Solutions for Practice Questions for Exam 3 **Math 132**

1. Determine whether or not the sequence $\{a_n\}$ converges and find its limit if it does converge.

a. $a_n = \dfrac{8n-7}{7n-8}$

Solution: $\lim_{n\to\infty} \dfrac{8n-7}{7n-8} = \dfrac{8}{7}$, therefore the sequence converges.

b. $a_n = \dfrac{n-e^n}{n+e^n}$

Solution: $\lim_{n\to\infty} \dfrac{n-e^n}{n+e^n} \overset{L}{=} \lim_{n\to\infty} \dfrac{1-e^n}{1+e^n} \overset{L}{=} \lim_{n\to\infty} \dfrac{-e^n}{e^n} = -1$, therefore, the sequence converges.

c. $a_n = \left(1+\dfrac{1}{n}\right)^n$

Solution: $\lim_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e^1$, therefore, the sequence converges

d. $a_n = \dfrac{n^3}{10n^2+1}$

Solution: $\lim_{n\to\infty} \dfrac{n^3}{10n^2+1} = \infty$, therefore the sequence diverges.

2. Find the Taylor Series for

a. $f(x) = \dfrac{1}{(x-4)^2}$ at $a = 5$.

Solution:

$f(x) = (x-4)^{-2}$, $f'(x) = -2(x-4)^{-3}, \ldots, f^{(n)}(x) = (-1)^n (n+1)!(x-4)^{-(n+2)}$

, so $f^{(n)}(5) = (-1)^n (n+1)!$ Thus

$$\frac{1}{(x-4)^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-5)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-5)^n =$$

b. $f(x) = \dfrac{1}{\sqrt{2x}}$ at $a = 2$

Solution:

$f(x) = \sqrt{2x}, f'(x) = -(2x)^{-3/2}, f''(x) = -1(-3)(2x)^{-5/2}, \ldots,$

$$f^{(n)}(x) = (-1)^n \left[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\right](2x)^{-(n+1/2)}$$

so $f^{(n)}(2)=(-1)^n\left[1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\right](4)^{-(n+1/2)}$ Thus

$$\frac{1}{\sqrt{2x}}=\sum_{n=0}^{\infty}\frac{(-1)^n\left[1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\right]}{n!2^{2n+1}}(x-2)^n$$

3. Evaluate the indefinite integral as an infinite series

a. $\int\frac{\sin x}{x}dx$

Solution: $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ thus $\frac{\sin x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$$\int\frac{\sin x}{x}dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}+C$$

b. $\int\frac{e^x-1}{x}dx$

Solution: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, so

$e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ and $\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$ Now

$$\int\frac{e^x-1}{x}dx=\int\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}dx=\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}+C$$

4. Find the sum of the following series:

a. $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$

Solution: This is a geometric series with $a=\frac{e}{\pi}$ and $r=\frac{e}{\pi}$ so

$$S=\frac{e/\pi}{1-e/\pi}=\frac{e}{\pi-e}$$

b. $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

Solution: this is a telescoping series $\frac{2}{n^2-1}=\frac{1}{n-1}-\frac{1}{n+1}$, so

$$S_n=a_2+a_3+a_4+\cdots+a_{n-1}+a_n$$

$$=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$=1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}$$

And $S=\lim_{n\to\infty}S_n=\frac{3}{2}$

c. $\sum_{n=1}^{\infty}\left[\left(\frac{7}{11}\right)^n-\left(\frac{3}{5}\right)^{n+1}\right]$

Solution:

$$\sum_{n=1}^{\infty}\left[\left(\frac{7}{11}\right)^n-\left(\frac{3}{5}\right)^{n+1}\right]=\sum_{n=1}^{\infty}\left(\frac{7}{11}\right)^n-\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n+1}=\sum_{n=1}^{\infty}\frac{7}{11}\left(\frac{7}{11}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^2\left(\frac{3}{5}\right)^{n-1}$$

$$=\frac{7/11}{1-7/11}-\frac{9/25}{1-3/5}=\frac{17}{20}$$

5. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. . Justify your answers by citing relevant tests or reason.

a. $\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{\ln n}$

Solution: $\lim_{n\to\infty}\frac{\sqrt{n}}{\ln n}\overset{L}{=}\lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{2\sqrt{n}}\overset{L}{=}\lim_{n\to\infty}\frac{\sqrt{n}}{2}=\infty$; Therefore,

$\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{\ln n}$ diverges by the nth term test for divergence.

b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$

Solution: $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ is divergent since it is a p-series with $p=1/3<1$.

However, $\frac{1}{n^{1/3}}>0$, $\frac{1}{n^{1/3}}>\frac{1}{(n+1)^{1/3}}$, and $\lim_{n\to\infty}\frac{1}{n^{1/3}}=0$ thus by the Alternating Series Test, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$ is convergent. Now since $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$ is convergent but $\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ diverges so $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$ is conditionally convergent.

c. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$

Solution: $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent since it is a p-series, $p=3>1$

therefore $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ is absolutely convergent.

6. Determine the interval of convergence for the following power series:

a. $\sum_{n=0}^{\infty}\frac{n!x^{2n}}{10^n}$

Solution: Use the ratio test to determine convergence of $\sum|a_n|$.

$\lim_{n\to\infty}\left|\frac{(n+1)!x^{2n+2}}{10^{n+1}}\cdot\frac{10^n}{n!x^{2n}}\right|=\frac{x^2}{10}\lim_{n\to\infty}(n+1)=\infty$ Thus $\sum_{n=0}^{\infty}\frac{n!x^{2n}}{10^n}$ converges only if $x=0$ otherwise it diverges.

b. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n3^n}$

Solution: Use the ratio test to determine convergence of $\sum|a_n|$.

$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-1)^n}\right|=\frac{|x-1|}{3}\lim_{n\to\infty}\frac{n}{n+1}=\frac{|x-1|}{3}$ thus the series converges absolutely if $\frac{|x-1|}{3}<1\Leftrightarrow|x-1|<3\Leftrightarrow-2<x<4$ Check the endpoints $x=-2$ and $x=4$ at $x=-2$ we get $\sum_{n=1}^{\infty}\frac{(-1)^n}{v}$ which converges conditionally it is the alternating harmonic series. At $x=4$ we get $\sum_{n=1}^{\infty}\frac{1}{n}$ which is a divergent p-series. Thus the IOC is $[-2,4)$

c. $\sum_{n=0}^{\infty}\frac{(2x-1)^n}{n^2+1}$

Solution: Use the ratio test to determine convergence of $\sum|a_n|$.

$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(2x-1)^n}\right|=|2x-1|\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}=|2x-1|$ thus the series converges absolutely if $|2x-1|<1\Leftrightarrow 0<x<1$. Check the endpoints $x=0$ and $x=1$. At $x=1$, $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$, compare to $\sum_{n=1}^{\infty}\frac{1}{n^2}$, now $\frac{1}{n^2+1}<\frac{1}{n^2}$ $\forall n\ge1$ and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series ($p=2$) then $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges (absolutely) so $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ also converges since it is eventually the same. At $x=0$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$, consider $\sum|a_n|=\sum\frac{1}{n^2+1}$ which is convergent from above thus $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ converges absolutely. Thus the IOC is $[0,1]$.

7. Determine whether the following series converge or diverge. Justify your answers by citing relevant tests or reason.

a. $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n+1}$

Solution: Look at $\sum |a_n| = \sum \frac{2^n}{3^n+1}$ this series looks like $\sum \frac{2^n}{3^n}$. Now

$3^n+1 > 3^n \Leftrightarrow \frac{1}{3^n+1} < \frac{1}{3^n} \Leftrightarrow \frac{2^n}{3^n+1} < \frac{2^n}{3^n} \ \forall n \geq 1$ and since $\sum \frac{2^n}{3^n}$ is a convergent geometric series, $r = \frac{2}{3}$, then $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n+1}$ is absolutely convergent by the DCT which implies $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n+1}$ is convergent.

b. $\sum_{n=0}^{\infty} \frac{n!}{e^{n^2}}$,

Solution: $\lim_{n\to\infty} \left(\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} \right) = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0$ therefore $\sum_{n=0}^{\infty} \frac{n!}{e^{n^2}}$ is convergent by the Ratio Test.

c. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Solution: Looks like $\sum \frac{\sqrt{n^2}}{n^3} = \sum \frac{1}{n^2}$. Now

$$\lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^2-1}}{n^3+2n^2+5} \right)}{\left(\frac{1}{n^2} \right)} = \lim_{n\to\infty} \frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5} = \lim_{n\to\infty} \frac{\sqrt{n^6-n^4}}{n^3+2n^2+5} = \lim_{n\to\infty} \frac{\sqrt{1-1/n^2}}{1+2/n+5/n^3} = 1$$

and since $\sum \frac{1}{n^2}$ is a convergent p-series (p=2) then by the LCT $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ is convergent.

d. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Solution: $\frac{1}{n \ln n} > 0 \ \forall n \geq 2$, $\frac{1}{(n+1)\ln(n+1)} > \frac{1}{n \ln n} \ \forall n \geq 2$, and

$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ thus by the AST $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is convergent.

e. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: Let $f(x) = \frac{1}{x\sqrt{\ln x}}$ for $[2,\infty)$. Now f is a positive and continuous function. Also $f'(x) = \frac{-1(2\ln x + 1)}{2x^2(\ln x)^{3/2}} < 0 \ \forall x \geq 2$, so f is a decreasing function.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx = \lim_{N\to\infty}\int_2^N \frac{1}{x\sqrt{\ln x}}dx = \lim_{N\to\infty} 2\sqrt{\ln x}\Big|_2^N = \lim_{N\to\infty}\left(2\sqrt{\ln N} - 2\sqrt{\ln 2}\right) = \infty$$

By the integral test since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$ diverges then $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges

f. $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

Solution: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{8n+1}\right)^n} = \lim_{n\to\infty} \frac{3n}{8n+1} = \frac{3}{8} < 1$, Thus by the Root Test $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$ converges.

g. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$

Solution: Note $\tan^{-1} n < \frac{\pi}{2} \ \forall n \geq 1$ which implies that $\frac{\tan^{-1} n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}} \ \forall n \geq 1$ and since $\frac{\pi}{2}\sum \frac{1}{n^{3/2}}$ converges (it is a constant times a convergent p-series) then $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$ is convergent by the DCT.

h. $\sum_{n-1}^{\infty} (-1)^n \frac{n}{n^2+25}$

Solution: $\frac{n}{n^2+25} > 0 \ \forall \text{n} \geq 1$ and $f'(x) = \frac{25 - x^2}{(x^2+25)^2} \leq 0 \ \forall x \geq 5$ so $u_n \geq u_{n+1} \ \forall n \geq 5$ thus by AST $\sum_{n=5}^{\infty} (-1)^n \frac{n}{n^2+25}$ is convergent and so $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$ is also convergent since it is eventually the same.