Calculation of Heat Determinant Coefficients
for Scalar Laplace type Operators

by

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ABSTRACT

In spectral geometry, one would like to know how much we can distinguish two manifolds by the spectrum of a differential operator. Heat invariants, such as the heat trace and heat content have been studied and isospectral manifolds have been found. We introduce a new heat invariant called the heat determinant which is a step foreword in the study of non-spectral invariants on manifolds. The heat determinant is not a pure spectral invariant in that it does not depend only on the eigenvalues of an operator, but also the eigenfunctions of that operator as well. We find the asymptotic expansion in $t$ of this new invariant and calculate the first 3 terms of the heat determinant for the scalar Laplacian on manifolds without boundary.

Keywords: Heat invariants; Heat kernel; Asymptotic expansions; Spectral geometry; Heat determinant
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Date
CHAPTER 1

INTRODUCTION

In 1966, Marc Kac popularized the question “Can you hear the sound of a drum?” [14]. In other words, can one determine the shape of a drum based on the frequencies one can hear? The area of spectral geometry was created to answer this question. We take a manifold and find the spectrum of some differential operator, such as the Laplacian, on this manifold. It is obvious that if you have a large drum and a small drum, the frequencies will be different. In spectral geometry, we want to answer the question “how much does the spectrum of an operator determine the geometry of our manifold”.

About the same time Kac popularized the question, two different manifolds (2 distinct 16-dimensional tori) were found to have the same eigenvalues [15]. The problem in 2-dimensions was not answered till 1992 [12]. Thus one cannot hear the shape of a drum. Spectral geometry then became the question of how much we can infer of the geometry of an object from the spectrum of an operator. Mathematicians then turned to spectral invariants, objects that are invariant on a manifold that depend on the spectrum of an operator. In practice, it is rather difficult to calculate the eigenvalues of a differential operator on a manifold, thus we study asymptotic regimes.

One of the most important operators on a manifold is the Laplacian, $\Delta$. It is the most “natural” second order elliptic partial differential operator on a manifold and it is seen in many important partial differential equations such as the wave and diffusion equations. The wave equations describes how waves propagate through space, while the diffusion equation describes how some material diffuses through space, such as particles or energy. For this paper, we will focus on the heat equation which describes how heat diffuses through a system. The heat equation is given by

$$\left(\frac{\partial}{\partial t} - \Delta\right)f(t, x) = 0,$$  \hspace{1cm} (1.1)

which we must give an initial condition

$$f(0, x) = g(x),$$  \hspace{1cm} (1.2)

and appropriate boundary conditions if present. The fundamental solution of the heat equation is called the heat kernel. The heat kernel, itself, has many applications to physics and mathematics, more specifically quantum gravity and spectral geometry.
From the heat kernel, we can define many invariants. Some spectral invariants that have been studied already include the heat trace and the heat content. In this paper we introduce a non-spectral invariant called the heat determinant. This new invariant is non-spectral since it depends on the eigenvalues and the eigenfunctions of our operator. We calculate the first 3 terms in the asymptotic expansion of the heat determinant.

This paper is organized as follows: Chapter 2 contains all the background information we need, focusing on differential geometry. It will introduce all the notation and objects we will be using throughout this paper. We also mention other heat invariants. In chapter 3, we define the heat determinant on vector bundles. We then go through calculation of the heat determinant, then calculating its asymptotics. In section 3.4, we calculate the first 3 coefficients in our asymptotic expansion. In chapter 4, we have our conclusion where we summarize our results. In chapter 5, we have our appendix where we have put many of our technical calculations.
CHAPTER 2

BACKGROUND INFORMATION

2.1 Differential Geometry

In this section, we will have a brief overview of differential geometry and the tools which we will be using. For a more in depth introduction, see [3], [8], or [4].

2.1.1 Basic Definitions

In this section, we will give the standard definitions of the mathematical objects we will be using throughout.

Definition. An n-dimensional manifold (without boundary), $\mathcal{M}$, is a Hausdorff, second countable, topological space such that for any point $p$ in $\mathcal{M}$, there exists a homeomorphism $\varphi_U : U \rightarrow V$, where $U$ is an open neighborhood of $p$ and $V$ is an open subset of $\mathbb{R}^n$.

We call the ordered pair $(U, \varphi_U)$ a coordinate patch and we define the local coordinate of a point $p \in \mathcal{M}$ as $\varphi_U(p) \in V \subset \mathbb{R}^n$. A manifold is called a differential manifold if for every pair of coordinate patches, $(U_1, \varphi_{U_1})$ and $(U_2, \varphi_{U_2})$ with $U_1 \cap U_2 \neq \emptyset$, the new map $\phi = \varphi_{U_2} \circ \varphi_{U_1}^{-1} : V_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable. We will only deal smooth manifolds, that is, $\phi$ is smooth (i.e. infinitely differentiable) for all pairs of coordinate patches with non-empty intersections. All manifolds herein are smooth manifolds of dimension $n$.

Definition. A vector (also called tangent vector or contravariant vector) at a point $p$ in a coordinate patch $(U_1, x_{U_1})$ on a manifold $\mathcal{M}$ is an element of $\mathbb{R}^n$ denoted as $X_p = (X_{U_1}^\mu) = (X_{U_1}^1, \ldots, X_{U_1}^n)$ such that for any other coordinate patch $(U_2, x_{U_2})$ with $p \in U_2$, then

$$X_{U_2}^\mu = \sum_{\nu=1}^n \frac{\partial x_{U_2}^\mu}{\partial x_{U_1}^\nu}(p) X_{U_1}^\nu. \quad (2.1)$$
A vector can also be represented as the differential operator

\[ X_p = \sum_{\nu=1}^{n} X^\nu_p \frac{\partial}{\partial x^\nu} \]

where \( \partial / \partial x^\nu \) is the coordinate basis of \( U \) in local coordinates.

**Definition.** The Jacobian matrix between two coordinate patches given by \( (U_1, x^\mu_{U_1}) \) and \( (U_2, x^\mu_{U_2}) \), is the matrix of partial derivatives

\[
\begin{pmatrix}
\frac{\partial x^\mu_{U_2}}{\partial x^\nu_{U_1}}
\end{pmatrix}
\]

(2.3)

and the Jacobian is

\[ J = \det \begin{pmatrix}
\frac{\partial x^\mu_{U_2}}{\partial x^\nu_{U_1}}
\end{pmatrix}. \]

(2.4)

Suppose we have another smooth manifold, \( B \), and a smooth mapping \( \pi : B \rightarrow M \). Then we define the triple \( (B, \pi, M) \) as a bundle where \( B \) is the bundle space and \( \pi \) is the projection. The we define a fiber as the image of \( \pi^{-1}(p) \), where \( p \in M \). A section of a bundle is a map \( s : M \rightarrow B \) such that \( s(p) \in \pi^{-1}(p) \).

We define the tangent space of \( M \) at a point \( p \) as the vector space containing all tangent vectors to \( M \) at \( p \) and is denoted as \( T_p M \). We define the tangent bundle as the set of all tangent spaces at every point of \( M \) and it is denoted as \( TM \). Then we define a vector field as a smooth mapping \( X : M \rightarrow TM \) which can also be represented in local coordinates as

\[ X(p) = \sum_{\nu=1}^{n} X^\nu(p) \frac{\partial}{\partial x^\nu}. \]

(2.5)

**Definition.** A covector (also called a cotangent or covariant vector) is a linear functional \( \alpha_p : T_p M \rightarrow \mathbb{R} \) and is referred to as the dual of a vector.

Thus the dual of \( T_p M \) is the set of all linear functionals on \( T_p M \) and is called the cotangent space at the point \( p \) and is denoted as \( T^*_p M \). Similarly, the cotangent bundle is the set of all cotangent spaces at every point of \( M \) and is denoted as \( T^* M \). Then we define a covector field (also called a 1-form) as a smooth mapping \( \alpha : M \rightarrow T^* M \). We also have a covector bundle which is the dual to the vector bundle.

We will henceforth use the notation \( \partial_\mu = \frac{\partial}{\partial x^\mu} \). The dual to the coordinate basis \( \partial_\mu \) is denoted as \( dx^\mu \) and satisfies the condition

\[ dx^\mu (\partial_\nu) = \partial_\nu (dx^\mu) = \delta_\nu^\mu, \]

(2.6)
where
\[ \delta_{\mu}^{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \] (2.7)
is the Kronecker delta. We will be using Einstein summation notation throughout this paper, thus we sum over repeated indices (i.e. \( \beta^{\mu} \alpha_{\mu} = \sum_{\mu} \beta^{\mu} \alpha_{\mu} \)).

### 2.1.2 Riemannian Metric

A Riemannian metric is the smooth assignment of a positive definite inner product in \( T_p M \) to \( p \in M \).

**Definition.** An inner product on a vector space \( v \) is defined as the mapping \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) that satisfies the properties that for all \( x, y, z \in V \) and \( \alpha, \beta \in \mathbb{C} \),

1. \( \langle x, y \rangle = \langle y, x \rangle \),
2. \( \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \),
3. \( \langle x, x \rangle \geq 0 \),
4. \( \langle x, x \rangle = 0 \) implies \( x = 0 \).

Then inner product defines a norm denoted as \( \|v\| = \sqrt{\langle v, v \rangle} \).

The Riemannian metric is denoted as \( g_{\mu\nu} \) and it is a function of \( x \) on \( M \). The metric is a positive definite matrix. The inverse of the metric is denoted as \( (g_{\mu\nu})^{-1} = g^{\mu\nu} \). We will also denote the determinant of the metric as \( g = \det g_{\mu\nu} \). Thus the inner product defined on \( T_p M \) is given by the metric as
\[ \langle u, v \rangle = g_{\mu\nu} u^{\mu} v^{\nu}. \] (2.8)
The inverse defines an inner product on \( T_p^* M \). Using \( g_{\mu\nu} \) and \( g^{\mu\nu} \), we are able to raise and lower indices, thus each vector has a corresponding covector and vice-versa given by
\[ v_{\mu} = g_{\mu\nu} v^{\nu} \text{ or } \alpha^{\mu} = g^{\mu\nu} \alpha_{\nu}. \] (2.9)

**Definition.** Suppose we have a smooth curve on a manifold \( M \) parametrized by a variable \( \tau \in [0, t] \). Then \( x(\tau) = (x^1(\tau), \ldots, x^n(\tau)) \) is the trajectory of the curve in \( M \). Then the \( \mu \)th component of the tangent vector of the curve is \( \frac{dx^{\mu}(\tau)}{d\tau} \). Therefore the arclength of the curve from \( \tau = 0 \) to \( \tau = t \) is
\[ s = \int_0^t d\tau \left\| \frac{dx(\tau)}{d\tau} \right\| = \int_0^t d\tau \sqrt{g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}}. \] (2.10)
The standard Euclidean volume form denoted as \( dx = dx^1 \cdots dx^n \) is not invariant under coordinate transformation. Thus we introduce the Riemannian volume form which is invariant.

**Definition.** The Riemannian volume element is

\[
d \text{vol}(x) = g^2(x)dx.
\]  

### 2.1.3 Tensors

**Definition.** A tensor of type \((p,q)\) of a vector space \(V\) is a multilinear map

\[
T : V \times \cdots \times V \times V^\ast \times \cdots \times V^\ast \rightarrow \mathbb{R}
\]

and is denoted as

\[
T_{\mu_1 \mu_2 \cdots \mu_p}^{\nu_1 \nu_2 \cdots \nu_q}(x),
\]

where each index goes from 1 to \(n\).

A tensor is invariant under coordinate changes with every upper index transforming as a vector and every lower index transforming as a covector. Then a tensor field is the smooth assignment of tensor to every point of \(M\) and is denoted as

\[
T^\mu_{\nu_1 \nu_2 \cdots \nu_q}(x).
\]

Suppose \(T\) is a tensor of type \((2,0)\) or \((1,1)\) or \((0,2)\), then we call \(T\) a matrix. Then we denote the trace of the matrix \(T\) as

\[
\text{tr } T = g^\mu_{\nu} T^\mu_{\nu} = T_{\mu}^{\mu}.
\]

**Definition.** Let \(S_q\) be the group of permutations of the set \(Z_q = \{1, 2, \cdots, q\}\). A permutation \(\varphi : Z_q \rightarrow Z_q\) is represented by

\[
\begin{pmatrix}
1 & \cdots & q \\
\varphi(1) & \cdots & \varphi(q)
\end{pmatrix}.
\]

An elementary permutation is a permutation that exchanges the order of only two elements. We say a permutation is even (odd) if it is the product of an even (odd) number of elementary permutations. We define the function

\[
\text{sign}(\varphi) = \begin{cases} 
1 & \text{if } \varphi \text{ is even} \\
-1 & \text{if } \varphi \text{ is odd}
\end{cases}.
\]
Using the above definition, we define the anti-symmetrization and symmetrization of tensors:

**Definition.** Let $T$ be a tensor of type $(0,q)$. the symmetrization of the indices of tensor $T$ is given by

$$T(\nu_1 \nu_2 \cdots \nu_q) = \frac{1}{q!} \sum_{\varphi \in S_q} T_{\nu_{\varphi(1)} \nu_{\varphi(2)} \cdots \nu_{\varphi(q)}}. \quad (2.18)$$

Secondly, we define the anti-symmetrization of tensor $T$ as

$$T[\nu_1 \nu_2 \cdots \nu_q] = \frac{1}{q!} \sum_{\varphi \in S_q} \text{sign}(\varphi) T_{\nu_{\varphi(1)} \nu_{\varphi(2)} \cdots \nu_{\varphi(q)}}. \quad (2.19)$$

We will use vertical lines to denote omission of an index from symmetrization and anti-symmetrization, i.e.

$$T[\nu_1 \nu_2 \cdots \nu_j - 1 | \nu_j | \nu_j + 1 \cdots \nu_q] = \frac{1}{(q-1)!} \sum_{\varphi \in S_{q-1}} \text{sign}(\varphi) T_{\nu_{\varphi(1)} \nu_{\varphi(2)} \cdots \nu_{\varphi(q-1)}}. \quad (2.20)$$

The Levi-Civita symbol is not a tensor, but is what we call a tensor density and it will be used often. The Levi-Civita symbol is

$$\varepsilon_{\mu_1 \mu_2 \cdots \mu_n} = \begin{cases} 
1 & \text{if } \{\mu_1, \mu_2, \cdots, \mu_n\} \text{ is an even perm. of } \{1, 2, \cdots, n\} \\
-1 & \text{if } \{\mu_1, \mu_2, \cdots, \mu_n\} \text{ is an odd perm. of } \{1, 2, \cdots, n\} \\
0 & \text{otherwise} 
\end{cases} \quad (2.21)$$

Throughout this paper, we will use the Levi-Civita density, which is just the Levi-Civita symbol scaled by the metric. The density is invariant and indices can be raised or lowered by the metric. The density will be denoted as

$$\varepsilon^{\mu_1 \mu_2 \cdots \mu_n} = g^{\frac{1}{2}} \varepsilon_{\mu_1 \mu_2 \cdots \mu_n} \quad \text{and} \quad \varepsilon^{\mu_1 \mu_2 \cdots \mu_n} = g^{-\frac{1}{2}} \varepsilon_{\mu_1 \mu_2 \cdots \mu_n}. \quad (2.22)$$

The Levi-Civita symbol is most helpful in writing the determinant of a matrix:

$$\det(A^\mu_{\nu}) = \varepsilon^{\mu_1 \cdots \mu_n} A^1_{\mu_1} \cdots A^n_{\mu_n} = \varepsilon_{\nu_1 \cdots \nu_n} A^{\nu_1}_{\mu_1} \cdots A^{\nu_n}_{\mu_n} = \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_n} \varepsilon_{\nu_1 \cdots \nu_n} A^{\nu_1}_{\mu_1} \cdots A^{\nu_n}_{\mu_n}. \quad (2.23)$$

We note that the Levi-Civita symbol satisfies

$$\varepsilon^{\mu_1 \cdots \mu_n} A^{\nu_1}_{\mu_1} \cdots A^{\nu_n}_{\mu_n} = \varepsilon^{\nu_1 \cdots \nu_n} \det(A^\mu_{\nu}), \quad (2.24)$$

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and
\[ \varepsilon^{\mu_1 \cdots \mu_k \alpha_1 \cdots \alpha_{n-k}} = (n - k)! \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} \varepsilon_{\nu_1 \cdots \nu_k \alpha_1 \alpha_{n-k}} \] (2.25)

where
\[ \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} = k! \delta^{\mu_1}_{\nu_1} \cdots \delta^{\mu_k}_{\nu_k}. \] (2.26)

These will be used in the following lemma.

**Lemma 1.** Suppose we have a matrix \( A^\mu_{\nu} \) with its inverse \((A^\mu_{\nu})^{-1}\) denoted as \( B^\nu_{\mu} \), and we define the object
\[ F^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} = \frac{1}{(n - k)!} \varepsilon^{\mu_1 \cdots \mu_k \alpha_1 \cdots \alpha_{n-k}} \varepsilon_{\nu_1 \cdots \nu_k \beta_1 \cdots \beta_{n-k}} A^{\alpha_1}_{\beta_1} \cdots A^{\alpha_{n-k}}_{\beta_{n-k}}, \] (2.27)

then
\[ F^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} = k! B^\nu_{\mu_1} \cdots B^\nu_{\mu_k} \det A. \] (2.28)

**Proof.** Let us multiply (2.27) by \( k \) \( A \)'s, then we obtain
\[ A^{\nu_1 \lambda_1} \cdots A^{\nu_k \lambda_k} F^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} = \frac{1}{(n - k)!} \varepsilon^{\mu_1 \cdots \mu_k \alpha_1 \cdots \alpha_{n-k}} \varepsilon_{\nu_1 \cdots \nu_k \beta_1 \cdots \beta_{n-k}} \]
\[ \times A^{\nu_1 \lambda_1} \cdots A^{\nu_k \lambda_k} A^{\beta_1}_{\alpha_1} \cdots A^{\beta_{n-k}}_{\alpha_{n-k}} \]
\[ = \frac{1}{(n - k)!} \varepsilon^{\mu_1 \cdots \mu_k \alpha_1 \cdots \alpha_{n-k}} \varepsilon_{\nu_1 \cdots \nu_k \lambda_1 \cdots \lambda_k \alpha_1 \cdots \alpha_{n-k}} \det A \]
\[ = \delta^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \det A. \] (2.30)

Now, by multiplying by \( k \) \( B \)'s, we have
\[ B^{\lambda_1}_{\kappa_1} \cdots B^{\lambda_k}_{\kappa_k} A^{\nu_1 \lambda_1} \cdots A^{\nu_k \lambda_k} F^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} = \delta^{\lambda_1}_{\kappa_1} \cdots \delta^{\lambda_k}_{\kappa_k} F^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k} \]
\[ = F^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k}. \] (2.32)

Therefore
\[ F^{\mu_1 \cdots \mu_k}_{\kappa_1 \cdots \kappa_k} = B^{\lambda_1}_{\kappa_1} \cdots B^{\lambda_k}_{\kappa_k} \delta^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \det A \]
\[ = k! B^\nu_{\mu_1} \cdots B^\nu_{\mu_k} \det A. \] (2.33)
2.1.4 Covariant Derivative

Partial derivatives are not invariant under a change of coordinates, therefore we must define an “invariant derivative”. First we need the affine connection which relates the tangent vectors at two different points on a manifold. The affine connection that is torsion-free and compatible with the metric is called the Levi-Civita connection and is denoted by the Christoffel symbols which are given by
\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}). \] (2.34)

Then the covariant derivatives of a vector field \( v \) and a covector field \( \alpha \) are
\[ \nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu_{\beta\nu} v^\beta \quad \text{and} \quad \nabla_\nu \alpha^\mu = \partial_\nu \alpha^\mu - \Gamma^\beta_{\mu\nu} \alpha^\beta. \] (2.35)

We also have covariant derivative of a tensor:
\[ \nabla_\gamma T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} = \partial_\gamma T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} + \Gamma^{\mu_1}_{\beta\gamma} T^{\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} + \cdots + \Gamma^{\mu_p}_{\beta\gamma} T^{\mu_1\ldots\mu_{p-1}}_{\nu_1\nu_2\ldots\nu_q} \]
\[ - \Gamma^\beta_{\nu_1\nu_2\ldots\nu_q} T^{\mu_1\mu_2\ldots\mu_p}_{\beta\gamma} + \cdots - \Gamma^\beta_{\nu_1\nu_2\ldots\nu_q} T^{\mu_1\mu_2\ldots\mu_p}_{\beta\gamma}. \] (2.36)

We also consider a vector bundle, \( \mathcal{V} \), on our manifold, \( \mathcal{M} \). Let \( A_\mu \) be a connection on this vector bundle, then the covariant derivative of a section \( \varphi \) of the bundle \( \mathcal{V} \) is given by
\[ \nabla_\mu \varphi = \partial_\mu \varphi + A_\mu \varphi. \] (2.37)

2.1.5 Curvature

The Riemann curvature tensor \( R^{\mu}_{\nu\alpha\beta} \) is found by finding the commutator of covariant derivatives evaluated on a vector. So we get
\[ [\nabla_\alpha, \nabla_\beta] v^\mu = R^{\mu}_{\nu\alpha\beta} v^\nu \] (2.38)

with
\[ R^{\mu}_{\nu\gamma\delta} = \partial_\gamma \Gamma^{\mu}_{\nu\delta} - \partial_\delta \Gamma^{\mu}_{\nu\gamma} + \Gamma^{\mu}_{\beta\gamma} \Gamma^\beta_{\nu\delta} - \Gamma^\beta_{\nu\gamma} \Gamma^\beta_{\mu\delta}. \] (2.39)

We can also evaluate the commutator of covariant derivatives on a covector and we have
\[ [\nabla_\gamma, \nabla_\delta] \alpha_\mu = -R^{\nu}_{\mu\gamma\delta} \alpha_\nu. \] (2.40)

Lastly, we can evaluate the commutator on a tensor of type \((p, q)\) and we get
\[ [\nabla_\gamma, \nabla_\delta] T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} = \sum_{a=1}^{p} R^{\nu_1}_{\beta_1\gamma\delta} T^{\mu_1\ldots\mu_{a-1}\beta_1\mu_{a+1}\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} - \sum_{b=1}^{q} R^{\beta_1}_{\nu_1\gamma\delta} T^{\mu_1\beta_1\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_{b-1}\beta_1\mu_{b+1}\ldots\nu_q} \] (2.41)
We can raise and lower indices on the Riemann tensor just like any tensor. The Riemann tensor is anti symmetric in the first two indices (i.e. $R_{\mu\nu\gamma\delta} = -R_{\nu\mu\gamma\delta}$), anti symmetric in the last two indices (i.e. $R_{\mu\nu\gamma\delta} = -R_{\mu\nu\delta\gamma}$), and is symmetric in the exchanging of these pairs (i.e. $R_{\mu\nu\gamma\delta} = R_{\gamma\delta\mu\nu}$).

By contracting over indices, we can define new tensors.

**Definition.** The Ricci tensor is given by

$$ R_{\mu\nu} = R_{\gamma\mu\gamma\nu} $$

and the scalar curvature is given by

$$ R = g^{\mu\nu} R_{\mu\nu} = R_{\mu\mu}. $$

The commutator of covariant derivatives of a section, $\varphi$, of a vector bundle, $\mathcal{V}$, defines the curvature 2-form, $\mathcal{R}$, of the bundle connection, $\mathcal{A}$,

$$ [\nabla_\mu, \nabla_\nu] \varphi = \mathcal{R}_{\mu\nu} \varphi, $$

where

$$ \mathcal{R}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. $$

### 2.1.6 Geodesics

**Definition.** A geodesic is a curve $x(\tau)$ on a manifold $\mathcal{M}$ such that the following equation holds:

$$ \frac{dx^\nu}{d\tau} \nabla_\nu \left( \frac{dx^\mu}{d\tau} \right) = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\gamma} \frac{dx^\nu}{d\tau} \frac{dx^\gamma}{d\tau} = 0 $$

for some parametrization $\tau$ which we call the affine parameter. A geodesic is a curve for which the tangent vector to the curve transported along the curve remains the tangent vector and is referred to as the shortest curve between two points.

Any two points on a manifold $\mathcal{M}$ that are sufficiently close can be connected by a single geodesic. The distance along the geodesic between two points $x$ and $x'$ is called the geodesic distance and is denoted as $d(x, x')$.

### 2.1.7 Synge Function

Before we define the Synge function, we need to introduce the injectivity radius of a manifold.
Definition. Let $x' \in \mathcal{M}$, then we define the ball of radius $r$ as

$$B_r(x') = \{ x \in \mathcal{M} \mid d(x, x') < r \}.$$  \hspace{1cm} (2.47)

Definition. Let $x, x' \in \mathcal{U} \subset \mathcal{M}$. We say $x$ and $x'$ are conjugate in $\mathcal{U}$ if they are connected by more than one geodesic.

Definition. Let $r_{\text{inj}}(x')$ be the largest radius of the ball $B_r(x')$ such $\forall x \in B_r(x')$, $x$ and $x'$ are not conjugate in $B_r(x')$. In other words, $r_{\text{inj}}(x')$ is the largest ball around $x'$ such that all points are connected to $x'$ by a single geodesic. We call $r_{\text{inj}}(x')$ the injectivity radius of the point $x'$.

Definition. The injectivity radius of the manifold, $\mathcal{M}$, is given by

$$\Gamma_{\text{inj}}(\mathcal{M}) = \inf_{x' \in \mathcal{M}} r_{\text{inj}}(x').$$  \hspace{1cm} (2.48)

Thus if $r < \Gamma_{\text{inj}}$, then $\forall x' \in \mathcal{M}$, $B_r(x')$ will not contain conjugate points.

Definition. The Synge function (also called a world function) is a bi-scalar function and is defined as half the square geodesic distance between points $x$ and $x'$ on $\mathcal{M}$. Thus

$$\sigma(x, x') = \frac{1}{2} d^2(x, x') = \frac{1}{2} \int_0^t d\tau \ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \hspace{1cm} (2.49)$$

We note $\sigma$ is multivalued for conjugate points. We will consider $\sigma$ as single valued when, along the shortest geodesic between $x$ and $x'$, $d(x, x') < \Gamma_{\text{inj}}$. We will then use this distance along this shortest geodesic as $d(x, x')$ in (2.49). Thus $\sigma$ is only defined “locally”.

Before we continue further, we will mention some of the notation we will be using. The coincidence limit of a bi-scalar function is denoted as

$$[f(x, x')] = \lim_{x \to x'} f(x, x').$$  \hspace{1cm} (2.50)

For the Synge function, we will use indices to denote covariant derivatives:

$$\sigma_\mu = \nabla_\mu \sigma(x, x') \quad \text{and} \quad \sigma_{\mu'} = \nabla_{\mu'} \sigma(x, x'),$$  \hspace{1cm} (2.51)

where $\nabla_\mu$ is the covariant derivative with respect to the $x$ coordinate and $\nabla_{\mu'}$ is the covariant derivative with respect to the $x'$ coordinate. Thus we can have multiple indices indicating multiple derivatives (i.e. $\sigma_{\mu'\nu} = \sigma_{\mu''} = \nabla_{\mu''} \nabla_\nu \sigma = \nabla_\nu \nabla_{\mu''} \sigma$).
The Synge function is very important in that it completely describes the local geometry near \(x\) and \(x'.\) The Synge function has the following properties:

\[
\begin{align*}
\sigma_{\mu} \sigma^{\mu} &= 2\sigma, & \sigma_{\nu} \sigma^{\nu'} &= 2\sigma, \\
\sigma_{\mu} \sigma^{\nu'} &= \sigma_{\nu}, & \sigma_{\mu} \sigma^{\nu'} &= \sigma_{\nu}, \\
\sigma_{\nu} \sigma^{\nu'} &= \sigma_{\mu}, & \sigma_{\mu} \sigma^{\nu'} &= \sigma_{\nu}'.
\end{align*}
\] (2.52)

We will denote the inverse matrix of \(\sigma_{\mu\nu}\) as \(\gamma^{\mu\nu}\), the inverse matrix of \(\sigma_{\mu\nu'}\) as \(\gamma^{\mu\nu'}\), and the inverse matrix of \(\sigma_{\mu'}\nu'\) as \(\gamma^{\mu'}\nu'\). Thus it is easy to see that \(\gamma\) matrices satisfy

\[
\begin{align*}
\gamma^{\mu\nu} \sigma_{\nu\alpha} &= \delta_{\alpha}^{\mu}, & \gamma^{\mu\nu'} \sigma_{\nu'\alpha} &= \delta_{\alpha}^{\mu}, \\
\sigma_{\mu} \gamma^{\mu\nu} &= \sigma^{\nu}, & \sigma_{\mu} \gamma^{\mu\nu'} &= \sigma^{\nu'}, \\
\sigma_{\nu} \gamma^{\nu\mu} &= \sigma^{\mu}, & \sigma_{\nu'} \gamma^{\nu'\mu} &= \sigma^{\mu}'.
\end{align*}
\] (2.55)

See Section 5.2 for more on the Synge function, its derivatives and their coincidence limits which will be used extensively.

We introduce the Van Fleck-Morette determinant which is defined as

\[
\Delta(x, x') = g^{-\frac{1}{2}}(x) g^{-\frac{1}{2}}(x') \det(-\sigma_{\mu\nu'}).
\] (2.58)

Again, see Section 5.2 for information on the derivatives and the coincidence limits of \(\Delta\).

We will use the following change of variables later:

\[
\xi^{\mu} = \frac{\sigma^{\mu}}{\sqrt{t}} \quad \text{and} \quad \xi^{\nu'} = \frac{\sigma^{\nu'}}{\sqrt{t}}.
\] (2.59)

These change of coordinates satisfy the following:

\[
|\xi|^2 = \xi^{\mu} \xi_{\mu} = \xi^{\nu'} \xi_{\nu'} = \frac{2\sigma}{t},
\] (2.60)

\[
\sigma_{\nu'} \xi_{\mu} = \xi_{\nu'} \quad \text{and} \quad \sigma_{\mu} \xi^{\nu'} = \xi_{\mu}.
\] (2.61)

Then the standard partial derivatives transform as

\[
\frac{\partial}{\partial x^{\mu}} = \frac{\sigma^{\nu'}}{\sqrt{t}} \frac{\partial}{\partial \xi^{\nu'}} \quad \text{and} \quad \frac{\partial}{\partial \xi^{\nu}} = \sqrt{t} \gamma^{\nu'} \frac{\partial}{\partial x^{\mu}}.
\] (2.62)

Thus we have

\[
\sigma_{\mu} \frac{\partial}{\partial x^{\mu}} = \xi^{\nu'} \frac{\partial}{\partial \xi^{\nu'}}.
\] (2.63)
and the volume element becomes
\[ d\text{vol}(x) = g^{\frac{1}{2}}(x) \, dx = l^2 \Delta^{-1}(x, x') g^{\frac{1}{2}}(x') \, d\xi. \] (2.64)

We also introduce the Gaussian average over the variable \( \xi \)
\[ \langle f(\xi) \rangle_a = \left( \frac{a}{\pi} \right)^\frac{n}{2} \int_{\mathbb{R}^n} d\xi \, g^{\frac{1}{2}}(x') \exp \left( -a \xi^2 \right) f(\xi), \] (2.65)

where \( a \) is a positive constant. In particular, we have
\[ \langle \xi_{\mu_1} \xi_{\mu_2} \cdots \xi_{\mu_{2k+1}} \rangle_a = 0, \] (2.66)
\[ \langle \xi_{\mu_1} \xi_{\mu_2} \cdots \xi_{\mu_{2k}} \rangle_a = \frac{(2k)!}{(2a)^k k!} g^{\mu_1 \mu_2 \cdots g^{\mu_{2k-1} \mu_{2k}}}. \] (2.67)

For more on the Gaussian integral and Gaussian average, see section 5.1.

### 2.1.8 Covariant Taylor Series

We consider a section, \( \varphi \), of a vector bundle, \( \mathcal{V} \), on the manifold \( \mathcal{M} \). To expand \( \varphi \) in covariant Taylor series, we will transport \( \varphi \) from \( x \) to \( x' \), then expand it in a Taylor series along the geodesic, then transport it back to \( x \). We first introduce the parallel transport operator \( \mathcal{P}(x, x') \), which parallel transports a field from the point \( x' \) to \( x \) along the connecting geodesic. Thus, the parallel transport operator is defined as
\[ \sigma^\mu \nabla_\mu \mathcal{P} = 0, \] (2.68)

with \( [\mathcal{P}] = 1 \). By taking derivatives of the above equation and symmetrizing, it is easy to see that
\[ \left[ \nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \mathcal{P} \right] = 0, \] (2.69)

with the same properties holding for primed derivatives and inverse of \( \mathcal{P} \) given by \( \mathcal{P}^{-1}(x, x') = \mathcal{P}(x', x) \). Therefore, let us define
\[ \bar{\varphi} = \mathcal{P}^{-1} \varphi. \] (2.70)

We expand this in a Taylor series along the geodesic using our affine parameter, \( \tau \). We find
\[ \bar{\varphi} = \sum_{k=0}^{\infty} \frac{1}{k!} \tau^k \left. \left( \frac{d^k}{d\tau^k} \varphi \right) \right|_{\tau=0}. \] (2.71)
We know that \( \frac{d}{d\tau} = U^\mu \partial_\mu \), where \( U^\mu = \frac{dx^\mu}{d\tau} \). Noting that \( \partial_\mu \phi = \nabla_\mu \phi \), and that \( U^\mu \nabla_\mu U^\nu = 0 \) since it is just the geodesic equation, we have

\[
\phi = \sum_{k=0}^\infty \frac{1}{k!} \tau^k \left( U^{\mu_1} \cdots U^{\mu_k} \nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \mathcal{P}^{-1} \phi \right) \bigg|_{\tau=0}.
\]

(2.72)

Noting that \( -\tau U^\mu(0) = \sigma^\mu \) (as derived in Section 5.2), and that evaluating \( \mathcal{P}^{-1} \) and its derivatives at \( \tau = 0 \) is the same as taking the coincidence limit, we find

\[
\phi = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \sigma^{\mu_1} \cdots \sigma^{\mu_k} \left[ \nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \phi \right].
\]

(2.73)

Therefore, we have the covariant Taylor expansion of a field \( \phi \) given by

\[
\phi = \mathcal{P} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \sigma^{\mu_1} \cdots \sigma^{\mu_k} \left[ \nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \phi \right].
\]

(2.74)

### 2.2 Laplace Type Operators

For this paper, we will only be looking at second-order elliptic partial differential operators, more specifically, the scalar Laplacian acting on manifolds without boundary. However, the heat determinant we introduce later can be applied any second-order elliptic partial differential operator.

**Definition.** Let \( \mathcal{H} = L^2(V) \) be the Hilbert space (complete inner product space) of square integrable sections of our vector bundle \( V \). The inner-product on \( \mathcal{H} \) is defined as

\[
(\phi, \psi) = \int d\text{vol}(x) \langle \phi, \psi \rangle,
\]

(2.75)

where \( \langle \cdot, \cdot \rangle \) is the inner product in the fiber of our bundle \( V \).

Suppose we have a second order partial differential operator given by

\[
L = -a^{\mu \nu} \nabla_\mu \nabla_\nu + b^\mu \nabla_\mu + c,
\]

(2.76)

where \( a, b, \) and \( c \) could all be functions of \( x \). Then we denote the leading symbol as \( \sigma_L(\xi, x) = a^{\mu \nu} \xi_\mu \xi_\nu \). The operator \( L \) is elliptic if \( \forall \xi \neq 0 \) and \( \forall x \in \mathcal{M}, \sigma_L(\xi, x) \geq C |\xi|^2 \) for some \( C > 0 \). i.e.

\[
a^{\mu \nu}(x) \xi_\mu \xi_\nu \geq C |\xi|^2 \quad \forall \xi \neq 0 \text{ and } \forall x \in \mathcal{M}.
\]

(2.77)

where \( |\xi|^2 = \delta^{\mu \nu} \xi_\mu \xi_\nu \).

Let \( C^\infty(V) \) be the space of smooth functions on \( V \).
**Definition.** A Laplace type operator is an elliptic operator $L : \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V})$ of the form

$$L = -\Delta + Q$$

(2.78)

where $Q$ is a smooth endomorphism of the bundle $\mathcal{V}$ and $\Delta$ is the standard Laplacian

$$\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu.$$  

(2.79)

The adjoint of an operator $L$ is denoted as $L^*$ which satisfies

$$\langle f, Lg \rangle = \langle L^* f, g \rangle$$

(2.80)

for all $f$ and $g$ in $H$. An operator is self-adjoint if $L^* = L$ and the domains of $L^*$ and $L$ are identical.

The Laplace type operators are symmetric (i.e. $\langle f, Lg \rangle = \langle Lf, g \rangle$). Laplace type operators have self-adjoint extensions, but they, themselves, are not self-adjoint. We will not distinguish between $L$ and its self-adjoint extension.

We only deal with self-adjoint elliptic operators on compact manifolds without boundary, since they have a countable set of eigenvalues and eigenfunctions. Recall that if $\lambda_i$ is an eigenvalue with the corresponding eigenfunction $\phi_i$, then they satisfy the equation

$$L\phi_i = \lambda_i \phi_i.$$  

(2.81)

If the set of eigenfunctions forms a complete orthonormal basis on our Hilbert space, then we can spectrally decompose a function of the operator $L$:

$$f(L)\phi = \sum_{k=0}^{\infty} f(\lambda_k) \langle \phi_k, \phi \rangle \phi_k.$$  

(2.82)

where $\phi$ is a section of our vector bundle. This makes life easy when dealing with elliptic operators.

We define the trace of a function of an operator as the sum of the function evaluated at our eigenvalues,

$$\text{Tr} f(L) = \sum_{k=0}^{\infty} f(\lambda_k).$$  

(2.83)

For a more indepth discussion on operators and spectral decomposition, see [6].
2.2.1 Heat Kernel

As we have already mentioned, we will only study compact manifolds without boundary. The heat kernel of an operator $L$ is denoted as $U(t,x,x')$ and satisfies the heat equation

$$(\partial_t + L) U(t,x,x') = 0,$$  \hspace{1cm} (2.84)

with the initial condition

$$U(0,x,x') = \delta(x,x'),$$ \hspace{1cm} (2.85)

where $\delta(x,x') = \sqrt{g} \sqrt{g'} \delta(x-x')$ is the covariant $\delta$ function. The operator $L$ acts on unprimed coordinates. The heat kernel gives us the general solution to an arbitrary initial condition. That is, the solution of the initial value problem

$$(\partial_t + L) f(t,x) = 0,$$ \hspace{1cm} (2.86)

$$f(0,x) = h(x),$$ \hspace{1cm} (2.87)

is given by

$$f(t,x) = \int_M d\text{vol}(x') \ U(t,x,x') h(x').$$ \hspace{1cm} (2.88)

We can spectrally decompose the heat kernel to

$$U(t,x,x') = \sum_{k=0}^{\infty} \exp(-t\lambda_k) \phi_k(x) \phi_k^*(x').$$ \hspace{1cm} (2.89)

2.3 Heat Invariants

In this section, we discuss a few heat invariants. We will assume $L$ is the general scalar Laplace type operator

$$L = -\Delta + Q,$$ \hspace{1cm} (2.90)

where $Q$ is a non-constant smooth endomorphism.
2.3.1 Heat Trace

One of the most important heat invariants is the heat trace. It has been studied extensively [11]. The heat trace is given by the trace of the heat semigroup \( \exp(-tL) \),

\[
\Theta(t) = \text{Tr} \left( \exp(-tL) \right) = \sum_{k=1}^{\infty} e^{-t\lambda_k}.
\] (2.91)

This is a pure spectral invariant, the asymptotics of which play an important role in spectral geometry and mathematical physics. It is well known that as \( t \to 0 \),

\[
\Theta(t) \sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} A_k t^k,
\] (2.92)

where \( A_k \) are called the heat trace invariants. They are given by integral of local invariants, more specifically, the first two are

\[
A_0 = \text{vol}(\mathcal{M}),
\] (2.93)

\[
A_1 = \int_{\mathcal{M}} d\text{vol} \left( Q - \frac{1}{6} R \right).
\] (2.94)

2.3.2 Heat Content

Another heat invariant is the heat content and is given by the integral of the heat kernel of the manifold. For scalar operators, the heat content is defined by

\[
\Pi(t) = \int_{\mathcal{M} \times \mathcal{M}} d\text{vol}(x) d\text{vol}(x') U(t, x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} |\phi_k|^2,
\] (2.95)

where

\[
\phi_k = \int_{\mathcal{M}} d\text{vol} \phi_k.
\] (2.96)

From (2.89), we find that

\[
\int_{\mathcal{M}} d\text{vol}(x') U(t; x, x') = (\exp(-tL) \cdot 1) (x).
\] (2.97)

Thus expanding the heat content as \( t \to \infty \) gives us

\[
\Pi(t) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Pi_k t^k,
\] (2.98)
where

\[ \Pi_0 = \text{vol}(\mathcal{M}), \quad (2.99) \]
\[ \Pi_1 = \int_{\mathcal{M}} d\text{vol} \, Q, \quad (2.100) \]
\[ \Pi_k = \int_{\mathcal{M}} d\text{vol} \, ( -\Delta + Q )^{k-2} Q, \quad k \geq 2. \quad (2.101) \]

We note that if \( Q = 0 \), then \( \Pi(t) = \text{vol}(\mathcal{M}) \).

### 2.3.3 Modified Heat Determinant

We introduce a modified heat determinant which we will not use since it vanishes for manifolds without boundary. We define a matrix \( \tilde{P} \) as

\[
\tilde{P}_{\mu\nu'}(t; x, x') = \nabla_\mu \nabla_{\nu'} U(t; x, x') = \sum_{l=1}^{\infty} e^{-t\lambda_l} \nabla_\mu \varphi_l(x) \nabla_{\nu'} \varphi_l(x').
\]

(2.102)

Thus we can define the modified heat determinant as

\[
\tilde{K}(t) = \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, \det \tilde{P}_{\mu\nu'}(t; x, x') = \frac{1}{n!} \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, \epsilon^{\mu_1 \cdots \mu_n} \epsilon_{\nu'_1 \cdots \nu'_n} \tilde{P}_{\mu_1 \nu'_1} \cdots \tilde{P}_{\mu_n \nu'_n}.
\]

(2.103)

We define a new object \( \tilde{E} \) as

\[
\tilde{E}_{k_1 \cdots k_n} = \int_{\mathcal{M}} dx \, \epsilon^{\mu_1 \cdots \mu_n} \nabla_{\mu_1} \varphi_{k_1} \cdots \nabla_{\mu_n} \varphi_{k_n}
\]

\[
= \int_{\mathcal{M}} d\varphi_{k_1} \wedge \cdots \wedge d\varphi_{k_n}.
\]

(2.104)

We note that \( d\varphi_{k_1} \wedge \cdots \wedge d\varphi_{k_n} = d(\varphi_{k_1} \wedge d\varphi_{k_2} \wedge \cdots \wedge d\varphi_{k_n}) \), thus using Stokes’ theorem, we have

\[
\tilde{E}_{k_1 \cdots k_n} = \int_{\partial \mathcal{M}} \varphi_{k_1} \wedge d\varphi_{k_2} \cdots \wedge d\varphi_{k_n}.
\]

(2.105)

Thus \( \tilde{E} = 0 \) if our manifold has no boundary.
We can write $\tilde{K}$ as

$$\tilde{K}(t) = \frac{1}{n!} \int_{M \times M} dx \, dx' \, e^{\mu_1 \cdots \mu_n} e^{\nu'_1 \cdots \nu'_n}$$

$$\times \left\{ \left( \sum_{k_1=1}^{\infty} e^{-t \lambda_{k_1}} \nabla_{\mu_{k_1}} \varphi_{k_1}(x) \nabla_{\nu'_{k_1}} \varphi_{k_1}(x') \right) \right. $$

$$\vdots$$

$$\times \left( \sum_{k_n=1}^{\infty} e^{-t \lambda_{k_n}} \nabla_{\mu_{k_n}} \varphi_{k_n}(x) \nabla_{\nu'_{k_n}} \varphi_{k_n}(x') \right) \right\}, \tag{2.106}$$

then

$$\tilde{K}(t) = \frac{1}{n!} \sum_{k_1, \ldots, k_n=1}^{\infty} e^{-t(\lambda_{k_1} + \cdots + \lambda_{k_n})} \tilde{E}_{k_1} \cdots \tilde{E}_{k_n} \overset{\leftrightarrow}{\tilde{E}}_{k_1}^{t} \cdots \overset{\leftrightarrow}{\tilde{E}}_{k_n}^{t} \tag{2.107}$$

From (2.105), it is clear that for manifolds without boundary $\tilde{K} = 0$. However if the boundary of our manifold is not empty, then the modified heat determinant is non-zero and we can study this new heat invariant that depends on the eigenvalues and eigenfunctions of our operator $L$. However, this only works for a scalar operator, as it is not invariant for elliptic second-order differential operators on vector bundles.
CHAPTER 3

HEAT DETERMINANT

In this section, we will define the heat determinant of an elliptic second order partial differential operator on a vector bundle, then look more specifically at the heat determinant for scalar Laplacian.

3.1 Definition of Heat Determinant

Define the tensor \( P \) as
\[
P_{\mu\nu}'(t; x, x') = \text{tr} \, U^*(t; x, x') \nabla_\mu \nabla_\nu' U(t; x, x'),
\]
where \( U(t; x, x') \) is the heat kernel of an operator \( L \). We note that the trace here is over the fiber of our vector bundle, thus \( P_{\mu\nu}' \) is a matrix with scalar entries. We also note that if \( L \) is self-adjoint, then \( U^*(t, x, x') = U(t, x', x) \). Then the heat determinant is defined as
\[
K(t) = \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, \det P_{\mu\nu}'(t, x, x'),
\]
which can also be written as
\[
K(t) = \frac{1}{n!} \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, \varepsilon^{\mu_1 \cdots \mu_n} \varepsilon_{\nu_1 \cdots \nu_n} P_{\mu_1 \nu_1}'(t, x, x') \cdots P_{\mu_n \nu_n}'(t, x, x').
\]

Let \( \phi_k \) be the eigenfunction corresponding to the eigenvalue \( \lambda_k \) of the operator \( L \). Then we define
\[
f_{kl}(x) = \langle \phi_k, \phi_l \rangle,
\]
then we have
\[
(\phi_k, \phi_l) = \int_{\mathcal{M}} d\text{vol}(x) \, f_{kl}(x) = \delta_{kl}.
\]
Now define
\[
B_{kl \, \mu} = \langle \phi_k, \nabla_\mu \phi_l \rangle,
\]
which then defines the one-forms
\[ B_{kl} = B_{kl} \mu dx^\mu = \langle \varphi_k, \nabla_\mu \varphi_l dx^\mu \rangle = \langle \varphi_k, D \varphi_l \rangle . \] (3.7)

Lastly, we define
\[ E_{k_1 \ldots k_n}^{l_1 \ldots l_n} = \int_\mathcal{M} dx^{\mu_1 \ldots \mu_n} \langle \varphi_{k_1}, \nabla_{\mu_1} \varphi_{l_1} \rangle \cdots \langle \varphi_{k_n}, \nabla_{\mu_n} \varphi_{l_n} \rangle \]
\[ = \int_\mathcal{M} B_{k_1 l_1} \wedge \cdots \wedge B_{k_n l_n} . \] (3.8)

Thus we can write the tensor \( P \) as
\[ P_{\mu \nu'}(t; x, x') = \text{tr} \left\{ \left( \sum_{k=1}^{\infty} e^{-t \lambda_k} \varphi_k(x') \varphi_k(x) \right) \left( \sum_{l=1}^{\infty} e^{-t \lambda_l} \nabla_\mu \varphi_l(x) \nabla_{\nu'} \varphi_l(x') \right) \right\} \]
\[ = \sum_{k, l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} \langle \varphi_k(x), \nabla_\mu \varphi_l(x) \rangle \langle \varphi_k(x'), \nabla_{\nu'} \varphi_l(x') \rangle \]
\[ = \sum_{k, l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} B_{kl} \mu(x) B^*_{kl} \nu'(x') . \] (3.9)

Then
\[ K(t) = \frac{1}{n!} \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, e^{H_{\mu_1 \ldots \mu_n} e^{\lambda_1 \ldots \lambda_n}} \]
\[ \times \left( \sum_{k, l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} B_{kl} \mu_1(x) B^*_{kl} \nu'_1(x') \right) \]
\[ \vdots \]
\[ \times \left( \sum_{k, l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} B_{kl} \mu_n(x) B^*_{kl} \nu'_n(x') \right) \]
\[ = \frac{1}{n!} \sum_{k_1 l_1 \ldots k_n l_n=1}^{\infty} \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \, e^{H_{\mu_1 \ldots \mu_n} e^{\lambda_1 \ldots \lambda_n}} \]
\[ \times \left( e^{-t(\lambda_{k_1} + \lambda_{l_1})} B_{k_1 l_1} \mu_1(x) B^*_{k_1 l_1} \nu'_1(x') \right) \]
\[ \vdots \]
\[ \times \left( e^{-t(\lambda_{k_n} + \lambda_{l_n})} B_{k_n l_n} \mu_n(x) B^*_{k_n l_n} \nu'_n(x') \right) \]
\[ = \frac{1}{n!} \sum_{k_1 l_1 \ldots k_n l_n=1}^{\infty} e^{-t(\lambda_{k_1} + \lambda_{l_1} + \ldots + \lambda_{k_n} + \lambda_{l_n})} E_{k_1 \ldots k_n}^{l_1 \ldots l_n} E^*_{k_1 \ldots k_n}^{l_1 \ldots l_n} . \] (3.10)
3.2 Calculation of Heat Determinant

Let’s fix a point \( x' \). Then in a ball \( B_r(x') \) of radius \( r < \Gamma_{inj} \) the heat kernel for an arbitrary elliptic second order differential operator is given by

\[
U(t; x, x') = \frac{1}{(4\pi t)^{n/2}} \exp \left\{ -\frac{\sigma(x, x')}{2t} \right\} \Delta^{1/2}(x, x') \mathcal{P}(x, x') \Omega(t; x, x'),
\] (3.13)

where \( \Delta(x, x') \) is the Van-Fleck-Morette determinant (2.58), \( \mathcal{P}(x, x') \) is the parallel transport operator, and \( \Omega(t, x, x') \) is the transport function. It is well known that at \( t \to 0 \), \( \Omega \) as the asymptotic expansion given by

\[
\Omega(t, x, x') \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k(x, x'),
\] (3.14)

where \( a_n(x, x') \) are the heat kernel coefficients which have been calculated (see [2] and sources therein). In this section, we calculate the mixed derivative of the heat kernel and find a general formula for the \( \det \mathcal{P} \).

3.2.1 Mixed Derivative of Heat Kernel

We look at the mixed derivative of the heat kernel first for vector valued operators. For an arbitrary vector-valued second order differential operator, let us define

\[
\Psi(t; x, x') = \Delta^{1/2}(x, x') \mathcal{P}(x, x') \Omega(t; x, x').
\] (3.15)

Then

\[
U(t; x, x') = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{\sigma(x, x')}{2t} \right) \Psi(t; x, x').
\] (3.16)

Taking two covariant derivatives of this, we have

\[
\nabla_\mu \nabla_\nu' U = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{\sigma}{2t} \right) \times \left( \frac{1}{4t^2} \sigma_\mu \sigma_\nu' \Psi - \frac{1}{2t} \left( \sigma_\mu \Psi + \sigma_\nu' \Psi + \sigma_\nu \Psi_\mu + \Psi_{\mu \nu'} \right) \right).
\] (3.17)

Then we get

\[
P_{\mu \nu'} = \text{tr} \ U^* (t, x', x) \nabla_\mu \nabla_\nu' U (t, x, x') = \frac{1}{(4\pi t)^n} \exp \left( -\frac{\sigma}{t} \right) \frac{1}{2t} \Psi_{\mu \nu'},
\] (3.18)
where
\[
Y_{\mu'\nu'} = \frac{1}{2t} \sigma_\mu \sigma_\nu' \text{tr} \Psi^* \Psi - \left( \sigma_\mu \text{tr} \Psi^* \Psi + \sigma_\nu \text{tr} \Psi^* \Psi' + \sigma_\nu' \text{tr} \Psi^* \Psi_\mu \right)
+ 2t \text{tr} \Psi^* \Psi_{\mu'\nu'}.
\]  
(3.19)

So if we define the following:
\[
S_{\mu'\nu'} = \sigma_\mu \sigma_\nu' \text{tr} \Psi^* \Psi,
\]  
(3.20)
\[
V_{\mu'\nu'} = -\sigma_\nu \text{tr} \Psi^* \Psi_\nu',
\]  
(3.21)
\[
W_{\mu'\nu'} = -\sigma_\nu' \text{tr} \Psi^* \Psi_\mu,
\]  
(3.22)
\[
Z_{\mu'\nu'} = -\sigma_\mu' \text{tr} \Psi^* \Psi + 2t \text{tr} \Psi^* \Psi_{\mu'\nu'}.
\]  
(3.23)

then
\[
Y_{\mu'\nu'} = Z_{\mu'\nu'} + W_{\mu'\nu'} + V_{\mu'\nu'} + \frac{1}{2t} S_{\mu'\nu'}.
\]  
(3.24)

Since $S$, $V$, $W$, and $Z$ are defined this way, they satisfy the following relations:
\[
S[^{\mu_1}_{\nu_1} V[^{\nu_2}_{\mu_2} v_2]} = V[^{\mu_1}_{\nu_1} W[^{\nu_2}_{\mu_2} v_2]},
\]  
(3.25)
\[
W[^{\mu_1}_{\nu_1} V[^{\nu_2}_{\mu_2} v_2]} = W[^{\mu_1}_{\nu_1} W[^{\nu_2}_{\mu_2} v_2]},
\]  
(3.26)
\[\]  
Thus
\[
\det P_{\mu'\nu'} = \left( \frac{1}{(4\pi t)^n} \frac{1}{2t} \exp \left( -\frac{\sigma}{t} \right) \right)^n \det Y_{\mu'\nu'},
\]  
(3.27)

where
\[
\det Y_{\mu'\nu'} = \frac{1}{n!} e^{\mu_1 \cdots \mu_n} v_1^{\nu_1} \cdots v_n^{\nu_n} \left\{ Z_{\mu_1 \nu_1'} \cdots Z_{\mu_n \nu_n'} + n \left( W_{\mu_1 \nu_1'} + V_{\mu_1 \nu_1'} + \frac{1}{2t} S_{\mu_1 \nu_1'} \right) Z_{\mu_2 \nu_2'} \cdots Z_{\mu_n \nu_n'} + n (n - 1) W_{\mu_1 \nu_1'} V_{\mu_2 \nu_2'} Z_{\mu_3 \nu_3'} \cdots Z_{\mu_n \nu_n'} \right\}.
\]  
(3.28)

So
\[
\det P_{\mu'\nu'}(x, x') = g^\frac{1}{2}(x) g^\frac{1}{2}(x') \frac{1}{(2t)^n (4\pi t)^n} \exp \left( -\frac{n}{t} \sigma(x, x') \right) H(t; x, x'),
\]  
(3.29)

where
\[
H(t; x, x') = g^{-\frac{1}{2}}(x) g^{-\frac{1}{2}}(x') \det Y_{\mu'\nu'}(t; x, x').
\]  
(3.30)
Therefore, our heat determinant is given by (3.2), which in all of its glory is

$$K(t) = \frac{1}{(2t)^n (4\pi t)^{n^2}} \int_{\mathcal{M} \times \mathcal{M}} dx \, dx' \exp \left( -\frac{n}{t} \sigma(x, x') \right) \det Y_{\mu\nu}'(t; x, x'). \quad (3.30)$$

Let \( r \) be a real number less than the injectivity radius of the manifold, \( \Gamma_{\text{inj}}(\mathcal{M}) \). Then we note that

$$K(t) = K_1(t) + K_2(t), \quad (3.31)$$

where

$$K_1(t) = \frac{1}{(2t)^n (4\pi t)^{n^2}} \int_{\mathcal{M}} dx' \int_{B_r(x')} dx \exp \left( -\frac{n}{t} \sigma(x, x') \right) \det Y_{\mu\nu}'(t; x, x'), \quad (3.32)$$

$$K_2(t) = \int_{\mathcal{M}} dx' \int_{\mathcal{M} - B_r(x')} dx \det P_{\mu\nu}'(t; x, x'). \quad (3.33)$$

By standard elliptic estimates, it has been shown [13] that \( \forall x \in \mathcal{M} - B_r(x') \) and \( 0 < t < 1 \) that the heat kernel has the estimate

$$|U(t; x, x')| \leq C_1 t^{-\frac{n}{2}} \exp \left( -\frac{r^2}{4t} \right), \quad (3.34)$$

where \( C_1 \) is a constant. From the heat equation, we note that each spatial derivative has the dimension of \( t^{-\frac{1}{2}} \). Thus we have an estimate for (3.1) given by

$$|P_{\mu\nu}'(t; x, x')| \leq C_2 t^{-(n+2)} \exp \left( -\frac{r^2}{2t} \right), \quad (3.35)$$

where \( C_2 \) is a constant. Thus we have the estimate

$$|\det P_{\mu\nu}'(t; x, x')| \leq C_3 t^{-n(n+2)} \exp \left( -\frac{n r^2}{2t} \right). \quad (3.36)$$

where \( C_3 \) is a constant. Thus as \( t \to 0 \),

$$K_2(t) \sim 0, \quad (3.37)$$

therefore

$$K(t) \sim K_1(t). \quad (3.38)$$

3.2.2 Heat Determinant Asymptotics

From above, we have as $t \to 0$,

$$K(t) \sim \frac{1}{(2t)^n (4\pi t)^{n^2}} \int d\text{vol}(x) \int_{B_r(x')} d\text{vol}(x') \exp \left( -\frac{n}{t} \sigma(x,x') \right) H(t;x,x').$$

(3.39)

Now using the coordinate transformation given by (2.60), the integral over $B_r(x')$ becomes the integral over the Euclidean ball $B_{r/\sqrt{t}}(\xi)$. As $t \to 0$, $\frac{r}{\sqrt{t}} \to \infty$, therefore we integrate $\xi$ over all $\mathbb{R}^n$. Thus

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{2^n (4\pi)^{n^2}} \int d\text{vol}(x') \int_{\mathbb{R}^n} d\xi \ g^{\frac{1}{2}}(x') \exp \left( -\frac{n}{2} \xi^2 \right) \Delta^{-1}(\xi,x') H(t;\xi,x').$$

(3.40)

Then using the notation for Gaussian average (2.65), we have

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{2^n (4\pi)^{n^2}} \left( \frac{2\pi}{n} \right)^{\frac{n}{2}} \int_{\mathcal{M}} d\text{vol}(x') \langle \Delta^{-1}H \rangle_{\frac{n}{2}},$$

(3.41)

where $\langle \Delta^{-1}H \rangle_{\frac{n}{2}}$ is a function of $x'$.

By using (3.14) and (3.15), we see that $\Psi(t;x,x')$ has an asymptotic expansion as $t \to 0$ and is given by

$$\Psi(t;x,x') \sim \sum_{k=0}^{\infty} t^k \psi_k(x,x'),$$

(3.42)

where, in general,

$$\psi_k(x,x') = \frac{(-1)^k}{k!} \Delta^{\frac{1}{2}}(x,x') P(x,x') a_k(x,x').$$

(3.43)

Thus $Z$, $V$, $W$, and $S$ all have asymptotic expansions in non-negative integer powers of $t$ since $\Psi$ can be expanded in non-negative integer powers of $t$. Therefore we can expand $H$ (3.29) in integer powers of $t$ greater than $-1$. Thus we get

$$H(t;x,x') \sim \Delta(x,x') \sum_{k=-1}^{\infty} t^k h_k(x,x'),$$

(3.44)

where $\Delta(x,x')$ was introduced for convenience. The sum begins from $k = -1$ since there is a $\frac{1}{t}$ term in $H$ which can be seen in the $S$ term in the det $Y$ (3.27). Then

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{(4\pi)^{n^2}} \left( \frac{\pi}{2n} \right)^{\frac{n}{2}} \sum_{k=-1}^{\infty} t^k \int_{\mathcal{M}} d\text{vol}(x') \langle h_k(x,x') \rangle_{\frac{n}{2}}.$$
If we define
\[ H_k = \int_M d\text{vol}(x') \langle h_k \rangle^n , \] (3.46)
then we have
\[ K(t) \sim t^{-n(n+\frac{1}{2})} \left( \frac{\pi}{4n} \right)^{\frac{n}{2}} \sum_{k=-1}^{\infty} t^k H_k. \] (3.47)

Next, we want to calculate \( \langle h_k \rangle^n \). We can expand \( h_k(x, x') \) in a covariant Taylor series about the point \( x' \). This gives us
\[ h_k(x, x') = \sum_{m=0}^{\infty} \sigma^{\mu_1} \cdots \sigma^{\mu_m} h_{k,\mu_1 \cdots \mu_m}(x'), \] (3.48)
where
\[ h_{k,\mu_1 \cdots \mu_m}(x') := \left( \frac{-1}{m!} \right)^m \left[ \nabla_{(\mu_1} \cdots \nabla_{\mu_m)} h_k(x, x') \right]. \] (3.49)

Then we have
\[ \langle h_k \rangle^n = \sum_{m=0}^{\infty} t^m \left( \frac{2^{m}}{m!} \right)^n g^{(\mu_1 \mu_2 \cdots \mu_{2m})} h_{k,\mu_1 \cdots \mu_{2m}}. \] (3.50)

Evaluating the Gaussian integral and relabeling indices, we obtain
\[ \langle h_k \rangle^n = \sum_{m=0}^{\infty} t^m \left( \frac{2^{m}}{m!} \right)^n g^{(\mu_1 \mu_2 \cdots \mu_{2m-1} \mu_{2m})} h_{k,\mu_1 \cdots \mu_{2m}}. \] (3.51)

Let us define
\[ h_{k,m}(x') = \frac{(2m)!}{(2n)^m m!} g^{(\mu_1 \mu_2 \cdots \mu_{2m-1} \mu_{2m})} h_{k,\mu_1 \cdots \mu_{2m}}. \] (3.52)

This gives us
\[ \langle h_k \rangle^n = \sum_{m=0}^{\infty} t^m h_{k,m}. \] (3.53)

If we define
\[ H_{k,m} = \int_M d\text{vol}(x') h_{k,m}(x'), \] (3.54)
then
\[ H_k = \sum_{m=0}^{\infty} t^m H_{k,m}. \] (3.55)
When we plug this all into (3.47), we get

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{(4\pi)^{n^2}} \left(\frac{\pi}{2n}\right)^{\frac{n}{2}} \sum_{k=-1}^{\infty} \sum_{m=0}^{\infty} t^{k+m} H_{k,m}.$$  

(3.56)

When we combine powers of $t$ and re-index the sum, we find

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{(4\pi)^{n^2}} \left(\frac{\pi}{2n}\right)^{\frac{n}{2}} \sum_{k=-1}^{\infty} t^k B_k,$$  

(3.57)

where

$$B_k = \sum_{m=-1}^{k} H_{m,k-m}.$$  

(3.58)

Note that we can express

$$B_k = \int_M d\text{vol}(x') b_k(x'),$$  

(3.59)

where

$$b_k(x') = \sum_{m=-1}^{k} h_{m,k-m}(x').$$  

(3.60)

It is the coefficients $B_k$ that we wish to calculate. In summary, to calculate $B_k$, we need to find $b_k$ (3.60) or $H_{k,m}$ (3.54), which are found from $h_{k,m}$ (3.52). To find $h_{k,m}$, we compute the covariant Taylor expansion coefficients $h_{k,\mu_1\cdots\mu_m}$ (3.49) of $h_k$ (3.44). The coefficients $h_k$ are calculated from $H$ (3.29) which is calculated from $Y_{\mu'\nu'}$ (3.19). To compute $Y_{\mu'\nu'}$, we only need the heat kernel and the Synge function (2.49).

### 3.3 Heat Determinant Asymptotics for Scalar Operators

For scalar operators, we do not have a trace in (3.1) (i.e. $\text{tr } \Psi^*\Psi = \Psi^2$). Therefore (3.20) - (3.23) simplify to

$$S_{\mu'\nu'} = \sigma_{\mu'\nu'}\Psi^2,$$  

(3.61)

$$V_{\mu'\nu'} = -\sigma_{\mu'\nu'}\Psi\Psi,$$  

(3.62)

$$W_{\mu'\nu'} = -\sigma_{\nu'}\Psi\Psi_{\mu'},$$  

(3.63)

$$Z_{\mu'\nu'} = -\sigma_{\mu'\nu'}\Psi^2 + 2t\Psi\Psi_{\mu'\nu'}.$$  

(3.64)
Since we have expanded the heat determinant in powers of $t$, let us also expand $\det Y$ (thus $H$) in powers of $t$. First we need to expand $S$, $V$, $W$, and $Z$ in powers of $t$. Thus from (3.61) - (3.64) and (3.42), we have

$$S_{\mu\nu'} = \sigma_{\mu} \sigma_{\nu'} \psi_{0}^{2} + 2 \sigma_{\mu} \sigma_{\nu'} \psi_{1} \psi_{0} t + \sigma_{\mu} \sigma_{\nu'} \left(2 \psi_{2} \psi_{0} + \psi_{1}^{2}\right) t^{2} + \cdots,$$

and

$$V_{\mu\nu'} = - \sigma_{\mu} \psi_{0} \psi_{0,\nu'} - \sigma_{\mu} \left(\psi_{1,\nu'} \psi_{0} + \psi_{0,\nu'} \psi_{1}\right) t - \sigma_{\mu} \left(\psi_{2,\nu'} \psi_{0} + \psi_{1,\nu'} \psi_{1}\right) t^{2} + \cdots,$$

$$W_{\mu\nu'} = - \sigma_{\nu'} \psi_{0} \psi_{0,\mu} - \sigma_{\nu'} \left(\psi_{1,\mu} \psi_{0} + \psi_{0,\mu} \psi_{1}\right) t - \sigma_{\nu'} \left(\psi_{2,\mu} \psi_{0} + \psi_{1,\mu} \psi_{1}\right) t^{2} + \cdots,$$

and

$$Z_{\mu\nu'} = - \sigma_{\mu\nu'} \psi_{0}^{2} + 2 \psi_{0} \left(\psi_{0,\mu\nu'} - \sigma_{\mu\nu'} \psi_{1}\right) t + \left(2 \psi_{1,\nu'} \psi_{0} + \psi_{0,\mu\nu'} \psi_{1}\right) - \sigma_{\mu\nu'} \left(2 \psi_{2} \psi_{0} + \psi_{1}^{2}\right) t^{2} + \left(2 \psi_{2,\nu'} \psi_{0} + \psi_{1,\nu'} \psi_{1} + \psi_{0,\mu\nu'} \psi_{2}\right) - \sigma_{\mu\nu'} \left(2 \psi_{3} \psi_{0} + 2 \psi_{2} \psi_{1}\right) t^{3} + \cdots,$$

where each additional index denotes the covariant derivative with respect to $x$ and $x'$ respectively. Thus

$$\psi_{k,\cdots,\mu} = \nabla_{\mu} \psi_{k,\cdots}$$

and

$$\psi_{k,\cdots,\nu'} = \nabla_{\nu'} \psi_{k,\cdots}$$

where $\psi_{k,\cdots}$ is $\psi_{k}$ with an arbitrary number of indices.

Expanding $H$ (3.29) in powers of $t$, we have the first three terms in (3.44):

$$h_{-1} = \Delta^{-1} \frac{1}{n!} \mathcal{E}_{\mu_{1}\cdots\mu_{n}} \mathcal{E}_{\nu_{1}'\cdots\nu_{n}'} \left\{ \frac{n}{2} (-1)^{n-1} \sigma_{\mu_{1}} \sigma_{\nu_{1}'} \sigma_{\mu_{2}} \sigma_{\nu_{2}'} \cdots \sigma_{\mu_{n}} \sigma_{\nu_{n}'} \psi_{0}^{2n} \right\},$$

$$h_{0} = \Delta^{-1} \frac{1}{n!} \mathcal{E}_{\mu_{1}\cdots\mu_{n}} \mathcal{E}_{\nu_{1}'\cdots\nu_{n}'} \left\{ (-1)^{n} \sigma_{\mu_{1} \nu_{1}'} \cdots \sigma_{\mu_{n} \nu_{n}'} \psi_{0}^{2n} + n (-1)^{n} \left(\sigma_{\mu_{1} \nu_{0} \nu_{1}'} + \sigma_{\nu_{1} \psi_{0} \nu_{1}}\right) \sigma_{\mu_{2}} \sigma_{\nu_{2}'} \cdots \sigma_{\mu_{n} \nu_{n}'} \psi_{0}^{2n-1} \right\} + \frac{n}{2} (-1)^{n-1} \sigma_{\mu_{1}} \sigma_{\nu_{1}'} \sigma_{\mu_{2} \nu_{2}'} \cdots \sigma_{\mu_{n} \nu_{n}'} 2 \psi_{1} \psi_{0}^{2n-1}.$$
We note that

\[ h_1 = \Delta^{-1} \frac{1}{n!} \mathcal{E}^{\mu_1 \cdots \mu_n} \mathcal{E}^{\nu_1 \cdots \nu_n} \left\{ \frac{n(n-1)}{2} \right. \]

\[ \left. \left( -1 \right)^{n-2} \sigma_{\mu_1 \nu_1} \left( \psi_{0,\mu_2 \nu_2} - \sigma_{\mu_2 \nu_2} \psi_1 \right) \sigma_{\mu_3 \nu_3} \cdots \sigma_{\mu_n \nu_n} 2\psi_0^{2n-1} \right. \]

\[ + n(n-1) \left( -1 \right)^n \sigma_{\mu_1 \nu_1} \psi_{0,\mu_1} \sigma_{\mu_2 \nu_2} \psi_{0,\mu_2} \sigma_{\mu_3 \nu_3} \cdots \sigma_{\mu_n \nu_n} \psi_0^{2n-2} \right\}, \]

\[ + \frac{n(n-1)}{2} \left( -1 \right)^{n-2} \sigma_{\mu_1 \nu_1} \left( \psi_{0,\mu_2 \nu_2} - \sigma_{\mu_2 \nu_2} \psi_1 \right) \sigma_{\mu_3 \nu_3} 4\psi_1 \psi_0 \]

\[ + \frac{n(n-1)}{2} \left( -1 \right)^{n-2} \sigma_{\mu_1 \nu_1} \sigma_{\mu_2 \nu_2} \left( \psi_{0,\mu_3 \nu_3} - \sigma_{\mu_3 \nu_3} \psi_0 \right) \]

\[ + \frac{n(n-1)}{2} \left( -1 \right)^{n-2} \sigma_{\mu_1 \nu_1} \sigma_{\mu_2 \nu_2} \left( \psi_{0,\mu_3 \nu_3} - \sigma_{\mu_3 \nu_3} \psi_0 \right) \]

\[ + \frac{n(n-1)(n-2)}{4} \left( -1 \right)^{n-3} \sigma_{\mu_1 \nu_1} \sigma_{\mu_2 \nu_2} \left( \psi_{0,\mu_3 \nu_3} - \sigma_{\mu_3 \nu_3} \psi_0 \right) \]

\[ + \frac{n(n-1)(n-2)}{4} \left( -1 \right)^{n-3} \sigma_{\mu_1 \nu_1} \sigma_{\mu_2 \nu_2} \psi_1 \left( \psi_{0,\mu_3 \nu_3} - \sigma_{\mu_3 \nu_3} \psi_0 \right) \]

\[ + n(n-1) \left( -1 \right)^n \sigma_{\mu_1 \nu_1} \psi_{0,\mu_1} \sigma_{\mu_2 \nu_2} \psi_{0,\mu_2} \sigma_{\mu_3 \nu_3} \]

\[ + n(n-1) \left( -1 \right)^n \sigma_{\mu_1 \nu_1} \psi_{0,\mu_1} \sigma_{\mu_2 \nu_2} \left( \psi_{0,\mu_3 \nu_3} - \sigma_{\mu_3 \nu_3} \psi_1 \right) \]

\[ \times \sigma_{\mu_4 \nu_4} \cdots \sigma_{\mu_n \nu_n} \psi_0^{2n-3}. \]  

(3.73)

For simplification, we define a matrix $F$ given by

\[ F^{\mu \nu} = \Delta^{-1} \left( -1 \right)^{n-1} \frac{1}{(n-1)!} \mathcal{E}^{\mu \alpha_1 \cdots \alpha_{n-1}} \mathcal{E}^{\nu \beta_1 \cdots \beta_{n-1}} \sigma_{\alpha_1 \beta_1} \cdots \sigma_{\alpha_{n-1} \beta_{n-1}}. \]  

(3.74)

Then by using Lemma 1 and letting $A_{\mu \nu} = -\sigma_{\mu \nu}$ and $B^{\mu \nu} = -\gamma^{\mu \nu}$, then

\[ F^{\mu \nu} = -\gamma^{\mu \nu}. \]  

(3.75)

We note that

\[ \sigma_{\mu \nu} F^{\mu \nu} = -2\sigma. \]  

(3.76)
Introducing
\[ F^{\mu_1\mu_2\nu_1\nu_2} = \Delta^{-1}(-1)^{n-2}(n-2)! \mathcal{E}_{12} \cdots \mathcal{E}_{n-1}^{a_{n-2}} \mathcal{E}_{\alpha_1^{a_1} \beta_1}^{a_{n-2}} \sigma_{\alpha_1^{a_1} \beta_1} \cdots \sigma_{a_{n-2}^{a_{n-2}}} \cdot \quad (3.77) \]

Then by using Lemma 1 and letting \( A_{\mu \nu} = -\sigma_{\mu \nu} \) and \( B^{\mu \nu} = -\gamma^{\mu \nu} \), then
\[ F^{\mu_1\mu_2\nu_1\nu_2} = \gamma^{\lambda_1 \nu_1} \gamma^{\lambda_2 \nu_2} \delta_{\lambda_1 \lambda_2} = \gamma^{\mu_1 \nu_1} \gamma^{\mu_2 \nu_2} - \gamma^{\mu_1 \nu_2} \gamma^{\mu_2 \nu_1}. \quad (3.78) \]

We note that
\[ \sigma_\mu \sigma_\nu F^{a_1 b_1} = 2 \sigma \gamma^{a b} - \sigma \gamma^{ab}, \quad (3.79) \]
\[ -\sigma_\mu F^{a_1 b_1} = (n-1) F^{a_1 b_1} = -(n-1) \sigma \gamma^{ab}, \quad (3.80) \]

and
\[ \sigma_\mu \sigma_\nu F^{a_1 b_1} = 2(n-1) \sigma. \quad (3.81) \]

Lastly, we introduce
\[ F^{\mu_1\mu_2\nu_1\nu_2} = \Delta^{-1}(-1)^{n-3}(n-3)! \mathcal{E}_{12} \cdots \mathcal{E}_{n-1}^{a_{n-3}} \mathcal{E}_{\alpha_1^{a_1} \beta_1}^{a_{n-3}} \sigma_{\alpha_1^{a_1} \beta_1} \cdots \sigma_{a_{n-3}^{a_{n-3}}} \cdot \quad (3.82) \]

Then by using Lemma 1 and letting \( A_{\mu \nu} = -\sigma_{\mu \nu} \) and \( B^{\mu \nu} = -\gamma^{\mu \nu} \), then
\[ F^{\mu_1\mu_2\nu_1\nu_2} = -\gamma^{\lambda_1 \nu_1} \gamma^{\lambda_2 \nu_2} \gamma^{\lambda_3 \nu_3} \delta_{\lambda_1 \lambda_2 \lambda_3} = -3 \gamma^{[\mu_1 [\nu_1 [\mu_2 [\nu_2 [\gamma \nu_3] \nu_3] \nu_3] \nu_3] \nu_3. \quad (3.83) \]

We note that
\[ -\sigma_\mu F^{a_1 a_2 b_1 b_2} = (n-2) F^{a_1 a_2 b_1 b_2}. \quad (3.84) \]

If we let
\[ G^{a_1 a_2 b_1 b_2} = \sigma_\mu \sigma_\nu F^{a_1 a_2 b_1 b_2}, \quad (3.85) \]

then
\[ G^{a_1 a_2 b_1 b_2} = -2 \sigma \gamma^{a_1 b_1} \gamma^{a_2 b_2} + 2 \sigma \gamma^{a_1 b_1} \gamma^{a_2 b_2} - \sigma^{b_1} \gamma^{a_2 \gamma^{a_1 b_2} - \sigma^{b_2} \gamma^{a_1 \gamma^{a_2 b_2}} + \sigma^{b_2} \gamma^{a_2 \gamma^{a_1 b_1} + \sigma^{a_1} \gamma^{a_2 b_2}}. \quad (3.86) \]

We will now simplify (3.71) - (3.73). We find that
\[ h_{-1} = \frac{1}{2} \sigma_\mu \sigma_\nu F^{\mu \nu} \psi_0^{2n}. \quad (3.87) \]

Hence, using (3.76),
\[ h_{-1} = -\sigma \psi_0^{2n}. \quad (3.88) \]
Simplifying $h_0$, we have

$$
h_0 = \psi_0^{2n} - (\sigma_\mu \psi_{0,\nu'} + \sigma_{\nu'} \psi_{0,\mu}) F^{\mu\nu'} \psi_0^{2n-1} + \sigma_\mu \sigma_{\nu'} F^{\mu\nu'} \psi_1 \psi_0^{2n-1} + \sigma_{\nu'} \psi_{0,\mu} F^{\mu\nu'} \psi_0^{2n-1}
$$

Plugging in (3.75), (3.76), (3.80), and (3.79), we have

$$
h_0 = \psi_0^{2n} + (\sigma_\mu \psi_{0,\nu'} + \sigma_{\nu'} \psi_{0,\mu}) \gamma^{\mu\nu'} \psi_0^{2n-1} - 2 \sigma_\nu \psi_1 \psi_0^{2n-1}
$$

Thus, we finally have

$$
h_0 = \psi_0^{2n} + (\sigma_\nu \psi_{0,\nu'} + \sigma_{\nu'} \psi_{0,\mu}) \gamma^{\mu\nu'} \psi_0^{2n-1} - 2 \sigma_\nu \psi_1 \psi_0^{2n-1}
$$

Simplifying $h_1$, we have

$$
h_1 = \left( \psi_{0,\mu\nu'} - \sigma_{\mu\nu'} \psi_1 \right) F^{\mu\nu'} \psi_0^{2n-1} - (\sigma_\nu \psi_{1,\mu} \psi_0 + \psi_{0,\mu} \psi_1) F^{\mu\nu'} \psi_0^{2n-2} - (\sigma_\nu \psi_{1,\mu} \psi_0 + \psi_{0,\mu} \psi_1) F^{\mu\nu'} \psi_0^{2n-2}
$$

(3.89)
Plugging in (3.75), (3.76), (3.78), (3.79), (3.80), and (3.84), we find

\[ h_1 = \left\{ \begin{array}{c}
(\sigma_{\mu'\nu'}\psi_1 - \psi_{0,\mu'\nu'}) \gamma^{\mu'\nu'} 2\psi_0^2 \\
+ (\sigma_{\nu'}(\psi_{1,\mu}\psi_0 + \psi_{0,\mu}\psi_1) + \sigma_{\mu}(\psi_{1,\nu'}\psi_0 + \psi_{0,\nu'}\psi_1)) \gamma^{\mu'\nu'} \psi_0 \\
+ (\sigma_{\mu}\psi_{0,\nu'} + \sigma_{\nu'}\psi_{0,\mu}) \gamma^{\mu'\nu'} 2(n - 1) \psi_1 \psi_0 \\
- (\sigma_{\mu_1}\psi_{0,\nu_1} + \sigma_{\nu_1}\psi_{0,\mu_1}) \psi_{0,\mu_2\nu_2} \left( \gamma^{\mu_1\nu_1}\gamma^{\mu_2\nu_2} - \gamma^{\mu_2\nu_2}\gamma^{\mu_1\nu_1} \right) 2\psi_0 \\
- \sigma \left( 2\psi_2 \psi_0 + \psi_1^2 \right) \psi_0 \\
+ \psi_{0,\mu'\nu'} \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) 2\psi_1 \psi_0 \\
- 4(n - 1) \sigma \psi_1^2 \psi_0 \\
+ (\psi_{1,\mu'\nu'}\psi_0 + \psi_{0,\mu'\nu'}\psi_1) \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) \psi_0 \\
- (n - 1) \sigma \left( 2\psi_2 \psi_0 + \psi_1^2 \right) \psi_0 \\
+ \sigma_{\mu_1}\sigma_{\nu_1}\psi_{0,\mu_2\nu_2} \psi_{0,\mu_3\nu_3} F_{\mu_1\mu_2\nu_1\nu_2\nu_3} \psi_0 \\
+ 2(n - 2) \psi_{0,\mu'\nu'} \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) \psi_1 \psi_0 \\
- 2(n - 1)(n - 2) \sigma \psi_1^2 \psi_0 \\
- (\psi_{1,\mu}\psi_0 + \psi_{0,\mu}\psi_1) \psi_{0,\nu'} \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) \\
- \psi_{0,\mu} (\psi_{1,\nu'}\psi_0 + \psi_{0,\nu'}\psi_1) \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) \\
- \psi_{0,\mu} \psi_{0,\nu'} \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) 2(n - 2) \psi_1 \\
- 2\sigma_{\mu_1}\sigma_{\nu_1}\psi_{0,\mu_2\nu_2} \psi_{0,\mu_3\nu_3} F_{\mu_1\mu_2\nu_1\nu_2\nu_3} \psi_0 \\
\right\} \psi_0^{2n-3}.
\]

Then simplifying further, we obtain

\[ h_1 = \left\{ \begin{array}{c}
2(n (\psi_1 - \sigma \psi_2)) \psi_0^2 - 2\psi_{0,\mu'\nu'} \gamma^{\mu'\nu'} \psi_0^2 \\
+ \psi_{1,\mu'\nu'} \left( 2\sigma \gamma^{\mu'\nu'} - \sigma^\mu \sigma^{\nu'} \right) \psi_0^2 \\
+ (\sigma^\mu (\psi_{1,\mu}\psi_0 + \psi_{0,\mu}\psi_1)) + \sigma^{\nu'} (\psi_{1,\nu'}\psi_0 + \psi_{0,\nu'}\psi_1) \right) \psi_0 \\
+ 2(n - 1) (\sigma^{\nu'} \psi_{0,\nu'} + \sigma^\mu \psi_{0,\mu}) \psi_1 \psi_0 \\
- 2 (\sigma^{\nu'} \psi_{0,\nu'} + \sigma^\mu \psi_{0,\mu}) \psi_{0,\mu'\nu'} \psi_0 \\
\right\} \psi_0^{2n-3}.
\]
3.4 Coefficients of Heat Determinant Asymptotic Expansion

We want to find the coefficients $B_k$ for scalar operators, for which we need the coefficients $b_k$ \((3.60)\). We will be using the coincidence limits derived in Section 5.2 throughout this section. For our first coefficient, we have

$$b_{-1} = h_{-1,0}. \quad (3.95)$$

From \((3.52)\) and \((3.88)\), and the fact that $[\sigma] = 0$,

$$h_{-1,0} = [h_{-1}] = \left[ -\sigma \psi_0^{2n} \right] = 0. \quad (3.96)$$

Thus

$$b_{-1} = 0. \quad (3.97)$$

Next, we want to calculate

$$b_0 = h_{-1,1} + h_{0,0}. \quad (3.98)$$

We will first calculate $h_{0,0}$. From \((3.52)\), we have

$$h_{0,0} = [h_0]. \quad (3.99)$$

From \((3.91)\), and noting the coincidence limits of $\sigma$ and its single derivatives are 0, then

$$h_{0,0} = [h_0] = \left[ \psi_0^{2n} \right] = 1. \quad (3.100)$$

Next, we will calculate $h_{-1,1}$. From equation \((3.52)\),

$$h_{-1,1} = \frac{1}{2n} g^{\mu_1 \mu_2} \left[ \nabla_{(\mu_1} \nabla_{\mu_2)} h_{-1} \right]. \quad (3.101)$$
Plugging in $h_{-1}$ from (3.88), we find
\[ h_{-1,1} = \frac{1}{2n} g^{\mu_1 \mu_2} \left[ \nabla_{\mu_1} \nabla_{\mu_2} \left( -\sigma \psi_0^{2n} \right) \right]. \] (3.102)

We will only have non-zero terms when both derivatives are on $\sigma$, thus since $[\sigma_{\mu \nu}] = g_{\mu \nu}$, we have
\[ h_{-1,1} = -\frac{1}{2}. \] (3.103)

Therefore, we have
\[ b_0 = \frac{1}{2}. \] (3.104)

Next, we want to calculate
\[ b_1 = h_{-1,2} + h_{0,1} + h_{1,0}. \] (3.105)

First, we will look at $h_{1,0}$. From (3.52),
\[ h_{1,0} = [h_1]. \] (3.106)

Since coincidence limits of $\sigma$ and its first derivatives are 0, from (3.94), we are left with only
\[ h_{1,0} = 2n \left[ \psi_1 + 2 [\psi_0, \mu] \right]. \] (3.107)

Next, looking at $h_{0,1}$, from (3.52),
\[ h_{0,1} = \frac{1}{2n} g^{\mu_1 \mu_2} \left[ \nabla_{\mu_1} \nabla_{\mu_2} h_0 \right]. \] (3.108)

Thus from (3.91),
\[ h_{0,1} = \frac{1}{2n} g^{\mu_1 \mu_2} \left[ \nabla_{\mu_1} \nabla_{\mu_2} \left\{ \psi_0^{2n} + \sigma^{\beta'} \psi_0, \beta' \psi_0^{2n-1} + \sigma^\alpha \psi_0, \alpha \psi_0^{2n-1} - 2n \sigma \psi_1 \psi_0^{2n-1} \right\} \right]. \] (3.109)

For the coincidence limit to not be 0, there must be two derivatives on each $\sigma$ and no derivatives or two on each $\psi_0$. Thus
\[ h_{0,1} = \frac{1}{2n} \left[ 2n \psi_0, \nu \psi_0^{2n-1} + 2 \sigma^{\beta \gamma} \psi_0, \beta \gamma \psi_0^{2n-1} + 2 \sigma^{\alpha \nu} \psi_0, \alpha \psi_0^{2n-1} + 2 \sigma^{\nu} \psi_0, \alpha \beta \gamma \psi_0^{2n-1} - 2 \sigma \psi_0, \nu \psi_0, \alpha \psi_0^{2n-1} - 2n \sigma \psi_1 \psi_0^{2n-1} \right]. \] (3.110)
Now taking the actual coincidence limit,

\[
h_{0,1} = \left(1 + \frac{1}{n}\right) [\psi_0,^v_v] - [\psi_0,^\mu_\mu'] - n[\psi_1]. \tag{3.111}
\]

Lastly, we need \(h_{-1,2}\).

\[
h_{-1,2} = \frac{1}{(2n)^2} 2! \delta^{(\mu_1 \mu_2 \mu_3 \mu_4)} \left[ \nabla_{\mu_1} \nabla_{\mu_2} \nabla_{\mu_3} \nabla_{\mu_4} h_{-1} \right]. \tag{3.112}
\]

Plugging in \(h_{-1}\), we find

\[
h_{-1,2} = -\frac{1}{8n^2} \delta^{(\mu_1 \mu_2 \mu_3 \mu_4)} \left[ \nabla_{\mu_1} \nabla_{\mu_2} \nabla_{\mu_3} \nabla_{\mu_4} \left( \sigma \psi_0^{2n} \right) \right]. \tag{3.113}
\]

The only derivatives that are not zero are the ones with 2 derivatives on \(\sigma\) and 2 on \(\psi_0^{2n}\) since the symmetrization of 4 indices on \(\sigma\) is 0. Thus we have

\[
h_{-1,2} = -\frac{1}{24n^2} \left( \delta^{(\mu_1 \mu_2 \mu_3 \mu_4)} + \delta^{(\mu_1 \mu_3 \mu_2 \mu_4)} + \delta^{(\mu_1 \mu_4 \mu_2 \mu_3)} \right) \left[ \sigma_{\mu_1 \mu_2} \nabla_{\mu_3} \nabla_{\mu_4} \psi_0^{2n} \right]. \tag{3.114}
\]

Taking the derivatives and coincidence limits, we have

\[
h_{-1,2} = -\frac{n + 2}{12n} [\psi_0,^v_v]. \tag{3.115}
\]

Therefore, we find that

\[
b_1 = n[\psi_1] + [\psi_0,^\mu_\mu'] + \left( \frac{11}{12} + \frac{5}{6n} \right) [\psi_0,^v_v]. \tag{3.116}
\]

We have calculated \(b_{-1}, b_0,\) and \(b_1\). Plugging in the coincidence limits from Section 5.2, we have

\[
b_{-1} = 0, \tag{3.117}
\]

\[
b_0 = \frac{1}{2}, \tag{3.118}
\]

\[
b_1 = \frac{1}{72n} \left( 12n^2 - n + 10 \right) R - nQ. \tag{3.119}
\]

Therefore, from (3.59), we obtain the first 3 global invariants

\[
B_{-1} = 0, \tag{3.120}
\]

\[
B_0 = \frac{1}{2} \text{vol}(\mathcal{M}), \tag{3.121}
\]

\[
B_1 = \int_{\mathcal{M}} d\text{vol} \left( \frac{1}{72n} \left( 12n^2 - n + 10 \right) R - nQ \right). \tag{3.122}
\]
CHAPTER 4

CONCLUSION

We have studied the asymptotic expansion of a new heat invariant, the heat determinant. The heat determinant seems like the next logical step in the study of heat invariants on a manifold. We have calculated the first three coefficients in the asymptotic $t$-expansion, thus we have

$$K(t) \sim \frac{t^{-n(n+\frac{1}{2})}}{(4\pi)^{n^2}} \left( \frac{\pi}{2n} \right)^{\frac{n}{2}} \left\{ B_{-1}t^{-1} + B_0 + B_1t + \cdots \right\}.$$  \hspace{1cm} (4.1)

We note that the first term is zero, the second term depends on the volume of the manifold, and the the third term depends on the dimension and the integral of the curvature over the entire manifold.

For future work, obviously, we can calculate more coefficients in our expansion, however they do get much more complicated rather quickly. The most important work to be done is to see what this new invariant can tell us about these manifolds, whether we can distinguish between any two manifolds, or if not, how much information about the manifold we can extract from the asymptotic expansion. One could also do more work with the modified heat determinant (2.103), more specifically, looking at manifolds with a boundary and seeing what kind of information we can extract from this other invariant. One could also study the diagonal of the heat kernel given by

$$K_{\text{diag}}(t) := \int_{\mathcal{M}} dx \det P_{\mu \nu}(t, x, x).$$  \hspace{1cm} (4.2)

This invariant contains much less information than the actual heat determinant, however, it is much easier to calculate higher $t$ terms in our asymptotic expansion.

The heat determinant is a first step in the study of non-spectral heat invariants. There is still a lot of work to be done in this area.
CHAPTER 5

APPENDIX

5.1 Gaussian Integrals

For a real positive symmetric matrix $A = (A_{ij})$, then for any vector $B = (B_i)$, we have

$$
\int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle + \langle B, x \rangle \right) = \pi^{\frac{n}{2}} (\det A)^{-\frac{1}{2}} \exp \left( \frac{1}{4} \langle B, A^{-1} B \rangle \right),
$$

(5.1)

where $A^{-1} = (A_{ij})$ is the inverse of $A$. Now we shall differentiate the above equation with respect to $B_i$ on both sides, for which we obtain

$$
\int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle + \langle B, x \rangle \right) x^i = \pi^{\frac{n}{2}} \left( \frac{1}{2} A_{ik} B_k \right) \exp \left( \frac{1}{4} \langle B, A^{-1} B \rangle \right).
$$

(5.2)

If we differentiate again with respect to $B_j$, we obtain

$$
\int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle + \langle B, x \rangle \right) x^i x^j = \pi^{\frac{n}{2}} \left( \frac{1}{2} A_{ij} + \frac{1}{2} A_{ik} B_k A_{jl} B_l \right) \exp \left( \frac{1}{4} \langle B, A^{-1} B \rangle \right).
$$

(5.3)

If we continue this process and set $B = 0$, we get the coefficients to the Taylor expansion of (5.1) in $B$, and we find that

$$
\int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle \right) x^{i_1} \ldots x^{i_{2k+1}} = 0
$$

(5.4)

and

$$
\int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle \right) x^{i_1} \ldots x^{i_{2k}} = \pi^{\frac{n}{2}} (\det A)^{-\frac{1}{2}} \frac{(2k)!}{2^{2k} k!} A^{(i_1 i_2 \ldots i_{2k-1} i_{2k})}.
$$

(5.5)
We define the Gaussian average as
\[
\langle f(x) \rangle = \pi^{-\frac{n}{2}} (\det A)^{\frac{1}{2}} \int_{\mathbb{R}^n} dx \exp \left( -\langle x, Ax \rangle \right) f(x).
\] (5.6)

Thus we have
\[
\langle x^{i_1} \cdots x^{i_{2k+1}} \rangle = 0,
\] (5.7)
and
\[
\langle x^{i_1} \cdots x^{i_{2k}} \rangle = \frac{(2k)!}{2^{2k}k!} A^{(i_1 i_2 \cdots A^{i_{2k-1} i_{2k})}.}
\] (5.8)

5.2 Evaluation of Coincidence Limits

The Synge function is defined in section 2.1.7. For more on the Synge function, see [16]. Here, we will evaluate the coincidence limits of the Synge function and its derivatives. We have
\[
\sigma = \frac{1}{2} \sigma_{\mu} \sigma^{\mu} = \frac{1}{2} \sigma_{\nu} \sigma^{\nu}.
\] (5.9)

From this, and noting that \([\sigma] = 0\), we find that
\[
[\sigma_{\mu}] = 0.
\] (5.10)

By differentiating (5.9), we get
\[
\sigma_{\mu} = \nabla_{\mu} \left( \frac{1}{2} \sigma_{\alpha} \sigma^{\alpha} \right) = \sigma_{\alpha \mu} \sigma^{\alpha},
\] (5.11)
\[
\sigma_{\mu}^{\nu} = \nabla_{\nu} \left( \frac{1}{2} \sigma_{\alpha} \sigma^{\alpha} \right) = \sigma_{\alpha \mu} \sigma^{\alpha}.
\] (5.12)

Before taking more derivatives, we shall find the coincidence limit of \(\sigma_{\mu \nu}\). From the definition of \(\sigma\) in (2.49), we have
\[
\sigma = \frac{1}{2} t \int_0^t d\tau g_{\mu \nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}.
\] (5.13)

Taking a covariant derivative, we have
\[
\sigma_{\alpha} = t \int_0^t d\tau g_{\mu \nu} \frac{dx^\mu}{d\tau} \nabla_\alpha \frac{dx^\nu}{d\tau}.
\] (5.14)
Evaluating the covariant derivative, and integrating by parts, we have

\[
\sigma_\alpha = t \int_0^t \frac{d\tau}{d\tau} g_{\mu\nu} \left( \frac{dx^\mu}{d\tau} + \Gamma_{\gamma\alpha}^\mu \frac{dx^\gamma}{d\tau} \right)
\]

\[
= t g_{\mu\alpha} \frac{dx^\mu}{d\tau} \delta_\alpha^\mu - t \int_0^t \frac{d\tau}{d\tau} g_{\mu\nu} \left( \delta_\alpha^\mu \frac{d^2x^\nu}{d\tau^2} - \Gamma_{\gamma\alpha}^\mu \frac{dx^\gamma}{d\tau} \frac{dx^\mu}{d\tau} \right)
\]

\[
= t U_\alpha - t \int_0^t \frac{d\tau}{d\tau} \left( \frac{d^2x^\alpha}{d\tau^2} - \Gamma_{\gamma\alpha}^\mu \frac{dx^\gamma}{d\tau} \frac{dx^\mu}{d\tau} \right). \tag{5.15}
\]

where \(U_\alpha = \frac{dx_\alpha}{d\tau}\). By definition, \(U_\alpha\) is the vector tangent to the geodesic connecting \(x\) and \(x'\). We note that the second term in the above equation is just the integral of (2.46) multiplied by \(g_{\mu\alpha}\). Thus we have

\[
\sigma_\alpha = t U_\alpha. \tag{5.16}
\]

Thus we have from (5.11),

\[
U_\mu = U_\alpha \sigma^\alpha_\mu. \tag{5.17}
\]

When we take the coincidence limit, we want the limit to not depend on direction, therefore

\[
[\sigma_{\mu\nu}] = \delta_\nu^\mu \tag{5.18}
\]

and

\[
[\sigma_{\mu\nu}] = g_{\mu\nu}. \tag{5.19}
\]

Now differentiating again (5.11) and (5.12), we have

\[
\sigma_{\mu\nu} = \nabla_\nu \left( \sigma_{\alpha\mu} \sigma^\alpha_\nu \right) = \sigma_{\alpha\mu\nu} \sigma^\alpha_\nu + \sigma_{\alpha\mu} \sigma^\alpha_\nu, \tag{5.20}
\]

\[
\sigma_{\mu'\nu'} = \nabla_\nu \left( \sigma_{\alpha\mu'} \sigma^\alpha_{\nu'} \right) = \sigma_{\alpha\mu'} \sigma^\alpha_{\nu'} + \sigma_{\alpha\mu} \sigma^\alpha_{\nu'}. \tag{5.21}
\]

We note that the derivatives with respect to \(x\) always commute with derivatives with respect to \(x'\). We also note that we can always commute the first two derivatives since \(\sigma\) is a scalar. Taking a derivative with respect to \(x\) of (5.20), we have

\[
\sigma_{\mu\nu\beta} = \sigma_{\alpha\mu\nu} \sigma^\alpha_\beta + \sigma_{\alpha\mu\nu} \sigma^\alpha_\beta + \sigma_{\alpha\mu} \sigma^\alpha_\nu + \sigma_{\alpha\mu} \sigma^\alpha_\nu. \tag{5.22}
\]

Taking the coincidence limit, we have

\[
[\sigma_{\nu\mu\beta}] = - [\sigma_{\beta\mu\nu}]. \tag{5.23}
\]

Since \(\sigma\) is symmetric in the first two indices, we have from (5.23),

\[
[\sigma_{\mu\nu\beta}] = 0. \tag{5.24}
\]
We also note that by commuting the last two indices in $\sigma_{\mu\nu\beta}$, we have
\[ \sigma_{\mu\nu\beta} - \sigma_{\mu\beta\nu} = R^\alpha_{\mu\nu\beta} \sigma_\alpha. \] (5.25)

Taking another derivative of (5.22), we have
\[ \sigma_{\mu\nu\beta\gamma} = \sigma_{\kappa\mu\nu\beta\gamma} \sigma^\kappa + \sigma_{\kappa\mu\nu\beta} \sigma^\kappa \gamma + \sigma_{\kappa\mu\nu\gamma} \sigma^\kappa \beta + \sigma_{\kappa\mu\nu} \sigma^\kappa \beta \gamma \\
+ \sigma_{\kappa\mu\beta\gamma} \sigma^\kappa \nu + \sigma_{\kappa\mu\beta} \sigma^\kappa \nu \gamma + \sigma_{\kappa\mu\gamma} \sigma^\kappa \nu \beta + \sigma_{\kappa\mu} \sigma^\kappa \nu \beta \gamma. \] (5.26)

Taking the coincidence limit gives us
\[ [\sigma_{\gamma\mu\nu\beta}] + [\sigma_{\beta\mu\nu\gamma}] + [\sigma_{\nu\mu\beta\gamma}] = 0. \] (5.27)

By commutation of derivatives, we have that
\[ \sigma_{\mu\nu\beta\gamma} - \sigma_{\mu\nu\gamma\beta} = R^\alpha_{\mu\beta\gamma} \sigma^\alpha_\nu + R^\alpha_{\nu\beta\gamma} \sigma^\alpha_\mu. \] (5.28)

Due to the symmetry of the Riemann tensor, we have
\[ [\sigma_{\mu\nu\beta\gamma}] = [\sigma_{\mu\nu\gamma\beta}] = [\sigma_{\nu\mu\beta\gamma}]. \] (5.29)

Taking a derivative of (5.25), we have
\[ \sigma_{\mu\nu\beta\gamma} - \sigma_{\mu\beta\nu\gamma} = R^\alpha_{\mu\nu\beta\gamma} \sigma^\alpha_\alpha + R^\alpha_{\mu\nu\beta} \sigma^\alpha \gamma. \] (5.30)

Taking this coincidence limit gives us
\[ [\sigma_{\mu\nu\beta\gamma}] - [\sigma_{\mu\beta\nu\gamma}] = R_{\mu\nu\beta}. \] (5.31)

If we switch $\beta$ and $\gamma$, then add this to the above equation, we find
\[ 2[\sigma_{\mu\nu\beta\gamma}] - [\sigma_{\mu\beta\nu\gamma}] - [\sigma_{\mu\gamma\nu\beta}] = R_{\mu\nu\beta} + R_{\beta\mu\nu\gamma}. \] (5.32)

Using (5.27) and (5.29), the above equations simplifies to
\[ [\sigma_{\mu\nu\beta\gamma}] = -\frac{1}{3} (R_{\mu\nu\beta} + R_{\beta\mu\nu\gamma}). \] (5.33)

To calculate the coincidence limits for primed indices, we just use the formula
\[ [f_{\cdots}]_\alpha = [f_{\cdots\alpha}] + [f_{\cdots\alpha}], \] (5.34)

which is true for any function of $x$ and $x'$. More specifically, we have
\[ \left[ f_{\mu\nu'} \right] = \nabla_v \left[ f_{\mu} \right] - \left[ f_{\mu\nu} \right], \] (5.35)
In summary, we have calculated the coincidence limits of $\sigma$ and its derivatives to be
\begin{align}
[\sigma] &= 0, \quad (5.36) \\
[\sigma_\mu] &= 0, \quad (5.37) \\
[\sigma_\nu] &= 0, \quad (5.38) \\
[\sigma_{\mu\nu}] &= -[\sigma_{\mu\nu'}] = [\sigma_{\mu'\nu}] = g_{\mu\nu}, \quad (5.39) \\
[\sigma_{\mu\nu\alpha}] &= [\sigma_{\mu'\nu\alpha}] = [\sigma_{\mu'\nu'\alpha}] = 0, \quad (5.40) \\
[\sigma_{\mu\nu\beta\gamma}] &= -\frac{1}{3} (R_{\mu\gamma\nu\beta} + R_{\mu\beta\nu\gamma}), \quad (5.41) \\
[\sigma_{\mu\nu\beta'\gamma'}] &= \frac{1}{3} (R_{\mu\gamma\nu\beta'} + R_{\mu\beta'\nu\gamma}), \\
[\sigma_{\mu\nu'\beta'\gamma'}] &= -\frac{1}{3} (R_{\mu'\gamma\nu\beta'} + R_{\mu'\beta'\nu\gamma}), \\
[\sigma_{\mu'\nu'\beta'\gamma'}] &= \frac{1}{3} (R_{\mu'\gamma\nu\beta'} + R_{\mu'\beta'\nu\gamma}).
\end{align}

We will now calculate the coincidence limits of the Van-Fleck determinant and its derivatives. By definition,
\begin{equation}
\Delta = g^{-\frac{1}{2}}(x) g^{-\frac{1}{2}}(x') \det(-\sigma_{\mu'\nu'}). \quad (5.42)
\end{equation}

Thus
\begin{equation}
[\Delta] = 1. \quad (5.43)
\end{equation}

Taking one derivative, we have
\begin{equation}
\Delta_\mu = \frac{1}{(n-1)!} \, (-1)^n \, \varepsilon^{\alpha_1 \cdots \alpha_n} \varepsilon^{\beta_1' \cdots \beta_n'} \, \sigma_{\alpha_1 \beta_1' \mu} \sigma_{\alpha_2 \beta_2'} \cdots \sigma_{\alpha_n \beta_n'}. \quad (5.44)
\end{equation}

Taking the coincidence limit, since there is a third derivative of $\sigma$, we have
\begin{equation}
[\Delta_\mu] = 0. \quad (5.45)
\end{equation}

Taking another derivative of (5.44), we have
\begin{equation}
\Delta_{\mu\nu} = \frac{1}{(n-1)!} \, (-1)^n \, \varepsilon^{\alpha_1 \cdots \alpha_n} \varepsilon^{\beta_1' \cdots \beta_n'} \sigma_{\alpha_1 \beta_1' \mu} \sigma_{\alpha_2 \beta_2'} \cdots \sigma_{\alpha_n \beta_n'} \\
+ \frac{1}{(n-2)!} \, (-1)^{n-2} \, \varepsilon^{\alpha_1 \cdots \alpha_n} \varepsilon^{\beta_1' \cdots \beta_n'} \sigma_{\alpha_1 \beta_1' \mu} \sigma_{\alpha_2 \beta_2'} \sigma_{\alpha_3 \beta_3'} \cdots \sigma_{\alpha_n \beta_n'}. \quad (5.46)
\end{equation}
Using (3.75) and (3.78), we have

$$\Delta_{\mu\nu} = \sigma_{\alpha\beta\mu}^{\prime} \gamma^{\alpha\beta} \Delta + \sigma_{\alpha_1\beta_1}^{\prime} \sigma_{\alpha_2\beta_2}^{\prime} \left( \gamma^{\alpha_1\beta_1} \gamma^{\alpha_2\beta_2} - \gamma^{\alpha_2\beta_1} \gamma^{\alpha_1\beta_2} \right) \Delta. \tag{5.47}$$

Taking the coincidence limit and using (5.41) gives us

$$[\Delta_{\mu\nu}] = -[\sigma_{\alpha\mu}^{\prime}] = \frac{1}{3} R_{\mu\nu}. \tag{5.48}$$

Next, we calculate the coincidence limits of $\psi_k$ and its derivatives. From (3.43),

$$\psi_k = \left( -1 \right)^k \frac{k!}{k!} \Delta^k a_k, \tag{5.49}$$

where $a_k$ can be calculated using the process given in [2]. We will only need the first couple coincidence limits which were originally calculated by [7], [5], and [9] and are given by

$$[a_0] = 1, \tag{5.50}$$
$$[a_1] = Q - \frac{1}{6} R, \tag{5.51}$$

for our operator $L = -\Delta + Q$. Thus we have

$$\psi_0 = \Delta^\frac{1}{2}, \tag{5.53}$$

and from (5.43), we have

$$[\psi_0] = 1. \tag{5.54}$$

Taking a single derivative yields

$$\psi_{0,\mu} = \frac{1}{2} \Delta_{\mu} \Delta^{-\frac{1}{2}}, \tag{5.55}$$

then using (5.45), we have

$$[\psi_{0,\mu}] = 0. \tag{5.56}$$

Taking one more derivative of $\psi_0$, we end up with

$$\psi_{0,\mu\nu} = \frac{1}{2} \Delta_{\mu\nu} \Delta^{-\frac{1}{2}} - \frac{1}{4} \Delta_{\mu} \Delta_{\nu} \Delta^{-\frac{3}{2}}. \tag{5.57}$$

Therefore, using (5.48), we have

$$[\psi_{0,\mu\nu}] = \frac{1}{6} R_{\mu\nu}. \tag{5.58}$$
To find the coincidence limits of the primed derivatives, we use the formula (5.34).

Next,
\[ \psi_1 = -\Delta^\perp a_1. \] (5.59)

Thus
\[ [\psi_1] = -[a_1] = \frac{1}{6} R - Q. \] (5.60)

In summary, we have the coincidence limits
\[
[\Delta] = 1, \quad (5.61) \\
[\Delta_\mu] = [\Delta_{\nu'}] = 0, \quad (5.62) \\
[\Delta_{\mu\nu}] = [\Delta_{\mu'\nu'}] = \frac{1}{3} R_{\mu\nu}, \quad (5.63) \\
[\Delta_{\mu\nu'}] = -\frac{1}{3} R_{\mu\nu}, \quad (5.64) \\
[\psi_0] = 1, \quad (5.65) \\
[\psi_{0,\mu}] = [\psi_{0,\nu'}] = 0, \quad (5.66) \\
[\psi_{0,\mu\nu}] = [\psi_{0,\mu'\nu'}] = \frac{1}{6} R_{\mu\nu}, \quad (5.67) \\
[\psi_{0,\mu\nu'}] = -\frac{1}{6} R_{\mu\nu}, \quad (5.68) \\
[\psi_1] = \frac{1}{6} R - Q. \quad (5.69)
\]
REFERENCES


Calculation of Heat Determinant Coefficients for Scalar Laplace type Operators

by

Benjamin Jerome Buckman

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