

Lecture 8: Variograms

Math 586

The wide-sense stationarity (WSS) assumption can be restrictive - generalize?

Instead of covariance $C(\mathbf{h})$, look at the differences $V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})$. Their variance may exist even if $Var[V(\mathbf{x})]$ does not exist.

Definition

$V(\mathbf{x})$ is an **Intrinsic Random Field** of order 0 (IRF-0) if

1. $\mathbb{E}[V(\mathbf{x})] = m$ (constant)
2. $Var[V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})]$ is only a function of lag \mathbf{h} .

Definition

$\gamma(\mathbf{h}) = \frac{1}{2}Var[V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})]$ is called the **(semi)variogram** for $V(\mathbf{x})$.

Example: Discrete case

The following is a simple example of a non-stationary random field that is still IRF-0:

Consider W_1, W_2, \dots, W_n independent, $\mathbb{E}(W_i) = 0$, $Var(W_i) = \sigma^2$.

Let $V(0) = 0$,

$$V(k) = \sum_{j=1}^k W_j.$$

Note that $Var[V(k)] = k\sigma^2$, therefore $V(k)$ is not WSS. (Why?)

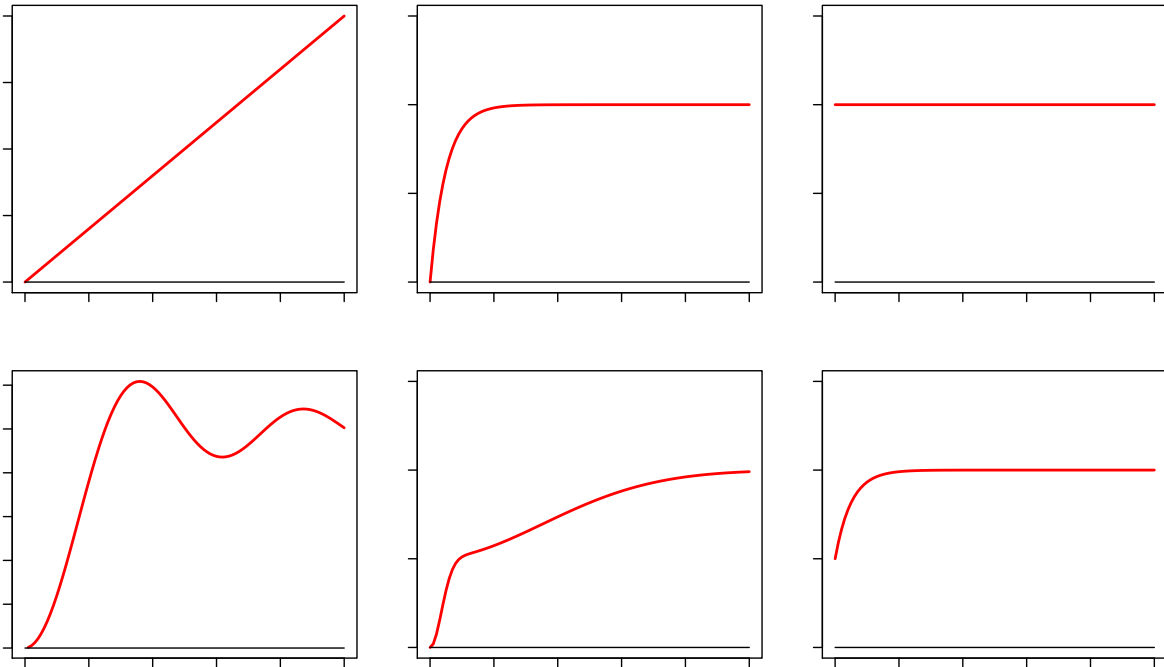
But

$$V(k+j) - V(k) = \sum_{i=k+1}^{k+j} W_i \Rightarrow$$

$Var[V(k+j) - V(k)] = j\sigma^2$ only depends on j , so V is IRF-0.

This V is called *random walk*.

Some variogram shapes:



Relation between variograms and covariances:

In case of WSS $V(\mathbf{x})$, with covariance function $C_V(\mathbf{h})$, what is $\gamma_V(\mathbf{h})$?

$$\begin{aligned} \gamma(\mathbf{h}) &= \frac{1}{2} \text{Var}[V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})] = \\ &= \frac{1}{2} \{ \text{Var}[V(\mathbf{x} + \mathbf{h})] + \text{Var}[V(\mathbf{x})] - 2\text{Cov}[V(\mathbf{x} + \mathbf{h}), V(\mathbf{x})] \} = \\ &= \frac{1}{2} [C_V(0) + C_V(0) - 2C_V(\mathbf{h})] = C_V(0) - C_V(\mathbf{h}) \end{aligned}$$

So, for a WSS process, the variogram is just the upside-down covariance.

As $|\mathbf{h}| \rightarrow \infty$, $\gamma_V(\mathbf{h}) \rightarrow C_V(0) = \text{Var}[V(\mathbf{x})]$, also called **the sill**.

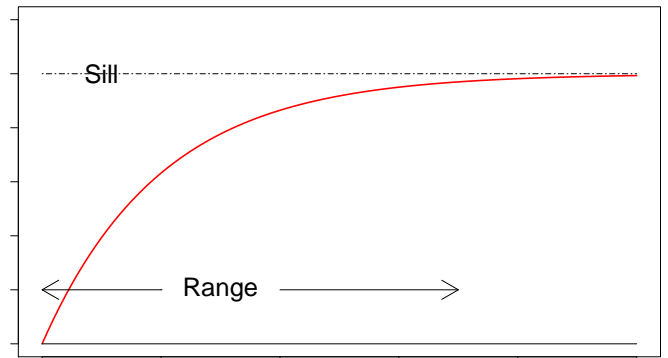
If $V(\mathbf{h})$ is stat. isotropic, and $\ell =$ smallest such that $\gamma_V(\ell) = C_V(0)$ then ℓ is **the range**.

Also (loosely) say that *practical range* \approx the distance beyond which covariance is negligible.

Correlation Scale =

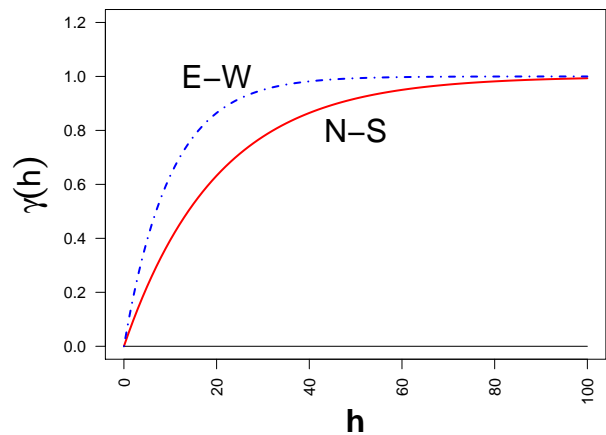
$$\lambda \text{ such that } \frac{C(\lambda)}{C(0)} \approx e^{-1}$$

Practical range and correlation scale are not rigorously defined.



Non-isotropic case could have several scales/ranges depending on direction.

Ror WSS case, only one sill, however. (Why?)



Variogram properties

- (i.) $\gamma(\mathbf{h}) \geq 0$ (Why?)
- (ii.) $\gamma(-\mathbf{h}) = \gamma(\mathbf{h})$ (symmetry)
- (iii.) $-\gamma(\mathbf{h})$ is a positive-semidefinite function (not just any function will work!)
- (iv.) Behavior at ∞ :

$$\lim_{|\mathbf{h}| \rightarrow \infty} \frac{\gamma(|\mathbf{h}|)}{|\mathbf{h}|^2} = 0$$

i.e. bounded by quadratic.

If the estimated variogram looks quadratic \Rightarrow trend not removed (i.e. $\mathbb{E}[V(\mathbf{x})]$ not constant).

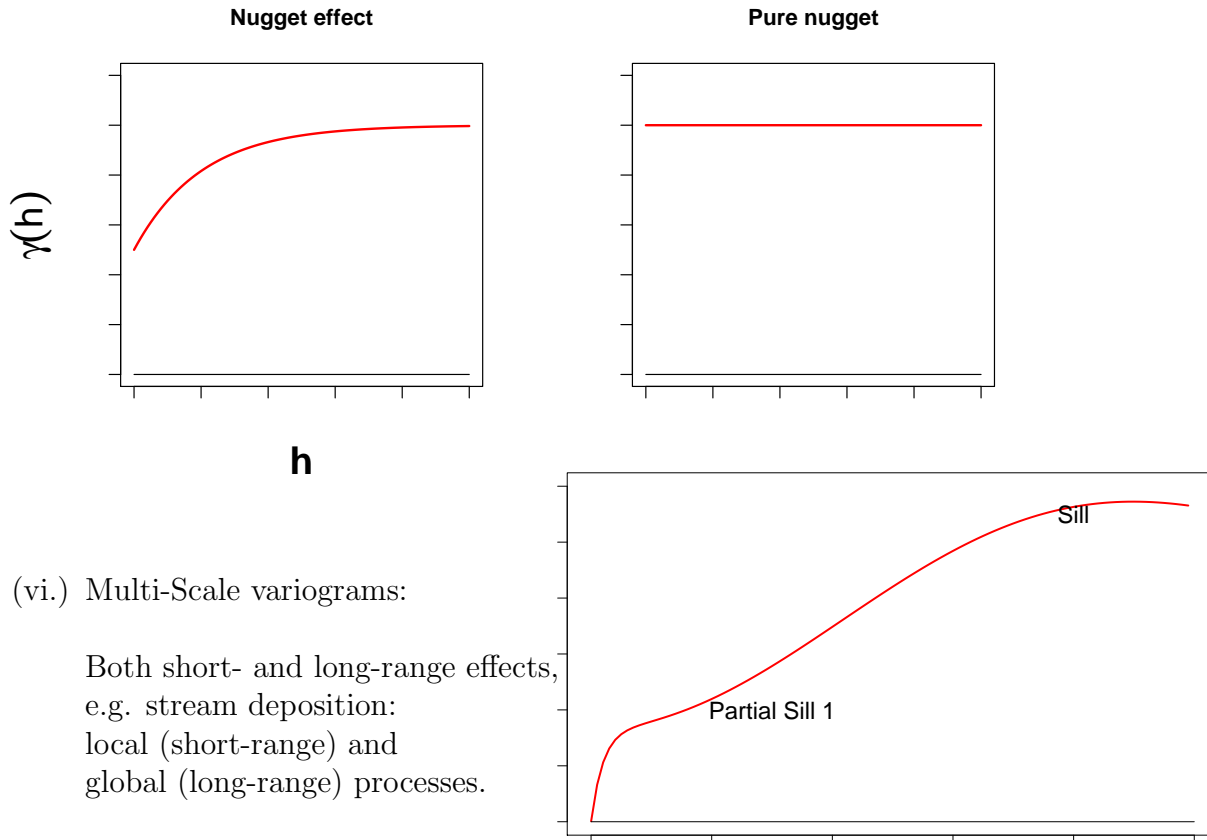
- (v.) Behavior near 0:

- (a) $\gamma(\mathbf{h}) \sim A|\mathbf{h}|^2$ near 0 $\Rightarrow V(\mathbf{x})$ is smooth (has a derivative in mean-square sense)

(b) $\gamma(\mathbf{h}) \sim A|\mathbf{h}| \Rightarrow V(\mathbf{x})$ is continuous but not differentiable

(c) $\gamma(\mathbf{h})$ is discontinuous at 0 \Rightarrow “nugget” effect.

Nugget: (independent) measurement error or fine-scale variation. Estimated variograms often show nuggets. Pure nugget: independent values.



(vi.) Multi-Scale variograms:

Both short- and long-range effects, e.g. stream deposition: local (short-range) and global (long-range) processes.

Some functional forms for variograms:

Let $h = |\mathbf{h}|$.

- Power (linear when $c = 1$; has no sill):

$$\gamma(\mathbf{h}) = Ah^c, \quad 1 \leq c < 2$$

- Exponential (A = “sill”, B = “scale”)

$$\gamma(\mathbf{h}) = A [1 - e^{-h/B}]$$

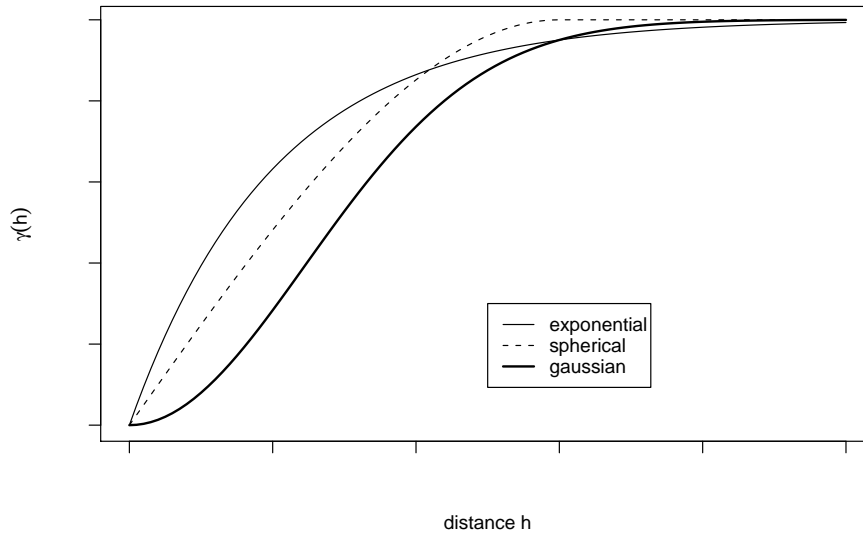
- Spherical

$$\gamma(\mathbf{h}) = \begin{cases} A [1.5(h/B) - 0.5(h/B)^3] & \text{if } h < B \\ A & \text{otherwise} \end{cases}$$

- Gaussian

$$\gamma(\mathbf{h}) = A \left[1 - e^{-(h/B)^2} \right]$$

variograms with equivalent "practical range"

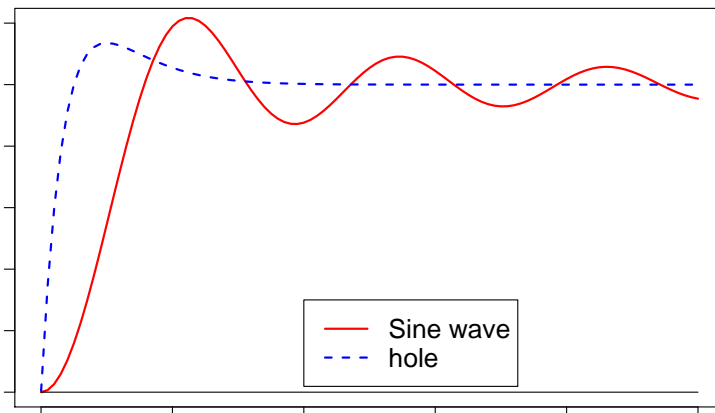


- “Hole” variogram (models layering, valid for 1d only)

$$\gamma(h) = A \left[1 - (1 - h/B)e^{-h/B} \right]$$

- Sine wave (models layering, periodicities)

$$\gamma(\mathbf{h}) = A \left[1 - (B/h) \sin(h/B) \right]$$



- Nested models

- (a.) Consider r.f. $V(\mathbf{x})$ with variogram $\gamma_V(\mathbf{h})$, and $\varepsilon(\mathbf{x})$ independent of $V(\mathbf{x})$, mean 0, variance σ^2 , and $\varepsilon(\mathbf{x}_1)$ independent of $\varepsilon(\mathbf{x}_2)$ for $\mathbf{x}_1 \neq \mathbf{x}_2$. (Models observational error, or pure nugget.)

Let $W(\mathbf{x}) = V(\mathbf{x}) + \varepsilon(\mathbf{x})$.

$\gamma_W(\mathbf{h}) = ?$

$$\begin{aligned} \gamma_W(\mathbf{h}) &= \frac{1}{2} \mathbb{E} [W(\mathbf{x} + \mathbf{h}) - W(\mathbf{x})]^2 = \frac{1}{2} \mathbb{E} [V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x}) + \varepsilon(\mathbf{x} + \mathbf{h}) - \varepsilon(\mathbf{x})]^2 = \\ &= \frac{1}{2} \mathbb{E} [V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})]^2 + \frac{1}{2} \mathbb{E} [\varepsilon(\mathbf{x} + \mathbf{h}) - \varepsilon(\mathbf{x})]^2 + \\ &\quad + \mathbb{E} \{ [V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})] [\varepsilon(\mathbf{x} + \mathbf{h}) - \varepsilon(\mathbf{x})] \} = \\ &= \gamma_V(\mathbf{h}) + \sigma^2 \Rightarrow \text{Nugget effect} \end{aligned}$$

- (b.) Generalization:

$V_1(\mathbf{x}), V_2(\mathbf{x}), \dots, V_k(\mathbf{x})$ independent, with variograms $\gamma_1(\mathbf{h}), \gamma_2(\mathbf{h}), \dots, \gamma_k(\mathbf{h})$.

If $W(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x}) + \dots + V_k(\mathbf{x})$ (independent), then
 $\gamma_W(\mathbf{h}) = \gamma_1(\mathbf{h}) + \gamma_2(\mathbf{h}) + \dots + \gamma_k(\mathbf{h})$.

Misspecification Example (1-d case):

Let $V(x)$ IRF-0

$W(x) = V(x) + \beta x$. Is W an IRF-0?

Suppose blindly calculate $\frac{1}{2} \mathbb{E} [W(x+h) - W(x)]^2 =$

$$\begin{aligned} &= \frac{1}{2} \mathbb{E} [V(x+h) - V(x) + \beta(x+h) - \beta x]^2 = \\ &= \frac{1}{2} \mathbb{E} [V(x+h) - V(x)]^2 + \frac{\beta^2 h^2}{2} \end{aligned}$$

Thus, the new variogram is $\gamma_V(h) + \frac{\beta^2 h^2}{2}$ (trend effect).

Note that W is not IRF-0: should detrend first!