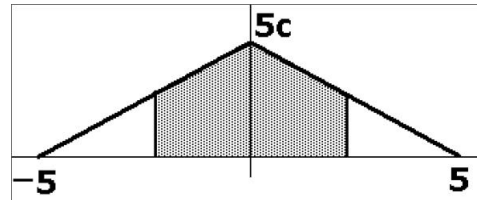


Math 382 Practice Exam 2 KEY

Consider the PDF of a random variable X

$$1. \quad f(x) = \begin{cases} c(5+x) & -5 \leq x \leq 0 \\ c(5-x) & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$



(a) Find c that makes $f(x)$ a legitimate density function. Plot $f(x)$.

Total area under the graph is $25c$, therefore $c = 1/25$

(b) Find the probability that X is between -2.5 and 2.5

Shaded area is $1 - 2 * (\text{area of triangle}) = 1 - 2(2.5 * 2.5c)/2 = 0.75$

(c) Find $\mathbb{E}(X)$

Since the graph is symmetrical about 0, the mean is 0

(d) Find $V(X)$

$$V(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{2}{25} \int_0^5 x^2(5-x) dx = \frac{2}{25} \left[\frac{5}{3}x^3 - \frac{1}{4}x^4 \right]_0^5 = 25/6$$

(e) Compute the CDF of X .

For $x < 0$:

$$F(x) = \frac{1}{25} \int_{-5}^x (5+t) dt = \frac{1}{25} \left[5t + \frac{t^2}{2} \right]_{-5}^x = \frac{1}{2} + x/5 + x^2/50$$

check that $F(-5) = 0$ and $F(0) = 0.5$

For $0 < x < 5$:

$$F(x) = 1/2 + x/5 - x^2/50 \quad (\text{Why?})$$

2. In the past few years, the average daily temperature in Socorro on October 10 was $58.5^\circ F$ with standard deviation $9.5^\circ F$. Assume Normal distribution.

(a) What are the chances that on an October 10, the average daily temperature will be above $40^\circ F$?

$$P(X > 40) = P\left(Z > \frac{40 - 58.5}{9.5}\right) = 0.974$$

(b) Find the interval $[a, b]$ symmetric around the mean, such that $P(a \leq X \leq b) = 0.98$

Find z^* such that $TA(z^*) = 0.98/2 + 0.5 = 0.99$, so that $z^* = 2.33$.

Then, $a = \mu - \sigma z^* = 36.4$ and $b = \mu + \sigma z^* = 80.6$.

3. The number of computer crashes at a Spare lab is believed to have Poisson distribution with the intensity of 0.2 crashes per hour.

(a) Find the distribution of times between consecutive crashes.

Exponential with mean = $1/\text{intensity} = 5$

(b) Find the probability that the lab will go 7 hours without a crash.

$$P(X > 7) = \exp(-7/5) = 0.2466$$

(c) Find the probability that the first crash will occur between 2 and 8 hours.

$$P(2 < X < 8) = \exp(-2/5) - \exp(-8/5) = 0.4684$$

(d) Describe the distribution of the time when 3rd crash occurs.

Gamma ($\alpha = 3, \beta = 5$) (sum of 3 exponentials, each with mean 5)

4. Using Normal approximation (even without continuity correction ± 0.5). First, $\mu = 0.15(200) = 30$ and $\sigma = \sqrt{200 * 0.15(1 - 0.15)} = 5.05$ Thus,

$$P(X \geq 25) \approx P\left(Z > \frac{25 - 30}{5.05}\right) = 1 - TA(-0.99) = 0.84$$

5. The following is a table of joint distribution of X and Y

	Y	0	1	
X = 0		0.25	0.35	0.6
X = 1		0.1	0.3	0.4
		0.35	0.65	

(a) Are X, Y independent? Explain.

They are not: for example, $P(X = 0, Y = 0) = 0.25$, but $P(X = 0) \cdot P(Y = 0) = 0.6 * 0.35 = 0.21$

(b) Fill in the marginal distributions of X, Y. **Done**

(c) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$

$$\mathbb{E}(X) = 0 * 0.6 + 1 * 0.4 = 0.4 \quad \mathbb{E}(Y) = 0.65$$

(d) Find $Cov(X, Y)$

$$\mathbb{E}(XY) = 0.3, \quad Cov(X, Y) = 0.3 - 0.4 * 0.65 = \mathbf{0.04}$$

(e) Find $\mathbb{E}(2X + 3Y)$, σ_{2X+3Y}^2

$$\mathbb{E}(2X + 3Y) = 2 * \mathbb{E}(X) + 3 * \mathbb{E}(Y) = 2.75$$

$$\sigma_{2X+3Y}^2 = 4\sigma_X^2 + 2 * 2 * 3 * Cov(X, Y) + 9\sigma_Y^2 = 4 * 0.24 + 12 * 0.04 + 9 * 0.2275 = 3.4875$$

6. An electrical circuit consists of 3 resistors connected in series, so that the total resistance is equal to $Y = R_1 + R_2 + R_3$. Each resistance R_i is random with the mean 900Ω and standard deviation 50Ω .

(a) Find the expected value of $Y = 3 * 900 = 2700$

(b) Find standard deviation of Y (assume that resistors are independent of each other).

$$\sigma_Y^2 = \sigma_{R_1}^2 + \sigma_{R_2}^2 + \sigma_{R_3}^2 + 0 = 3 * 50^2, \quad \sigma_Y = \sqrt{7500} = 86.6$$

(c) Suppose that the resistors are no longer independent, but for each i and j , $Cov(R_i, R_j) = 1000 \Omega^2$

i.

$$\sigma_Y^2 = Cov(R_1 + R_2 + R_3, R_1 + R_2 + R_3) = \sigma_{R_1}^2 + \sigma_{R_2}^2 + \sigma_{R_3}^2 + 2Cov(R_1, R_2) + 2Cov(R_1, R_3) + 2Cov(R_2, R_3) = 3 * 50^2 + 6 * 1000 = 13500$$

$$\sigma_Y = 116.2$$

ii. Find correlation between R_1 and R_2

$$\rho = \frac{Cov(R_1, R_2)}{\sigma_R^2} = \frac{1000}{50^2} = 0.4$$

7. Suppose that X and Y have joint density $f(x, y) = ky$, for $0 \leq y \leq x \leq 2$

(a) find the constant k that makes f a legitimate density function.

$$1 = \int_0^2 \int_y^2 f(x, y) dx dy$$

is the volume of pyramid with its base a triangle in (X, Y) plane, height $h = 2k$.

$$1 = Volume = \frac{Ah}{3} = \frac{4}{3}k, \quad \text{thus } k = 3/4$$

(b) find $P(X + Y < 2)$

See small plot. Volume of pyramid over shaded triangular region = $1/4$

(c) find the conditional density of X given that Y equals 0.5

See large plot. When $y = 0.5$, cross-section gives the rectangle $0.5 \leq x \leq 2$.

Thus, X is uniform on $0.5 \leq x \leq 2$, and its density $f(x) = 1/(2 - 0.5) = 2/3$.

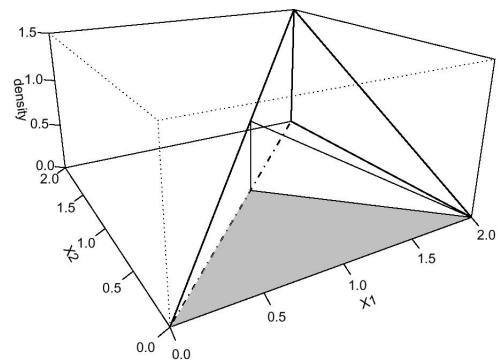
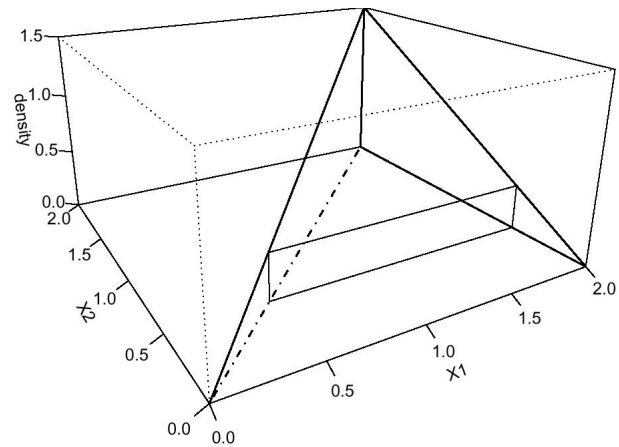
(d) Find the marginal density of X . Are X and Y independent?

$$f_X(x) = \int_0^x 0.75y dy = 3x^2/8, \quad 0 \leq x \leq 2$$

Clearly X and Y are not independent because $Y \leq X$.

(e)

$$\int_0^2 \int_y^2 x \sin(y) * 0.75y dx dy$$



8. The life length of fuses is Exponential with mean = 300 hours. Assume that fuses behave independently.

(a) Find the joint density for lifetimes of two such fuses.

By independence,

$$f(x_1, x_2) = f_1(x_1)f_2(x_2) = \left(\frac{1}{300}e^{-x_1/300}\right) \left(\frac{1}{300}e^{-x_2/300}\right)$$

(b) Find $P(\text{Total life of 2 such fuses}) \leq 500$ hours.

Let's work in 100 hour units.

$$\begin{aligned} P(X_1 + X_2 \leq 5) &= \int_0^5 \left[\int_0^{5-x_2} \frac{1}{3}e^{-x_1/3} dx_1 \right] \frac{1}{3}e^{-x_2/3} dx_2 = \\ &= \int_0^5 \frac{1}{3}e^{-x_2/3} \left(1 - e^{-5/3 - x_2/3}\right) dx_2 = \int_0^5 \frac{1}{3}(e^{-x_2/3} - e^{-5/3 - x_2/3}) dx_2 = 1 - e^{-5/3} - \frac{5}{3}e^{-5/3} = 0.496 \end{aligned}$$

We could also use Poisson process, since $X_1 + X_2$ is *Gamma*($\alpha = 2, \beta = 300$). Let N be number of fuse burnouts on $[0, 500]$ interval. It is Poisson with the mean $\mu = \lambda t = 5/3$.

$$P(X_1 + X_2 < 500) = P(N \geq 2) = 1 - f(0) - f(1) = 1 - e^{-5/3} - \frac{5}{3}e^{-5/3}$$