Final Exam Review Solution **Math 132**

Evening:.

1. Let R be the region bounded by the curve $y=(x-2)^2$ and the line $y=4$.
   a. Find the volume of the solid generated by revolving R about the x-axis.

   $$V=\int_0^4\left(\pi 4^2-\pi\left((x-2)^2\right)^2\right)dx=\frac{256}{5}\pi$$

   b. Find the volume of the solid generated by revolving R about the y-axis.

   $$V=\int_0^4 2\pi x\left(4-(x-2)^2\right)dx=\frac{128}{3}\pi$$

   c. Find the volume of the solid generated by revolving R about the line $x=-1$.

   $$V=\int_0^4 2\pi(x+1)\left(4-(x-2)^2\right)dx=64\pi$$

2. Find the area of the region in the first quadrant bounded by $y=5x-8$, $y=16-x^2$ and the x-axis.

   $$A=\int_0^{8/5}\left(16-x^2\right)dx+\int_{8/5}^{3}\left(16-x^2-(5x-8)\right)dx$$

3. Find the arc length of the curve $y=\frac{1}{3}(x^2+2)^{3/2}$ for $0\le x\le 1$.

   $$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\sqrt{1+\left(\frac{1}{2}\left(x^2+2\right)^{1/2}(2x)\right)^2}dx=\sqrt{x^4+2x^2+1}dx=\left(x^2+1\right)dx$$

   $$L=\int_0^1\left(x^2+1\right)dx=\frac{4}{3}$$

4. A cylindrical storage tank of diameter 4 m and length 8 m is lying on its side. If the tank is half full of water (with density 1000 kg/m^3), find the force exerted by the water on one end of the tank.

   $$F=\int_{-2}^{2}\rho g\left(2\sqrt{4-y^2}\right)(2-y)dy$$

5. A conical tank 5 m in diameter and 10 m in height is resting on its base. The tank is filled with water, how much work is required to pump all the water over the top of the tank?

   $$W=\int_0^{10}\frac{\pi}{16}\rho g(10-y)^2(10-y)dy$$

6. Evaluate the following:
   a. Let $u=\arctan x,\ du=\frac{1}{1+x^2}dx,\ dv=xdx,\ v=\frac{x^2}{2}$ So

$$\int x\arctan x dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$
$$= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C$$

b. $\displaystyle\int \frac{x^2+8x-3}{x^3+3x^2}dx = \int\left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right)dx = 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$

c. Let $x = \tan\theta,\ \ dx = \sec^2\theta d\theta$, So

d. Let $x = 2\sin\theta,\ dx = 2\cos\theta d\theta$,

$$\int \frac{x^2}{\left(4-x^2\right)^{3/2}}dx = \int \frac{4\sin^2\theta}{\left(4-4\sin^2\theta\right)^{3/2}}2\cos\theta d\theta = \tan\theta - \theta = \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$$

e.

$$\int_1^3 \frac{1}{\sqrt[3]{x-2}}dx = \int_1^2 \frac{1}{\sqrt[3]{x-2}}dx + \int_2^3 \frac{1}{\sqrt[3]{x-2}}dx$$
$$= \lim_{N\to 2^-}\left[\frac{3(N-2)^{2/3}}{2} - \frac{3}{2}\right] + \lim_{N\to 2^-}\left[\frac{3}{2} - \frac{3(N-2)^{2/3}}{2}\right] = 0$$

so Thus $\displaystyle\int_1^3 \frac{1}{\sqrt[3]{x-2}}dx$ converges.

7. Using calculus, find the surface area of a sphere of radius r. Rotate the top half of a circle, $y = \sqrt{r^2 - x^2}$ about the x-axis.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\frac{x^2}{r^2-x^2}}dx = \frac{r}{\sqrt{r^2-x^2}}dx,$$

So $\displaystyle S = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}dx = 4\pi r^2$

8. Write the Taylor series for $f(x) = \dfrac{1}{2x-5}$ at $a = 3$.

$f(x) = (2x-3)^{-1}$, $f'(x) = -1(2x-3)^{-2}(2), \ldots, f^{(n)}(x) = (-1)^n (n)!(2x-5)^{-(n+1)}(2^n)$, so

$f^{(n)}(1) = (-1)^n n!2^n$ Thus

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty}\frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{(-1)^n n!2^n}{n!}(x-3)^n = \sum_{n=0}^{\infty}(-1)^n 2^n (x-3)^n$$

9. Find the radius and interval of convergence for the power series $\displaystyle\sum_{k=1}^{\infty}\frac{(x+2)^k}{k4^k}$.

$\lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{(k+1)4^{k+1}}\cdot\frac{k4^k}{(x+2)^k}\right| = \frac{|x+2|}{4}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x+2|}{4}$, Then by the Ratio test of the absolute value of the series, it will converge absolutely if $\frac{|x+2|}{4} < 1 \Leftrightarrow -6 < x < 2$ Check the endpoints: at $x = 2$, $\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent p-series.. And at $x = -6$, $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ is convergent because it is the Alternating Harmonic Series. Thus the interval of convergence is $[-6,2)$ with radius of convergence 4..

10. Determine whether the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt[4]{k}}$ converges absolutely, converges conditionally or diverges.
$\sum_{k=1}^{\infty}|a_k| = \sum_{n=1}^{\infty}\frac{1}{k^{1/4}}$ is divergent since it is a p-series with $p = 1/4 < 1$. However, $\frac{1}{k^{1/4}} > 0$, $\frac{1}{k^{1/4}} > \frac{1}{(k+1)^{1/4}}$, and $\lim_{k\to\infty}\frac{1}{k^{1/4}} = 0$ thus by the Alternating Series Test, $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{1/3}}$ is convergent. Now since $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{1/4}}$ is convergent but $\sum_{k=1}^{\infty}\frac{1}{k^{1/4}}$ diverges so $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{1/4}}$ is conditionally convergent.

11. Determine whether the series $\sum_{k=1}^{\infty}\frac{k^2-1}{3k^4+1}$ converges or diverges. This series looks like $\sum_{k=1}^{\infty}\frac{k^2}{k^4} = \sum_{k=1}^{\infty}\frac{1}{k^2}$ which is a convergent p-series (p=2) and since $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{k^2-1}{3k^4+1}}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{k^4-k^2}{3k^4+1} = \frac{1}{3}$, then by the Limit Comparison test, $\sum_{k=1}^{\infty}\frac{k^2-1}{3k^4+1}$ is also convergent.

12. Find the six roots of $\sqrt[6]{-64i} = \sqrt[6]{64\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)} = 2(\cos\phi + i\sin\phi)$ where $\phi = -\frac{\pi}{12},\frac{3\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\frac{15\pi}{12},\frac{19\pi}{12}$.

13. Find the area of the region inside $r = -3\cos\theta$ and outside $r = 1-\cos\theta$.

$$A = 2\left(\int_{\pi}^{4\pi/3}\frac{1}{2}(-3\cos\theta)^2 d\theta - \int_{\pi}^{4\pi/3}\frac{1}{2}(1-\cos\theta)^2 d\theta\right) = \pi$$

14. Replace the polar equation $r = 3\cos\theta$ with the Cartesian equation. Identify or describe the graph.

$r=3\cos\theta \Leftrightarrow r^2=3r\cos\theta \Leftrightarrow x^2+y^2=3x \Leftrightarrow \left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$, Circle centered at $\left(\frac{3}{2},0\right)$ with radius 3/2

15. For the parametric curve $x=e^{\sqrt{t}}$, $y=t-\ln t^2$, write the equation of the line tangent to the curve at $t=1$.

$$m_{\tan}=\left.\frac{dy}{dx}\right|_{t=1}=\left.\frac{dy/dt}{dx/dt}\right|_{t=1}=\left.\frac{1-\frac{1}{t^2}2t}{e^{\sqrt{x}}\frac{1}{2\sqrt{t}}}\right|_{t=1}=-\frac{2}{e} \text{ and } \left(x=e^1, y=1\right)$$

So tan line is: $y-1=-\frac{2}{e}(x-2)$

16. Evaluate the expression

a. $\frac{3+2i}{1+i}\cdot\frac{1-i}{1-i}=\frac{5}{2}+\frac{i}{2}$

b. $\left(\frac{1}{2}+\frac{1}{2}i\right)^{15}=\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{15}=\frac{\sqrt{2}}{2^8}\left(\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right)=\frac{1}{2^8}(1-i)$

c. $\left|-1+2\sqrt{2}i\right|=\sqrt{1+8}=3$