Some Discussion on Weibull Statistics MATE445_2009

Experimental observations on the strength of brittle materials such as ceramics suggest that their failure statistics can be often described by Weibull statistics. The statistics goes back to the original paper of Weibull [1], where the basis for the failure equation is derived, and where it is applied to such diverse phenomenon as the survival life of light bulbs, the fatigue life of steels, the strength of ceramics, etc.

The Weibull equation we will be referring to is:

$$P_f = 1 - e^{-\left(\frac{V}{V_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m} \qquad (1)$$

where P_f is the **cumulative** probability of failure, V is the volume of the body under the uniform stress σ, and σ_o is the "reference" strength of the body at the "reference" volume V_o. Thus, the ceramic material will be designated by Weibull shape factor **m** and these two reference parameters, rather than a mean strength and a standard deviation of strength. The cumulative probability of **survival** is:

$$P_s = 1 - P_f = e^{-\left(\frac{V}{V_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m} \qquad (2)$$

The cumulative probability implies that if, say, N_o pieces of a material of identical volume V are stressed uniformly (same everywhere) to a stress of σ, then the estimated number failed will be $N_o{*}P_f$, and the number survived will be $N_o{*}(1-P_f)$. Note that failures can occur at any stress less than this stress σ, but that the **cumulative** number of failures will be P_f times the original number. This is the meaning of cumulative probability.

It is critical that the reference volume be specified when the reference strength σ_o is specified, otherwise the reference strength has no meaning. A material supplier can choose any reference volume when specifying the reference strength. However, often the strength refers to the volume of the material that is used for the laboratory tests. Thus, if tension tests were run on a ceramic volume of, say, 2 ml, then that would be the reference volume. But of course, one can take that data and the m-value, to recalculate a reference strength based on a volume of, say, 0.1 ml, as long as the product $V_o\sigma_o^m$ remained constant; see equation (1).

Physically, a high m-value implies that the probability of failure is either zero or one, depending on whether the stress applied is higher or lower than the reference strength σ_o. This occurs in materials of high reliability, such as metals, where the m-value can be anywhere from 20 to 50. Such materials also exhibit minimal dependence on volume, since the stress term in the exponential expression dominates for high m. At the other extreme, low m-values, such as 5 and below, imply that strengths can vary over a wide range, and that the strength is quite dependent on volume. Green ceramics (not sintered) or inadequately sintered ceramics exhibit low values. Typical structural ceramics such as SiC or Si_3N_4 exhibit m in the range 4 to 10, the latter being quite good. In the case of fibers, typical values lie between 10 and 20. The lower values come about because of poor handling and bundling on reels, when extremely small cracks can be nucleated. Note that it only takes cracks of the order of few tens of nanometers to render a low strength compared to the average strength of the fibers; that is why it is important to maintain good practice during handling of fibers.

Equation (1) has been modified into a number of other forms. For example one can find equations of the form:

$$P_f = 1 - e^{-\left(\frac{V}{V_o}\right)\left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m} \qquad (3)$$

where σ_u is another positive parameter, such that $(\sigma-\sigma_u)$ is zero when the magnitude of stress is less than σ_u. In other words, the probability of failure is zero when the stress is compressive or is less than the stress σ_u. While this form has some physical validity, equation (1) is the more common equation, and is the one we shall adopt for this discussion.

If two volumes V_1 and V_2 were subjected to the same stress σ, then their cumulative survival probability must be the <u>product</u> of the two individual survival probabilities. However, when one multiplies two exponential terms (see equation 2), the power term is added, so that the cumulative probability that both volumes would survive is then $(V=V_1+V_2)$. In other words the volume effect is additive, and it was this important consideration that went into developing the Weibull equation.

If the stress were to vary within a component, then one would have to find the survival probability for each volume separately under different stresses σ_1 and σ_2, and then multiply the individual cumulative survival probabilities to obtain the final total probability of survival.

In the case of fiber reinforced composites, $V = (\pi D^2/4)L$, where D is the diameter of fibers, which remains constant between lab scale and component scale, and L is the length that can change from one sample to another; e.g., from laboratory scale specimens to component level. That is why V/Vo is replaced L/Lo, where L is the actual length of fibers, and Lo is a reference length, often the length of samples used to generate the Weibull statistics; e.g. 50 mm.

$$P_f = 1 - e^{-\left(\frac{L}{L_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m} \qquad (4)$$

Determination of Weibull Parameters

In determining Weibull parameters from laboratory tests, the probability of failure is equated to the expected value, which is itself determined by the ranking of strength with reference to all tests conducted under identical conditions (same volume or length of material). Thus, strengths are ranked in increasing order, starting from the lowest strength. If ‘i’ is the rank among N tests, then the probability of failure (P_f) is equated to i/(N+1), since the ‘i’ samples have broken up to and including stress σ, out of a total of N samples; (N-i) samples have not broken, or have survived stress σ. The rest of the technique relies on taking double logarithm of equation (1) to obtain the linearized form of relationship as follows:

$$P_f = 1 - e^{-\left(\frac{V}{V_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m}$$

$$\frac{1}{1-P_f} = e^{\left(\frac{V}{V_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m}$$

$$Ln\left(\frac{1}{1-P_f}\right) = \left(\frac{V}{V_o}\right)\left(\frac{\sigma}{\sigma_o}\right)^m \tag{5}$$

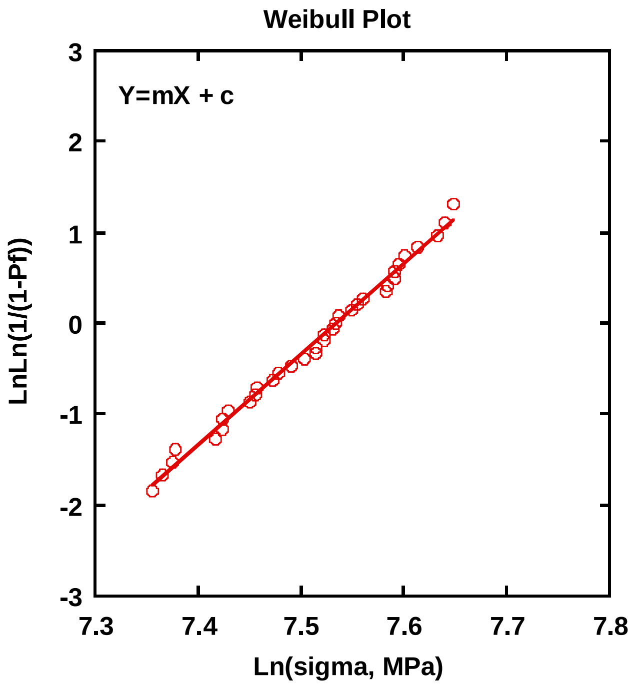
$$Ln\left[Ln\left(\frac{1}{1-P_f}\right)\right] = Ln\left(\frac{V}{V_o}\right) + mLn\sigma - mLn\sigma_o$$

A plot of LnLn{1/(1-P_f)} versus Lnσ would thereby provide a straight line with slope **m**, if the data do indeed follow Weibull statistics. When tests are conducted in the laboratory, the volume tested is equated to V_o, and thus the intercept of the line will simply be mLn(σ_o), and in this way **σ_o** will be obtained corresponding to volume V_o. For fibers, the corresponding terms will be L and Lo. In some materials, there might be two slopes corresponding to two regimes of stresses or boundaries. For example, surface and bulk defects may represent quite different sizes of flaws, and the former may in general be bigger than interior ones. In such cases, it is not uncommon to see low m value at lower strengths, and higher m values at higher strengths.

Typically a Table is made following testing of N samples (typically 20 or higher)

Rank, i	Strength, σ, MPa	Pf = i/(N+1)	LnLn {1/(1-Pf)}	Ln(σ)
1	1250	1/41	-3.70	
2	1301	2/41	..	
3	1325	3/41	..	
..	..		..	
..	..		..	
N=40	1980	40/41	1.31	

The resultant plot will look something like this:



The slope and intercept provide m and σ_o (for the volume V_o), respectively.