

1. Find the linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$. Use it to give an approximate value for $\sqrt[3]{7.98}$.

Solution: $f(x) \approx 2 + \frac{1}{12}(x-8)$, thus $f(7.98) \approx 2 + \frac{1}{12}(7.98-8) = 1.9983$

2. Does the function $f(x) = |x|$ on $[-2, 2]$ satisfy the conditions of the Mean Value Theorem? Why or why not?

Solution: No, because $f'(x)$ does not exist at $x = 0$, therefore it is not differentiable on $(-2, 2)$.

3. For $f(x) = \frac{3+x}{3-x}$, find the differential dy .

Solution: $dy = \frac{(3-x) - (3+x)(-1)}{(3-x)^2} dx = \frac{6}{(3-x)^2} dx$

4. Determine the vertical and horizontal asymptotes of $f(x) = \frac{2x^2 + 6x}{x^2 - 9}$

Solution: $f(x) = \frac{2x(x+3)}{(x+3)(x-3)}$

$\lim_{x \rightarrow 3^+} \frac{2x(x+3)}{(x-3)(x+3)} = \infty$ and $\lim_{x \rightarrow 3^-} \frac{2x(x+3)}{(x-3)(x+3)} = -\infty$, thus we have a vertical asymptote at $x = 3$

$\lim_{x \rightarrow \pm\infty} \frac{2x^2 + 6x}{x^2 - 9} = 2$, thus we have a horizontal asymptote at $y = 2$.

5. Evaluate $\lim_{x \rightarrow 0} \frac{e^{-5x} - 1 + 5x}{x^2}$

Solution: $\lim_{x \rightarrow 0} \frac{e^{-5x} - 1 + 5x}{x^2} \stackrel{L}{=} \lim_{x \rightarrow 0} \frac{-5e^{-5x} + 5}{2x} \stackrel{L}{=} \lim_{x \rightarrow 0} \frac{25e^{-5x}}{2} = \frac{25}{2}$

6. Evaluate $\lim_{x \rightarrow \pi/2^-} \frac{3 \sec x}{2 + \tan x}$

Solution: $\lim_{x \rightarrow \pi/2^-} \frac{3 \sec x}{2 + \tan x} \stackrel{L}{=} \lim_{x \rightarrow \pi/2^-} \frac{3 \sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow \pi/2^-} 3 \sin x = 3$

7. Evaluate $\lim_{x \rightarrow \pi/2^-} (1 - \sin x) \tan x$

Solution: $\lim_{x \rightarrow \pi/2^-} (1 - \sin x) \tan x = \lim_{x \rightarrow \pi/2^-} \frac{(1 - \sin x) \sin x}{\cos x} \stackrel{L}{=} \lim_{x \rightarrow \pi/2^-} \frac{\cos x - 2 \sin x \cos x}{-\sin x} = 0$

8. Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{x^2 \sec x} \right)$

Solution: $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{x^2 \sec x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sec x - 1}{x^2 \sec x} \right) \stackrel{L}{=} \lim_{x \rightarrow 0} \left(\frac{\sec x \tan x}{2x \sec x + x^2 \sec x \tan x} \right)$
 $= \lim_{x \rightarrow 0} \left(\frac{\tan x}{2x + x^2 \tan x} \right) \stackrel{L}{=} \lim_{x \rightarrow 0} \left(\frac{\sec^2 x}{2 + 2x \tan x + x^2 \sec^2 x} \right) = \frac{1}{2}$

9. Evaluate $\lim_{x \rightarrow \infty} \left(1 - \frac{4}{x} \right)^x$

Solution: $\lim_{x \rightarrow \infty} \left(1 - \frac{4}{x} \right)^x = \exp \left[\lim_{x \rightarrow \infty} x \ln \left(1 - \frac{4}{x} \right) \right] = e^{-4}$

10. Evaluate $\lim_{x \rightarrow 0^+} (1+x)^{4/x}$

Solution: $\lim_{x \rightarrow 0^+} (1+x)^{4/x} = \exp \left\{ \lim_{x \rightarrow 0^+} \frac{4}{x} \ln(1+x) \right\} = e^4$

11. Find the local and absolute extreme values of the function $f(x) = x - \sqrt{x}$ on $[0, 4]$.

Solution: $f'(x) = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}}$ so $f'(x) = 0$ when $x = \frac{1}{4}$, so $f(0) = 0$, $f\left(\frac{1}{4}\right) = -\frac{1}{4}$ and $f(4) = 2$.

Therefore absolute maximum occurs at $(4, 2)$ and absolute minimum occurs at $\left(\frac{1}{4}, -\frac{1}{4}\right)$.

12. Given $f'(x) = (x-1)(x+2)(x+4)$, determine the critical points of $f(x)$ and use the second derivative test to determine whether they correspond to local maxima, local minima, or the test is inconclusive.

Solution: $f'(x) = 0$ at $x = 1, -2, -4$. Now $f''(x) = (x-1)(x+2) + (x-1)(x+4) + (x+2)(x+4)$ and $f''(1) = 15$, so at $x = 1$ there is a local minimum, $f''(-2) = -6$, so at $x = -2$, there is a local maximum and $f''(-4) = 10$, so at $x = -4$ there is a local minimum.

13. Determine where $f(x) = (1000-x)^2 + x^2$ is increasing. Use it to decide which is larger 1000^2 or $998^2 + 2^2$.

Solution: $f'(x) = 2(1000-x)(-1) + 2x = -2000 + 4x$ now $f'(x) = 0$ when $x = 500$ and $f(x)$ is increasing on $(500, \infty)$ (and decreasing on $(-\infty, 500)$). Also, $f(0) = 1000^2$ and $f(2) = 998^2 + 2^2$ so since f is decreasing then $f(0) > f(2)$, i.e. $998^2 + 2^2 < 1000^2$.

14. For each function, $f(x) = (x^2 - 1)^3$ and $f(x) = x\sqrt{3-x}$

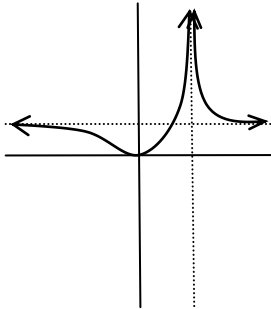
Solution: CP at $(0, -1)$, $(1, 0)$, and $(-1, 0)$, f is decreasing on $(-\infty, -1) \cup (-1, 0)$ and increasing on $(0, 1) \cup (1, \infty)$, local min at $(0, -1)$, and f is concave up on $(-\infty, -1) \cup \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \cup (1, \infty)$ and concave down on

$$\left(-1, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, 1\right)$$

Solution: Note the domain is $(-\infty, 3]$, CP $(2, 2)$, $(3, 0)$, f is increasing on $(-\infty, 2)$ and decreasing on $(2, 3)$, local max at $(2, 2)$ and the function is concave down on $(-\infty, 3)$

15. Sketch the graph of a function that satisfies all the conditions given below.

Solution:



16. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal? (The volume of a sphere is $\frac{4\pi r^3}{3}$.)

Solution: Volume: $V = \pi r^2 h + \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right) \Rightarrow h = \frac{V}{\pi r^2} - \frac{2r}{3}$, surface area is the sum of the area of the bottom, side, and

top. $S(r) = \pi r^2 + 2\pi r h + \frac{1}{2}(4\pi r) = \frac{5}{3}\pi r^2 - \frac{2V}{r}$, for $r > 0$ and $S'(r) = \frac{10\pi r^3 - 6V}{3r^2} = 0$ when $r = \sqrt[3]{\frac{3V}{5\pi}}$, and

$S''(r) = \frac{10\pi}{3} - \frac{4V}{r^3} < 0$ when $r = \sqrt[3]{\frac{3V}{5\pi}}$ thus the dimensions that minimize surface area are $r = \sqrt[3]{\frac{3V}{5\pi}} = h$

17. A closed box with square base is to be built to house an ant colony. The bottom of the box and all four sides are to be made of material costing \$1/ft² and the top is to be constructed of glass costing \$5/ft². What are the dimensions of the box of greatest volume that can be constructed for \$72? Verify your answer yields a maximum.

Solution: Cost $C = x^2 + 4xy + 5x^2 = 6x^2 + 4xy = 72$, so $y = \frac{72 - 6x^2}{4x}$ for $x > 0$. Now

$V = x^2 y = x^2 \left(\frac{72 - 6x^2}{4x} \right) = 18x - \frac{9}{2}x^3$ and $V' = 18 - \frac{9}{2}x^2 = 0$ when $x = 2$. To verify that $x = 2$ maximizes volume,

use the second derivative test, $V'' = -9x$ so $V''(2) = -18$ thus volume is maximized when $x = 2, y = 6$.