Final Exam Review Solutions **Math 132** **L. Ballou**

1. Let R be the region in the first quadrant bounded by the curve $y = x^3$ and $y = 2x - x^2$. Determine the volume of the solid obtained by revolving R about
   a. The *x*-axis. Use the disk/washer method.
   $$\int_0^1 \left(\pi\left(2x - x^2\right)^2 - \pi\left(x^3\right)^2\right)dx = \frac{41\pi}{105}$$
   b. The y-axis. Use the shell method.
   $$\int_0^1 2\pi x\left(2x - x^2 - x^3\right)dx = \frac{13\pi}{30}$$
   c. The line $y = -2$. Use the disk/washer method
   $$\int_0^1 \left(\pi\left(2x - x^2 + 2\right)^2 - \pi\left(x^3 + 2\right)\right)dx = \frac{72\pi}{35}$$

2. Using calculus, find the volume of a cone with height *h* and radius of the base *r*.
   $$V = \int_0^r 2\pi x\left(\frac{-h}{r}x + h\right)dx = 2\pi\left(\frac{-h}{3r}x^3 + \frac{h}{2}x^2\right)\Bigg|_0^r = \frac{\pi r^2 h}{3}$$

3. Find the area of the region in the first quadrant bounded by $y = x^2$, $y = 2x^2 - 4x$ and $y = 0$.
   $$A = \int_0^2 x^2 dx + \int_2^4 \left(x^2 - \left(2x^2 - 4x\right)\right)dx$$

4. Find the length of the curve $y = x^{1/2} - \frac{x^{3/2}}{3}$ for $1 \le x \le 3$
   $L = \int ds$, where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\right)^2}dx = \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)dx$
   so $L = \int_1^3 \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{3/2}\right)dx = 2\sqrt{3} - \frac{4}{3}$

5. A cylindrical tank of radius 3 m and length of 10 m is lying on its side on horizontal ground. If this tank is initially full of water (density 1000 kg/m^3), how much work is done to pump all this gasoline to a point 5 ft above the top of the tank?
   $$W = \int_{-3}^3 20\rho g\sqrt{9 - y^2}\left(8 - y\right)dy = 720\rho g\pi$$

6. A trough has vertical ends that are equilateral triangles (downward pointing) with sides of length 2 m. If the trough is filled will water, find the force exerted by the water on one end of the trough.

$$F=\int_0^{\sqrt{3}} \rho g\left(\frac{2}{\sqrt{3}}y\right)\left(\sqrt{3}-y\right)dy$$

7. Evaluate the following:

a. $\int \frac{\sin^3 x}{\sqrt{\cos x}}dx=\int \frac{\left(1-\cos^2 x\right)\sin x}{\sqrt{\cos x}}dx$, now let $u=\cos x,\ \ du=-\sin x dx$ so.

$$\int \frac{\left(1-\cos^2 x\right)\sin x}{\sqrt{\cos x}}dx=-\int \frac{1-u^2}{\sqrt{u}}du=-\left(2\sqrt{u}-\frac{2}{5}u^{5/2}\right)+C=\frac{2}{5}\left(\cos x\right)^{5/2}-2\sqrt{\cos x}+C$$

b. Let $u=x,\ du=dx,\ dv=\sin x dx$ and $v=-\cos x$

then $\int x\sin x dx=-x\cos x-\int -\cos x dx=-x\cos x+\sin x+C$.

c. $\frac{1}{(x-2)\left(x^2+4\right)}=\frac{1/8}{x-2}+\frac{-x/8-1/4}{x^2+4}$ so

$$\int \frac{1}{(x-2)\left(x^2+4\right)}dx=\frac{1}{8}\int \frac{1}{x-2}dx-\frac{1}{4}\int \frac{1}{x^2+4}dx-\frac{1}{8}\int \frac{x}{x^2+4}dx$$

$$=\frac{1}{8}\ln|x-2|-\frac{1}{8}\arctan\left(\frac{x}{2}\right)-\frac{1}{16}\ln\left|x^2+4\right|+C$$

d. Let $x=4\tan\theta,\ \ dx=4\sec^2\theta d\theta$, then

$$\int \frac{dx}{\sqrt{x^2+16}}=\int \frac{4\sec^2\theta}{\sqrt{16\tan^2\theta+16}}d\theta=\int \sec\theta d\theta=\ln\left|\sec\theta+\tan\theta\right|=\ln\left|\frac{\sqrt{x^2+4}}{4}+\frac{x}{4}\right|+C$$

e. Let $x=2\sin\theta,\ \ dx=2\cos\theta d\theta$,

then

$$\int \frac{x^2}{\sqrt{4-x^2}}dx=\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}2\cos\theta d\theta=\int 4\sin^2\theta d\theta=4\int \frac{1}{2}\left(1-\cos 2\theta\right)d\theta$$

$$=2\theta-\sin 2\theta=2\theta-2\sin\theta\cos\theta=2\arcsin\left(\frac{x}{2}\right)-x\left(\frac{\sqrt{4-x^2}}{2}\right)+C$$

.

f. $\int_0^2 \frac{x}{x^2-1}dx = \int_0^1 \frac{x}{x^2-1}dx + \int_1^2 \frac{x}{x^2-1}dx$, Now

$\int_0^1 \frac{x}{x^2-1}dx = \lim_{N\to 1^-}\int_0^N \frac{x}{x^2-1}dx = \lim_{N\to 1^-}\frac{1}{2}\ln\left|x^2-1\right|\Big|_0^N = \lim_{N\to 1^-}\frac{1}{2}\ln\left|N^2-1\right| = \infty$, thus

$\int_0^2 \frac{x}{x^2-1}dx$ diverges.

g. $\int_2^\infty \frac{dx}{x\ln x} = \lim_{N\to\infty}\int_2^N \frac{dx}{x\ln x} = \lim_{N\to\infty}\ln\left(\ln x\right)_2^N = \lim_{N\to\infty}\left(\ln\left(\ln N\right)-\ln\left(\ln 2\right)\right) = \infty$ Thus the integral diverges.

8. Find the Taylor series for $f(x) = \frac{1}{4x-3}$ at $a=1$.

$f\left(x\right) = \left(4x-3\right)^{-1}$, $f\,'\left(x\right) = -1\left(4x-3\right)^{-2}\left(4\right),\ldots, f^{(n)}\left(x\right) = \left(-1\right)^n\left(n\right)!\left(4x-3\right)^{-(n+1)}\left(4^n\right)$, so $f^{(n)}\left(1\right) = \left(-1\right)^n\left(n\right)!4^n$ Thus

$$\frac{1}{4x-3} = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(1\right)}{n!}\left(x-1\right)^n = \sum_{n=0}^{\infty}\frac{\left(-1\right)^n n!4^n}{n!}\left(x-1\right)^n = \sum_{n=0}^{\infty}\left(-1\right)^n 4^n\left(x-1\right)^n$$

9. Find the radius and interval of convergence for the power series $\sum_{k=1}^{\infty}\frac{(2x+3)^k}{4^k}$.

$\lim_{k\to\infty}\left|\frac{\left(2x+3\right)^{k+1}}{\left(k+1\right)^2 4^{k+1}}\cdot\frac{k^2 4^k}{\left(2x+3\right)^k}\right| = \lim_{k\to\infty}\left|\frac{2x+3}{4}\right| = \frac{\left|2x+3\right|}{4}$, Then by the Ratio test of the absolute value of the series, it will converge absolutely if $\frac{\left|2x+3\right|}{4} < 1 \Leftrightarrow -\frac{7}{2} < x < \frac{1}{2}$

Check the endpoints: at $x = 1/2$, $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent, it is a p-series with $p=2$.

And at $x = -7/2$, $\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{k^2}$ converges by the AST. Thus the interval of convergence is $\left[-\frac{7}{2},\frac{1}{2}\right]$ and the radius of convergence is $R=2$..

10. Determine whether the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^2}{k^3+1}$ converges absolutely, converges conditionally or diverges.

$\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\frac{k^2}{k^3+1}$, compare to $\sum_{k=1}^{\infty}\frac{k^2}{k^3} = \sum_{k=1}^{\infty}\frac{1}{k}$ is divergent since it is a p-series with $p=1$ and since $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{k^2}{k^3+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k^3}{k^3+1} = 1$, then by the Limit Comparison test $\sum_{k=1}^{\infty}|a_k| = \sum_{n=1}^{\infty}\frac{k^2}{k^3+1}$ is divergent. However, $|a_k| = \frac{k^2}{k^3+1} > 0\ \forall k \ge 1$, let $f(x) = \frac{x^2}{x^3+1}$ so $f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} < 0 \ \ \forall x \ge 2$ thus $|a_k| = \frac{k^2}{k^3+1}$ is a decreasing sequence and $\lim_{n\to\infty}\frac{k^2}{k^3+1} = 0$, so by the Alternating Series Test, $\sum_{k=1}^{\infty}\frac{(-1)^n k^2}{k^3+1}$ is convergent. Now since $\sum_{k=1}^{\infty}\frac{(-1)^{n+1}k^2}{k^3+1}$ is convergent but $\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\frac{k^2}{k^3+1}$ diverges so $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^2}{k^3+1}$ is conditionally convergent.

11. Determine whether the series $\sum_{k=1}^{\infty}\frac{2k+3}{k^2+3k+1}$ converges or diverges.

Looks like $\sum_{k=1}^{\infty}\frac{k}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k}$, now $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{2k+3}{k^2+3k+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{2k^2+3k}{k^2+3k+1} = 2$ Therefore, by the Limit Comparison Test since $\sum_{k=1}^{\infty}\frac{1}{k}$ is a divergent p-series, then $\sum_{k=1}^{\infty}\frac{2k+3}{k^2+3k+1}$ is also divergent.

12. Determine whether the series $\sum_{k=1}^{\infty}\frac{\sin(5k)}{1+3^k}$ converges or diverges.

$0 \le |\sin(5k)| \le 1 \Rightarrow 0 \le \frac{|\sin(5k)|}{1+3^k} \le \frac{1}{1+3^k} < \frac{1}{3^k}$ and since $\sum_{k=1}^{\infty}\frac{1}{3^k}$ is a convergent geometric series with $r = \frac{1}{3}$, then by the Direct Comparison Test $\sum_{k=1}^{\infty}\frac{|\sin(5k)|}{1+3^k}$ converges, so $\sum_{k=1}^{\infty}\frac{\sin(5k)}{1+3^k}$ converges absolutely and by the absolute convergence theorem it converges.

13. Find and sketch the four roots of $3+3i$.

$\sqrt[4]{3+3i} = \sqrt[4]{\sqrt{18}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$, so $\phi = \frac{1}{4}\left(\frac{\pi}{4} + 2k\pi\right) = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, , \frac{25\pi}{16}$

Thus

$$w_1 = 18^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$$
$$w_2 = 18^{1/8}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$$
$$w_3 = 18^{1/8}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right)$$
$$w_4 = 18^{1/8}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right)$$

14. Find the area of the region enclosed by one loop of the curve $r = 3\sin 3\theta$.

$$A = \int_0^{\pi/3} \frac{1}{2}(3\sin 3\theta)^2\, d\theta = \frac{3\pi}{4}$$

15. Find the area of the region inside $r = 4\cos\theta$ and outside $r = 2$.

$$A = 2\left(\int_0^{\pi/3} \frac{1}{2}(4\cos\theta)^2\, d\theta - \int_0^{\pi/3} \frac{1}{2}(2)^2\, d\theta\right) = 2\sqrt{3} + \frac{4\pi}{3}$$

16. For the parametric curve $x = t^2$, $y = 3\ln t + 2$, write the equation of the line tangent to the curve at $t = 1$.

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{\frac{3}{t}}{2t}\right|_{t=1} = \frac{3}{2} \text{ and } (x = 1, y = 2)$$

So tan line is: $y - 2 = \frac{3}{2}(x - 1)$

17. Evaluate the expression

a. $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

b. $\frac{5+3i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{7}{10} + \frac{11}{10}i$

c. $(-1-i)^{24} = \left(\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)\right)^{24} = 2^{12}\left(\cos(-18\pi) + i\sin(-18\pi)\right) = 2^{12}$

d. $|-3-4i| = \sqrt{9+16} = 5$