Four-dimensional Fourier transform derivation of the Green function for the wave equation

In this set of notes, we will see how to derive the retarded and advanced Green functions for the wave equation [these are equations (6.44) in Jackson's "Classical Electrodynamics"], by using a four-dimensional Fourier transform in spacetime and contour integration in the complex ω plane. If the techniques of contour integration (Cauchy's integral formula) are familiar to you, then this will probably be more transparent than Jackson's derivation – otherwise, you might learn some cool new things if you can follow along! Let's take a deep breath first, and thus not make any mistakes (ha!). Ready?

The Green functions for the wave equation are defined as those functions $G(\mathbf{x}, t; \mathbf{x}', t')$ which satisfy

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\alpha\delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{1}$$

where the constant α is 1 in most references, but is 4π in Jackson's conventions. Jackson writes ∇ explicitly as $\nabla_{\mathbf{x}}$, to distinguish it from $\nabla_{\mathbf{x}'}$; I will write the latter as ∇' instead, if the need to use it arises. This is a definition of several Green functions (plural), since we haven't specified boundary conditions for our partial differential equation yet.

Define the four-dimensional Fourier transform of G and its inverse by the symmetrical convention:

$$g(\mathbf{k}, \omega; \mathbf{x}', t') = \frac{1}{(2\pi)^2}\int e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} G(\mathbf{x}, t; \mathbf{x}', t') d^3\mathbf{x}\, dt \tag{2}$$

$$G(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{(2\pi)^2}\int e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} g(\mathbf{k}, \omega; \mathbf{x}', t') d^3\mathbf{k}\, d\omega. \tag{3}$$

Next, we substitute the above equation for G into the wave equation to find an equation for g. On the left side of the equation, the operators ∇^2 and $\partial^2/\partial t^2$ become the algebraic factors $-k^2$ and $-\omega^2$, respectively. On the right side, use the following trick: write each delta function as the Fourier transform of a constant function, using the formula

$$\delta(t - t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega \tag{4}$$

so that

$$\delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t') = \left(\frac{1}{2\pi}\right)^4 \int e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t')]} d^3\mathbf{k}\, d\omega \tag{5}$$

Now, we have an integral over $\mathbf{k}$ and ω on both sides of the equation. For all spacetime points $(\mathbf{x}, t, \mathbf{x}', t')$, the integrals are equal; this cannot occur unless the integrands on each side are equal.

In other words, we can now cancel the integral signs and the common exponential factor to get an equation for g:

$$\left(\frac{1}{2\pi}\right)^2 \int e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left(\frac{\omega^2}{c^2}-k^2\right) g(\mathbf{k},\omega;\mathbf{x}',t') d^3\mathbf{k}\, d\omega$$
$$= \frac{-\alpha}{(2\pi)^4}\int e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]} d^3\mathbf{k}\, d\omega \tag{6}$$

becomes

$$g(\mathbf{k},\omega;\mathbf{x}',t') = \left(\frac{-c^2\alpha}{4\pi^2}\right)\frac{e^{-i(\mathbf{k}\cdot\mathbf{x}'-\omega t')}}{\omega^2-c^2k^2} \tag{7}$$

This shows the real utility of Fourier transforms for partial differential equations: derivative operators become simple algebraic factors, and we can solve the Fourier transformed equations with regular algebra. Now, all we need to do is take the inverse transform of g to get G back. Combining equations (3) and (7), we get

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{(2\pi)^2}\left(\frac{-c^2\alpha}{4\pi^2}\right)\int \frac{e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]}}{\omega^2-c^2k^2} d^3\mathbf{k}\, d\omega. \tag{8}$$

However, there is a big subtlety when doing the integrals! The expression for g blows up at the points $\omega = ck$ and $\omega = -ck$, due to the term in the denominator. Let's do the integral over ω first; that turns out to be the simplest way to deal with the singularities.

Our integral for G involves integrating g from $\omega = -\infty$ to $\omega = +\infty$, but that takes us right through the singularities at $\omega = \pm ck$. The integral is therefore undefined without some prescription for handling these. Our solution will be to treat ω as a complex variable; then we shall avoid each of the poles by simply going around them in the complex plane. This is ambiguous also, since we could pass either above or below each pole.

Each particular prescription for avoiding the poles gives a valid Green function for the wave equation! As we shall see below, the differences between them are due to the usual ambiguity in handling partial differential equations: the boundary conditions are as yet unspecified, so we should expect arbitrariness in our solution. How many solutions should there be? Since we are solving a second order equation, we should expect two linearly independent solutions (which differ by a solution of the homogeneous wave equation – the wave equation without the delta function source term).

We shall be interested mostly in two particular paths: one which passes below both poles, and also in a path which passes above both of them. Closing the contours appropriately by adding a large semicircle (of zero contribution to the integral) then allows us to evaluate the ω integral simply, using the Cauchy integral formula.

For those who need a quick refresher: the Cauchy integral formula says that the integral around a closed loop of any function $f(\omega)$ of the complex variable ω which is analytic except perhaps at a few isolated singularities (called a meromorphic function in math terms) can be evaluated in a simple way. (An analytic function is one whose Taylor series actually converges to the function; most any well-behaved function of the real variable ω will automatically be an analytic function of the complex variable ω if written in the exact same way; singularities are where the function is undefined.)

Here is the Cauchy integral formula the integral of $f(\omega)$ along a closed contour C (traversed *counterclockwise*) is given by $2\pi i$ times the sum of the 'residues' of f which are enclosed by C:

$$\oint_C f(\omega) d\omega = 2\pi i \sum_{R_i \in C} R_i \tag{9}$$

(If the path is integrated in a clockwise sense, then the integral becomes the negative of this.) The residues of f, R_i, are the coefficients of the term $1/(\omega - \omega_i)$ in the full Laurent series for f around each singularity ω_i (this is like the Taylor expansion around $(\omega - \omega_i)$, but it includes negative integer exponents of $(\omega - \omega_i)$ also).

Singularities which can be written in the form $f(\omega) = g(\omega)/(\omega - \omega_i)$, where $g(\omega)$ is well behaved at ω_i, are known as *simple poles*. The residue of a simple pole is easy to find: it's just $g(\omega_i)$, as should be clear after staring at the way f behaves near ω_i in the above (provided you keep in mind the definition of the residue!). These are the type of singularities in our formula for the Green function, so we are almost done with our ω integral.

So far, we have only an integral from $-\infty$ to ∞ along the real ω axis (while avoiding the two poles along the way). To make use of the Cauchy integral formula above, we must first close the path of integration by adding a semicircle of radius $r \to \infty$, in either the upper half or lower half of the complex ω plane.

Of course, the only way to add another piece to our path of integration without changing the answer is if the integral over the new piece vanishes. This is guaranteed to happen when the integrand itself vanishes. As far as the variable ω is concerned, the integrand in equation (8) is just a constant times the factor

$$\frac{e^{-i\omega(t-t')}}{(\omega - ck)(\omega + ck)}. \tag{10}$$

For each of our proposed new contours, the magnitude $|\omega|$ goes to ∞, while its imaginary part is purely positive (the upper choice) or purely negative (the lower choice). If we write split ω into its real and imaginary parts, $\omega = \omega_R + i\omega_I$, then the exponential factor in the numerator is seen to behave as $e^{-i\omega_R(t-t')} e^{\omega_I(t-t')}$. The first factor is purely oscillatory, while the second gives either exponential growth or decay for large ω_I, depending upon whether $\omega_I(t - t')$ is positive or negative, respectively.

Since we wish to close our path where the integrand vanishes, we always need the case of exponential decay. Thus, whether we close the contour in the upper or lower plane depends upon the sign of $(t - t')$. If $t > t'$, then we close the contour in the lower half of the complex ω plane, and if $t < t'$, we add the semicircle in the upper half plane. The Cauchy integral formula does the rest, once we compute the residues. Ignoring the terms which do not depend upon ω for the moment, the residue of the relevant factor, the term in (9), at $\omega = -ck$ is given by

$$R_1 = \frac{e^{ick(t-t')}}{-2ck} \tag{11}$$

while that at $\omega = ck$ is

$$R_2 = \frac{e^{-ick(t-t')}}{2ck}. \tag{12}$$

These are easily found from the residue formula for simple poles: just plug in $\omega = -ck$ to $(\omega + ck)$ times the term (9) to get R_1, for example.

Now, we can find G for our two choices of paths which avoid the poles. For the choice which passes under both poles, which we shall label with a '$-$' sign, there are *no* poles inside the contour if we also use the semicircle in the lower half plane to close the contour. Since this happens when $t > t'$, the Cauchy integral formula thus gives 0 for our integral if $t > t'$. However, if $t < t'$, then we close the contour in the upper half plane, and then the contour encloses both poles. The integral then becomes $2\pi i$ times the sum of the two residues above:

$$\begin{aligned}\oint_{-} \frac{e^{-i\omega(t-t')}}{(\omega - ck)(\omega + ck)} d\omega &= (2\pi i)\left(\frac{-i\sin[ck(t-t')]}{ck}\right)\Theta(t'-t) \\ &= \frac{2\pi}{ck}\sin[ck(t-t')]\Theta(t'-t)\end{aligned} \tag{13}$$

where we have used the Heaviside step function, $\Theta(t'-t)$, to express the fact that the integral vanishes for $t > t'$.

A similar strategy gives the integral when the prescription to avoid the poles says to go over each of them (we shall label this path with a '$+$' sign). Now, the integral is 0 if we use the upper semicircle to close the contour, and $-2\pi i$ times the sum of the two residues when we use the lower semicircle (the negative sign comes from the fact that the contour is being traversed in a clockwise sense). Thus, this path gives

$$\oint_{+} \frac{e^{-i\omega(t-t')}}{(\omega - ck)(\omega + ck)} d\omega = \frac{-2\pi}{ck}\sin[ck(t-t')]\Theta(t-t'). \tag{14}$$

Already, we see that the Θ functions impose some notion of *causality* onto our Green functions: the integral evaluated along the $-$ contour vanishes for $t > t'$, while the integral along the $+$ contour vanishes for $t < t'$. To recover the Green functions in regular space, we now need to perform the inverse spatial Fourier transform (the integral over $d^3\mathbf{k}$ in eq. [8]). The integrand contains the factor $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$, so we shall use spherical coordinates for $\mathbf{k}$, with the north pole along the direction of $(\mathbf{x}-\mathbf{x}')$. Equation (8), which depends upon the choice of ω contour, now becomes (using eq. [13])

$$G_{-}(\mathbf{x}, t; \mathbf{x}', t') = \frac{-c\alpha}{8\pi^3}\iiint k^2 \sin\theta \, dk \, d\theta \, d\phi e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta}\frac{\sin[ck(t-t')]}{k}\Theta(t'-t)$$

$$= \frac{c\alpha}{4\pi^2} \int\!\!\int k \sin\theta \, dk \, d\theta e^{ikR\cos\theta} \sin[ck(t-t')]\Theta(t'-t)$$

where we have defined $R \equiv |\mathbf{x} - \mathbf{x}'|$, integrated over ϕ, and absorbed the minus sign into the argument of the sin function in the second line. The integral over θ is now easily done, giving

$$G_-(\mathbf{x}, t; \mathbf{x}', t') = \frac{c\alpha}{4\pi^2} \int k \, dk \frac{2\sin kR}{kR} \sin[ck(t'-t)]\Theta(t'-t).$$

$$= \frac{2c\alpha}{4\pi^2 R} \int_0^\infty dk \sin(kR) \sin[ck(t'-t)]\Theta(t'-t). \tag{15}$$

To evaluate the integral over k, it helps to use the following trigonometric identity: $\sin A \sin B = [\cos(A-B) - \cos(A+B)]/2$. Then, G_- is found to be

$$G_-(\mathbf{x}, t; \mathbf{x}', t') = \frac{c\alpha}{4\pi^2 R} \int_0^\infty dk \Big[\cos(k[R - c(t'-t)]) - \cos(k[R + c(t'-t)])\Big]\Theta(t'-t).$$

Since $\cos(-kx) = \cos(kx)$ while $\sin(-kx) = -\sin(kx)$, we can rewrite the integral as half the integral from $k = -\infty$ to $k = \infty$, and replace each $\cos(kx)$ term by e^{ikx}. We then recognize each of the terms as the standard Fourier integral representation of a delta function:

$$G_-(\mathbf{x}, t; \mathbf{x}', t') = \frac{c\alpha}{8\pi^2 R} \int_{-\infty}^\infty dk \Big[e^{ik[R - c(t'-t)]} - e^{ik[R + c(t'-t)]}\Theta(t'-t)$$

$$= \frac{c\alpha}{4\pi R} \Big[\delta\big(R - c[t'-t]\big) - \delta(R + c[t'-t])\Big]\Theta(t'-t). \tag{16}$$

However, R is a positive quantity, and the Heaviside function forces $(t'-t)$ to be positive also, so only the first δ function is ever nonzero.

This defines the so-called ‘advanced’ Green function:

$$G_-(\mathbf{x}, t; \mathbf{x}', t') = \frac{c\alpha}{4\pi R} \delta\big(R - c[t'-t]\big) = \frac{\alpha}{4\pi R} \delta\left(t' - \left[\frac{R}{c} + t\right]\right). \tag{17}$$

Similarly, we can use the + contour to define the ‘retarded’ Green function. Combining equations (14) and (8) will give the negative of what we found for G_-, but with $\Theta(t'-t)$ becoming $\Theta(t-t')$:

$$G_+(\mathbf{x}, t; \mathbf{x}', t') = \frac{-c\alpha}{4\pi R} \Big[\delta\big(R - c[t'-t]\big) - \delta(R + c[t'-t])\Big]\Theta(t-t'). \tag{18}$$

Now, the $\Theta(t-t')$ forces us to keep the *second* δ function instead, since R is positive but $c[t'-t]$ will be negative. So,

$$G_+(\mathbf{x}, t; \mathbf{x}', t') = \frac{c\alpha}{4\pi R} \delta(R + c[t'-t])$$

$$= \frac{\alpha}{4\pi R} \delta\left(t' - \left[t - \frac{R}{c}\right]\right) \tag{19}$$

which, together with equation (17), is what Jackson has in equation (6.44).

This exact same procedure is used throughout physics! For example, the Feynman diagrams one encounters in quantum field theory typically have some 'legs' on them representing the propagation of some sort of particle. For a spinless particle, the wave equation it satisfies is exactly the wave equation we are dealing with here! So this procedure gives the so-called 'propagator' for a scalar particle to go from one spacetime point, $(\mathbf{x}', t')$, to another, $(\mathbf{x}, t)$. The factor which the leg in the diagram represents is exactly a wave equation Green function!

In older textbooks on field theory, they frequently made use of the exact Green functions we have derived here, namely either G_- and G_+, depending upon whether $t' > t$ or vice versa (respectively). In more modern treatments, a 'covariant' approach is used where the time ordering does not matter. The Green function used for this approach is called the Feynman or 'causal' propagator, G_F. The contour used to get G_F goes under one pole, and over the other. It then reduces to G_+ when $t > t'$, and to G_- when $t < t'$, so that 'time ordering' is done automatically. (The second, independent solution of the wave equation that goes with G_F is anticausal: only the future events influence those in the past, so it is seldom used. Its Fourier contour over ω goes over the pole G_F went under, and vice versa.)