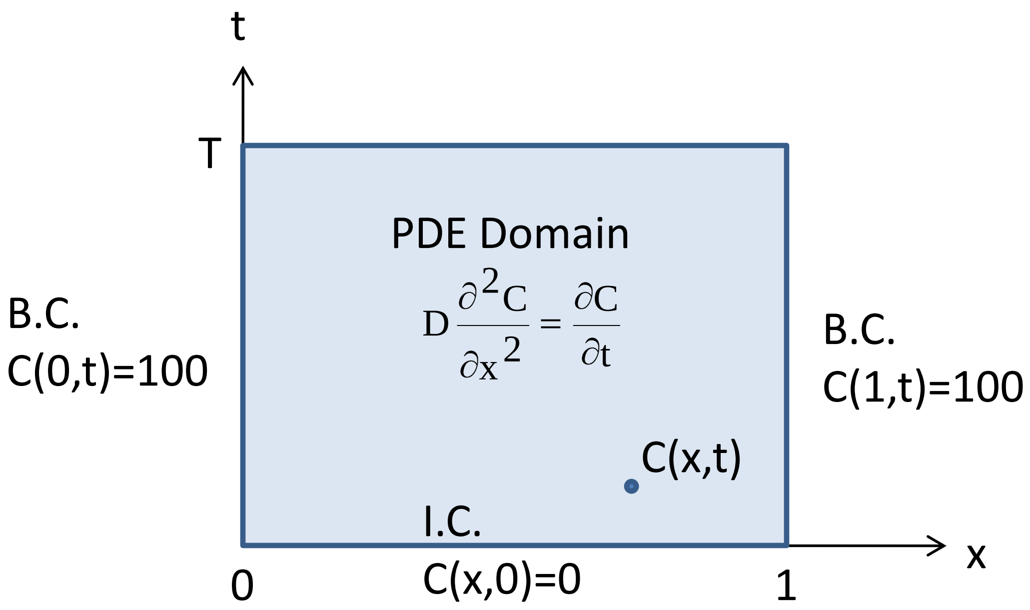
$$0 \leq \lambda \leq \frac{1}{2}$$

$$\lambda = \frac{D\Delta t}{\Delta x^2}$$

- Necessary and sufficient condition for stability!

Stability Analysis

- Example of stable and unstable explicit formulation



- Case 1: D = 1, $\Delta x = 0.2$, $\Delta t = 0.01$

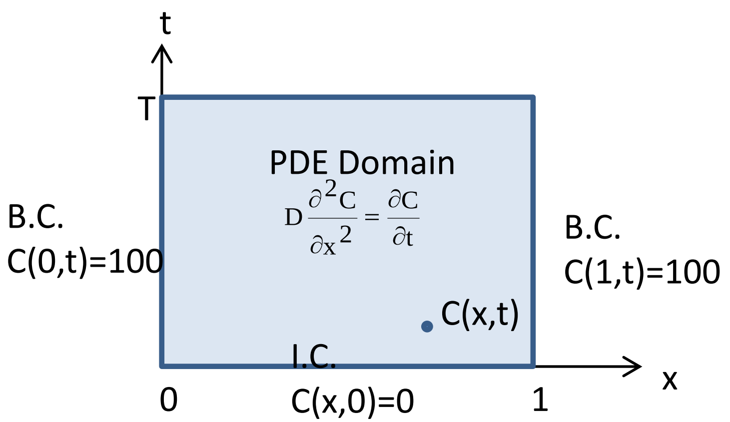
Stability Analysis

- Example of stable and unstable explicit formulation

$$C_i^{n+1} = \lambda C_{i+1}^n + (1-2\lambda) C_i^n + \lambda C_{i-1}^n$$

where $\lambda = \frac{1}{4}$, then

$$C_i^{n+1} = \frac{C_{i+1}^n + 2C_i^n + C_{i-1}^n}{4}$$



time step,n	space					
	0	1	2	3	4	5
0	0	0	0	0	0	0
1	100	0	0	0	0	100
2	100	25	0	0	25	100
3	100	38	6	6	38	100
4	100	45	14	14	45	100
5	100	51	22	22	51	100
6	100	56	29	29	56	100

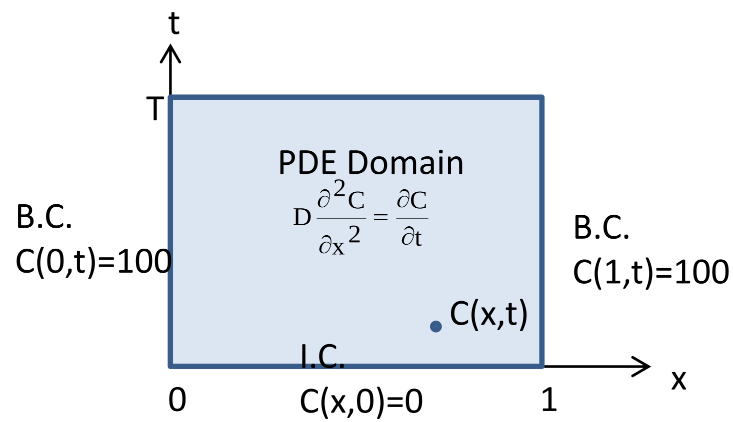
Stability Analysis

- Example of stable and unstable explicit formulation
- Case 2: D = 1, $\Delta x = 0.2$, $\Delta t = 0.04$

$$C_i^{n+1} = \lambda C_{i+1}^n + (1-2\lambda) C_i^n + \lambda C_{i-1}^n$$

where $\lambda = 1$, then

$$C_i^{n+1} = C_{i+1}^n - C_i^n + C_{i-1}^n$$



time step,n	space					
	0	1	2	3	4	5
0	0	0	0	0	0	0
1	100	0	0	0	0	100
2	100	100	0	0	100	100
3	100	0	100	100	0	100
4	100	200	0	0	200	100
5	100	-100	200	200	-100	100
6	100	400	-100	-100	400	100

- Conclusion: Whenever using explicit method, the time interval should be determined by gridsize and diffusion constant.

Stability Analysis

- For θ = 1, **Implicit Method**

$$\left[D \frac{C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}}{\Delta x^2} \right] = \frac{C_i^{n+1} - C_i^n}{\Delta t}$$

- Substitute, rearrange and simplify,

$$A = \frac{\psi(t + \Delta t)}{\psi(\Delta t)} = \frac{1}{1 + 4\lambda \sin^2\left(\frac{\beta \Delta x}{2}\right)}$$

- Since |A| ≤ 1, for all λ, the procedure is unconditionally stable.
- If $\sin^2(y) = 1$,

$$A = \frac{1}{1 + 4\lambda}$$