8.1 Transient Flow

The transient condition is only applicable for a relatively short period after some pressure disturbance has been created in the reservoir. In practical terms, if pressure is reduced at the wellbore, reservoir fluids will begin to flow near the vicinity of the well. The pressure drop of the expanding fluid will provoke flow from further, undisturbed regions in the reservoir. The pressure disturbance and fluid movement will continue to propagate radially away from the wellbore. The gradually extending region affected by flow is seen in Figure 8.1. In the time for which the transient condition is applicable it is assumed that the pressure response in the reservoir is not affected by the presence of the outer boundary, thus the reservoir appears infinite in extent.

![Figure 8.1 Conceptual view of transient flow prior to reaching outer boundary](image)

The condition is mainly applied to the analysis of well tests in which the well's production rate is deliberately changed and the resulting pressure response in the
wellbore is measured and analyzed during a brief period of a few hours after the rate change has occurred. Then, unless the reservoir is extremely small, the boundary effects will not be felt and the reservoir is, mathematically, infinite. In this case, both the pressure and pressure derivative, with respect to time, are functions of both position and time, i.e., \( p = f(r,t) \).

Two modes of transient production are constant production rate and constant flowing wellbore pressure. Both are shown in Figure 8.1. The figure illustrates that constant rate implies a decline in wellbore pressure and that constant pressure implies a decline in production rate. Mathematical solutions of both conditions are well documented in the literature.

The continuity equation for transient flow, in Cartesian coordinates, can be expressed as:

\[
- \nabla (\rho \mathbf{v}) = \frac{\partial}{\partial t} (\rho \phi) \tag{8.1}
\]

As described in Section 6.2, combining Darcy’s Law

\[
\mathbf{v} = -\frac{K}{\mu} \nabla \psi \tag{6.60}
\]

where

\[
\psi = \int \frac{p}{\rho} \frac{dp}{p_o} + gz \tag{6.61}
\]

with an equation of state, for example, assume an “ideal liquid” where compressibility is constant,

\[
c = \frac{1}{\rho} \frac{\partial \rho}{\partial P} \tag{6.66}
\]

results in a diffusivity equation. For radial geometry, isotropic and homogeneous porous media, the diffusivity equation is,

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{\phi \mu c \partial p}{k \partial t} \tag{8.2}
\]

To solve this problem requires initial and boundary conditions. For an infinite-acting reservoir, with constant rate production at the wellbore, the following conditions apply.
\[
P(r,0) = P_i
\]
\[
P(\infty,t) = P_i
\] (8.3)

\[
\frac{\partial p}{\partial r} \bigg|_{r \to 0} = -\frac{q\mu}{2\pi k h}
\]

The wellbore is assumed to be infinitesimally small, i.e. \( r \to 0 \), or line source. A mathematical technique to solve the transient, radial flow of a single-phase, slightly compressible fluid can be accomplished applying the Boltzmann transform.

\[
s = \frac{q\mu cr}{4kt}
\] (8.4)

The solution is:

\[
p(r,t) = p_i - \frac{q\mu}{4\pi k h} Ei(-s)
\] (8.5)

where \( Ei(-s) \) is known as the exponential integral, \( Ei(-x) = \int_{-\infty}^{x} e^{-s} ds \) and \( x \) is defined as:

\[
x = \frac{q\mu cr}{4kt}
\]

A widely applied approximation occurs for \( x < 0.01 \), such that

\[
Ei(x) \equiv -\ln(x) = -\ln(1.781x)
\] (8.6)

Figure 8.2 Comparison of Ei function with natural log approximation
8.2 Superposition

The principle of superposition states the total pressure drop at any point in the reservoir is the sum of the pressure drops at any point caused by flow in each of the wells in the reservoir. As an example, consider a three well system illustrated in Figure 8.3.

![Figure 8.3. Application of superposition in space for a three well system](image)

The total pressure drop in well A is the summation of the pressure drops caused by each well.

\[
\Delta p_{\text{total A}} = \Delta p_A + \Delta p_B + \Delta p_C
\]  

(8.7)

Substituting in Eq. (8.7) the line source solutions, and converting to field units, results in the following equation for total pressure drop in Well A.

\[
\Delta p_{\text{total A}} = \frac{141.2 q_a \mu BO}{kh} \left[ \frac{1}{2} \ln \left( \frac{kt}{1688 \phi_{ac} r_{ac}^2} \right) + S_a \right] - \frac{141.2 q_b \mu BO}{kh} \frac{1}{2} E_i \left( \frac{948 \phi_{ac} r_{ab}^2}{kt} \right) - \frac{141.2 q_c \mu BO}{kh} \frac{1}{2} E_i \left( \frac{948 \phi_{ac} r_{ac}^2}{kt} \right)
\]

(8.8)

Common applications of this method include: evaluating multiwell pressure transient tests; e.g., particularly for interference or pulse tests, simulating pressure behavior in reservoirs with boundaries, and for identifying well locations.
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In compact dimensionless form:

\[ \Delta p(t,r) = \frac{141.2 \mu B}{kh} \sum_{j=1}^{n} q_j P_D(r_{Dj},t_D) \]  
(8.9)

**Superposition of variable rate and time**

The objective of superposition of variable rate and time is to model variable rate behavior in wells to correctly analyze pressure test measurements. Figure 8.4 illustrates a simple single well, two-rate test. The problem can be defined as at some time, \( t, t > t_1 \), what is \( p_{wf} \)? The well produces at a rate, \( q_1 \) from \( t = 0 \) to \( t_1 \) at which time the production rate changes to \( q_2 \) from \( t_1 \) to \( t \).

![Figure 8.4. Two-rate test](image)

The solution using the principle of superposition is:

\[
\Delta p_T = \frac{141.2 \mu B}{kh} \left[ \frac{1}{2} \ln \left( \frac{k(t-t_o)}{1688 \phi \mu c t_w^2} \right) + S \right] \\
+ \frac{141.2(q_2 - q_1) \mu B}{kh} \left[ \frac{1}{2} \ln \left( \frac{k(t-t_1)}{1688 \phi \mu c t_w^2} \right) + S \right] 
\]  
(8.10)

Expressed in compact form,

\[
\Delta p_T = \frac{141.2 \mu B}{kh} \sum_{j=1}^{n} \left[ \frac{1}{2} \ln \left( \frac{k(t-t_{j-1})}{1688 \phi \mu c t_w^2} \right) + S \right] \cdot (q_j - q_{j-1}) 
\]  
(8.11)

The general form of equation for superposition in space and time/rate is given by,
\[ \Delta p(t, r) = \frac{141.2 \mu B}{kh} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( q_j - q_{j-1} \right) \rho \left( D_i + \left( t - t_{j-1} \right) D_j \right) \]  

(8.12)

where \( m \) is the number of wells and \( n \) is the number of rates for each well. A special case of Eq. (8.10) occurs if \( q_2 = 0 \) (shut in). The result is the Horner Approximation to pressure buildup testing.

**Method of Images**

Employing a technique called the “method of images” derives the pressure behavior of a well near a sealing linear fault or other flow barrier. In this formulation the effect of a fault is simulated by assuming the presence of another identical well producing at a symmetrical position across the fault, as shown in Figure 8.5. The image well interacts with the actual well so that no flow occurs across the fault. The resulting pressure drop at the real well due to its own production and the "interference drop" from the image well add together to simulate correctly the pressure behavior of the real well as though it were in the proximity of the fault.

![Figure 8.5](image)

Figure 8.5. Schematic of a no flow boundary with a real and image well located a distance \( d \) from the boundary.

By applying superposition we can calculate the pressure at any point in the xy plane. For the special case at the wellbore and constant production rate,
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\[
P_{wf} = P_i - \frac{141.2qMB}{kh} \left[ \frac{r^2}{2} - \frac{Ei(-x)}{t} \right]
\]

\[
or \text{ if assume ln approximation,}
\]

\[
P_{wf} = P_i - \frac{141.2qMB}{kh} \left[ \frac{1}{2} \ln \left( \frac{kt}{168 \phi \mu c r_w^2} \right) + S \right] - \frac{1}{2} \left[ Ei \left( \frac{948 \phi \mu c (2d)^2}{kt} \right) \right]
\]

With the previous definition of \( x \), the second part of Eq. (8.13) is abbreviated notation where the subscript refers to the time function and the superscript the spatial distance.

Example

Consider a single sealing fault located 250 feet away from the active well. What is the bottom hole flowing pressure after flowing 350 stbd for 8 days, given the following information.

\( k = 25 \text{ md} \quad c_i = 2 \times 10^{-5} \text{ psia}^{-1} \)

\( \mu = 0.50 \text{ cp} \quad B_o = 1.13 \text{ res bbl/STB} \)

\( \phi = 16.0\% \quad h = 50 \text{ ft} \)

\( p_i = 3000 \text{ psi} \quad r_w = 0.333 \text{ ft} \quad \text{Skin} = +5 \)

1. Check if the logarithmic approximation is valid

\[
x = \frac{948 \phi \mu c}{k} \cdot \frac{t^2}{r^2} = \frac{948 \cdot (16) \cdot (0.5) \cdot (2 \times 10^{-5})}{25 \cdot (8) \cdot (24)} = 3.16 \times 10^{-7} \cdot \frac{t^2}{r^2}
\]

For \( r = r_w \), then \( x_{rw} = 3.5 \times 10^{-8} << 0.01 \), thus In approximation applies

For \( r = 2L \) then \( x_{2L} = 0.079 >> 0.01 \), thus Ei function applies

2. Calculate the pressure drop

\[
\Delta P = \frac{141.2qMB}{kh} \left[ \frac{1}{2} \ln \left( \frac{kt}{168 \phi \mu c r_w^2} \right) + S \right] - \frac{1}{2} \left[ Ei \left( \frac{948 \phi \mu c (2d)^2}{kt} \right) \right]
\]

\[
= 22.337 \left[ \frac{1}{2} \ln \left( \frac{25 \cdot (8) \cdot (24)}{168 \cdot (16) \cdot (0.5) \cdot (2 \times 10^{-5}) \cdot (0.333)} \right) + 5 \right] + \frac{1}{2} \left( 2.039 \right)
\]

\[
= 22.337 \cdot [(8.2945 + 1.0195)]
\]

\[
= 297 + 23
\]

\[
= 320 \text{ psi}
\]
3. Calculate the bottom hole flowing pressure

\[ p_{wf} = p_i - \Delta p = 3000 - 320 = 2680 \text{ psi} \]

If a pressure buildup test is run in the well then both superposition in space and time must be applied to develop a solution. In abbreviated form, the resulting solution can be expressed as,

\[
p_{ws} = p_i - \frac{141.2q\mu B}{kh} \frac{1}{2} \left[ - \frac{r_w^2}{t+\Delta t} \text{Ei}(-x) + \frac{r_w^2}{t+\Delta t} \text{Ei}(-x) \frac{(2d)^2}{\Delta t} + \text{Ei}(-x) \frac{(2d)^2}{\Delta t} \right] \quad (8.14)
\]

As an approximation, consider time (\(\Delta t\)) sufficiently large such that \(x << 0.01\) and thus at the wellbore the logarithmic approximation is reasonable. Also, early in the buildup when \(\Delta t\) is small and provided \((d)\) is large, then the arguments of the \(\text{Ei}\) function becomes large and thus \(\text{Ei}(\infty) \to 0\). Subsequently the last two terms in Eq. (8.14) approach zero, and the equation reduces to;

\[
p_{wf} = p_i - \frac{141.2q\mu B}{kh} \ln \left( \frac{\Delta t + t}{\Delta t} \right) \quad (8.15)
\]

Thus early in the pressure buildup the slope of the buildup plot will remain unchanged from the conventional analysis. As the shut-in time (\(\Delta t\)) increases, the logarithmic approximation becomes reasonable for the image well terms, thus

\[
p_{wf} = p_i - \frac{141.2q\mu B}{kh} \ln \left( \frac{\Delta t + t}{\Delta t} \right) + \ln \left( \frac{\Delta t + t}{\Delta t} \right) \quad (8.16)
\]
The result is a doubling of the slope. An example is shown in Figure 8.6.

![Horner pressure buildup plot with linear discontinuity](image)

**Figure 8.6 Example of Horner pressure buildup plot with linear discontinuity (Earlougher, 1977)**

To estimate the distance to the linear discontinuity for a pressure buildup test requires the intersection point of the two straight lines as shown in Figure 8.6. If we equate Eqs. (8.15) and (8.16), the result is an expression for the distance to the fault.

\[
d = \sqrt{\frac{0.0002637k t}{\gamma \phi \mu c \left( t + \Delta t \right)}}
\]

(8.17)

where \( g \) is the exponential of Euler’s constant \( = 1.781 \). Eq. (8.17) is valid for large \( t \).

**Example**

The following well and reservoir data was acquired for the test data of Figure 8.6.

\[
\begin{align*}
t_p &= 530 \text{ hrs} \\
\mu &= 0.20 \text{ cp} \\
k &= 40.6 \text{ md} \\
\phi &= 0.09 \\
c_t &= 22.6 \times 10^{-6} \text{ psi}^{-1}
\end{align*}
\]

From the figure the intersection point = 285, therefore from Eq. (8.15) the distance to the fault, \( d = 166 \text{ ft} \).
8.3 Unsteady state water influx model

The transient nature of many aquifers suggests a time dependent term is necessary to
calculate the water influx into a reservoir. Consider a circular reservoir of radius \( r_b \), as
shown in figure 8.7, in a horizontal, circular aquifer of radius, \( r_e \), which is isotropic and
homogeneous.

![Figure 8.7 Schematic of a radial reservoir/aquifer system](image)

In this case the inner boundary is defined as the interface between the reservoir
and the aquifer. It thus becomes useful to require pressure at the inner boundary to be
constant and observe the flow rate as it enters the reservoir from the aquifer. The
solution to this problem is known as the \textit{constant terminal pressure solution} (CTPS) and
was solved by Van Everdingen and Hurst, 1949. The conditions necessary are:

\[
p(r,0) = p_i \\
p(\infty, t) = p_i \quad \text{Infinite} \\
or \quad \frac{\partial p}{\partial r}
\bigg|_{r_e} = 0 \\
\text{bounded}
\]

\[p = \text{constant } @ r = r_b \quad (8.18)\]

Darcy’s Law gives the rate of fluid influx,

\[
q(t) = \frac{2\pi kh}{\mu} \left( r \frac{\partial p}{\partial r} \right) \quad (8.19)
\]

Define dimensionless radius, \( r_D = r/r_b \), then at the interface,
\[ q(t) = \frac{2\pi kh}{\mu} \left( \frac{\partial p}{\partial r_D} \right)_{r_D=1} \]  

(8.20)

The cumulative influx is given by,

\[ W_e = \int_{0}^{t} q(t) \, dt \]

(8.21)

\[ = \frac{2\pi kh}{\mu} \int_{0}^{t} \left( \frac{\partial p}{\partial r_D} \right)_{r_D=1} \, dt \]

Define dimensionless time, \( t_D \), as

\[ dt_D = \frac{k}{\phi \mu c \, r_b^2} \, dt \]  

(8.22)

where \( k, \phi, \mu, \) and \( c \) are properties of the aquifer. Substitute into Eq. (8.21),

\[ W_e = 2\pi h c \, r_b^2 \int_{0}^{t} \left( \frac{\partial p}{\partial r_D} \right)_{r_D=1} \, dt_D \]

(8.23)

\[ = 2\pi h c \, r_b^2 \cdot W_D(t_D) \]

where \( W_D(t_D) \) is the dimensionless cumulative water influx per unit pressure drop imposed at the reservoir/aquifer boundary. Define the water influx constant, \( B \), as:

\[ B = 2\pi h c \, r_b^2 \cdot f \]  

(8.24)

where \( f \) is fraction of the aquifer which subtends the reservoir circumference; i.e., for a full circle as shown in Fig. 8.7, \( f = \theta/360^\circ = 360^\circ/360^\circ = 1 \). Therefore for any pressure drop, \( \Delta p = p_i - p \) at the boundary, the water influx can be expressed as:

\[ W_e = B \cdot \Delta p \cdot W_D(t_D) \]  

(8.25)

The rigorous solution for \( W_D(t_D) \) was developed through Laplace transforms,

\[ W_D(t_D) = \frac{4}{\pi} \int_{0}^{\infty} \left( 1 - \frac{u}{t_D} \right)^{2} \left( J_0^2(u) + Y_0^2(u) \right) \, du \]  

(8.26)
where the integral must be solved numerically. In practice, \( W_D(t_D) \), is presented in tabular or graphical form. Figure 8.8 is an example for bounded and infinite radial aquifers.

![Figure 8.8 Dimensionless water influx, constant terminal pressure case, radial flow](image)

It is possible to extend this theory to calculate the cumulative water influx corresponding to a continuous pressure decline at the reservoir/aquifer interface. To do so, the pressure history is divided into a series of discrete pressure steps. For each pressure drop, \( \Delta p \), the corresponding water influx can be calculated using the superposition of the separate influxes with respect to time to give the cumulative water influx. Figure 8.9 illustrates the procedure for approximating the continuous pressure decline. For example, the cumulative fluid produced at time, \( t \), by the pressure drop is:

\[
W_e(t) = B \cdot \Delta p \cdot W_D(t_D)
\]  \hspace{1cm} (8.27)

Likewise, for the next step,

\[
W_e(t) = B \cdot \Delta p_1 \cdot W_D(t_D - t_{D1})
\]  \hspace{1cm} (8.28)
Figure 8.9 Development of discrete pressure steps for continuous reservoir/aquifer boundary pressure.

Summing the terms,

$$W_e(t) = B \cdot \sum_{j=0}^{n-1} \Delta p_j \cdot W_D(t_D - t_D^j)$$

(8.29)

or if the pressure plateaus are taken as infinitesimally small,

$$W_e(t) = B \cdot \int_0^{t_D} \frac{\partial \Delta p}{\partial \tau} W_D(t_D - \tau) d\tau$$

(8.30)

Example

Determine the water influx during the first year for the following reservoir/aquifer system.

$$\phi = 0.209 \quad c_t = 6 \times 10^{-6} \text{ psi}^{-1} \quad A_r = 1216 \text{ acres}$$

$$\mu = 0.25 \text{ cp} \quad k = 275 \text{ md} \quad A_a = 250,000 \text{ acres}$$

$$h = 19.2 \text{ ft} \quad \theta = 180 \text{ deg}$$
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Solution

- Calculate water influx constant

\[ B = 1.119 \phi h c r_b^2 \left( \frac{t}{t_b} \right) \]

\[ = 455 \text{ bbl/psi} \] \hspace{1cm} (8.24)

- Calculate \( t_D \)

\[ t_D = \frac{0.00632kt}{\phi \mu c r_b^2} \] \hspace{1cm} (8.22)

\[ = 0.1643 \text{ days} \]

- Pressure history

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<th>Time period</th>
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<th>( P_r, ) psi</th>
<th>( \Delta p, ) psi</th>
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<td>3790.5</td>
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<td>4</td>
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<td>3728.5</td>
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</tr>
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- Calculate \( W_D(t_D) \) from graph or tables

<table>
<thead>
<tr>
<th>Time period</th>
<th>Time, days</th>
<th>( t_D )</th>
<th>( W_D(t_D) )</th>
<th>( \Delta p, ) psi</th>
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- Calculate $W_e$

<table>
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<th>Summation term</th>
<th>$W_e$ (bbls)</th>
</tr>
</thead>
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</tr>
<tr>
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<td>62,200</td>
</tr>
<tr>
<td>2</td>
<td>2.5(22.9) + 9.5(16.7) + 20(10) =</td>
<td>189,200</td>
</tr>
<tr>
<td>3</td>
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