Chapter 6 Applications Integration

Section 6.2 Region Between Curves

Recall:

Theorem 5.3 (Continued) The Fundamental Theorem of Calculus, Part 2:
If $f$ is continuous at every point of $[a,b]$ and $F$ is any antiderivative of $f$ on $[a,b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a).$$

Definition:
Let $f(x)$ be a function defined on a closed interval $[a,b]$. Then the definite integral of $f$ over $[a,b]$ is

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{x}_k) \Delta x$$

if the limit exists.

Comments:
- $\int_{a}^{b} f(x) \, dx$ is read the integral from $a$ to $b$ of $f$ of $x$ with respect to $x$.
- Recall from section 4.8, $\int$ is the integral sign, $f(x)$ is the integrand of the integral and $x$ is the variable of integration.
- $a$ is the lower limit of integration and $b$ is the upper limit of integration.
- When the definition is satisfied, we say the Riemann sums of $f$ on $[a,b]$ converge to the definite integral $I = \int_{a}^{b} f(x) \, dx$ and that $f$ is integrable over $[a,b]$.
- The value of the definite integral depends on the function not the letter we choose to represent the variable of integration. So if $\int_{a}^{b} f(x) \, dx = I$, then $\int_{a}^{b} f(t) \, dt = I$ or $\int_{a}^{b} f(u) \, du = I$. The variable of integration is called a dummy variable.
Area
Find the area between the curve \( y = 4 - x^2 \), the \( y \)-axis \((x = 0)\) and the \( x \)-axis \((y = 0)\).

Areas Between Curves

Example: Find the area enclosed by \( y = x^2 - 2x \) and \( y = x \).

Example: Find the area of the region bounded by \( y = |x| \) and \( y = x^2 - 6 \).
Example: Find the area of the region enclosed by $y = x^3$ and $y = x$.

Example: Find the area enclosed by $x = y^2$ and $x = y + 2$.

Section 6.3 Volumes by Slicing

Three Types of Volume Problem:

A. **Area of the cross section is constant.** This occurs for right cylinders. A right cylinder is obtained when a plane region is moved along a line perpendicular to plane region through a distance $h$.

   ![Diagram of a right cylinder, a rectangular solid, and a cone]

   In this case, the volume is $V = (\text{area } R)h = Ah$ when $A$ is constant.

B. **Area of a cross section is not constant.**
To find the volume of the solid $S$ that extends across the $x$-axis from $x = a$ to $x = b$ with cross-sectional areas perpendicular to $x$-axis, where each cross-section has area that varies from point to point.

1. Divide $S$ into slices perpendicular to $x$-axis, having width $\Delta x = \frac{b-a}{n}$.

2. Volume, $V = V_1 + V_2 + \cdots + V_n = \sum_{i=1}^{n} V_i$, $V_i$ is approximately the area of the rectangular face of the slice times the width of the slice. In other words, $V_i \approx A_i \Delta x$.

3. So $V \approx \sum_{i=1}^{n} A_i \Delta x$ where $A_i$ is a function of $x$.

4. Thus $V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i) \Delta x = \int_{a}^{b} A(x) \, dx$.

**Example 1:** The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the $x$-axis is a semicircle with its diameter across the base of the enclosed region. Find the volume of the solid.
**C: Volumes of revolution:** Volume obtained by revolving a plane region about an axis.

**Example 2:** Find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ about the $x$-axis.

**Example 3:** Find the volume of the solid obtained by rotating the region bounded by $y = 6 - x^2$, $x = 0$, and $y = 2$ about the $y$-axis.

**Example 4:** Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = 3$ about the line $y = -1$. 
Example 5: Find the volume of the solid obtained by rotating the region bounded by
\( y = x \) and \( y = \sqrt{x} \) about the \( x = 2 \).

Section 6.4 Volumes by Shells

Suppose we consider revolving the following region about the \( y \)-axis. Notice that the disk/washer method does not provide a nice solution here. But consider the technique we used. In section 6.3, volume is

\[
V = \int_a^b dV = \int_a^b A(x) \, dx,
\]

where \( dV \) is the volume of a slice of the solid. Using this idea, let’s consider a different way to cut the solid into pieces. Instead of slicing the solid into disks, let’s cut the solid into rings/shells. Just as in section 6.3, the volume \( V = \sum_{i=1}^{n} V_i \) so let’s find the volume of a shell.
So the volume of a cylindrical shell is \( \Delta V = 2\pi r \cdot h \cdot \Delta r \). Note, \( 2\pi r \) is the circumference of the shell, \( h \) is the height, and \( \Delta r \) is the thickness of the shell.

For our problem then \( dV = 2\pi r \cdot h \cdot \Delta r \) and \( r \) is the distance from the shell to the axis of rotation, i.e. \( r = x \), and height is \( h = f(x) \), and \( \Delta r = \Delta x = dx \). So the volume of the solid is \( V = \int_a^b 2\pi x \cdot f(x) \, dx \).

Notice we can also just use our original formula from section 6.3. Suppose we slice a shell and “unroll” it so that it is flat.

Now \( V = \int_a^b A(x) \, dx \) where \( A(x) = 2\pi x f(x) \) and \( A(x) \) is the area of the face of the “unrolled” shell.

**Example 1:**
Find the volume of the solid generated when the region bounded by \( y = x^2 \), \( y = 0 \), and \( x = 1 \) is rotated about the \( y \)-axis.
Example 2:
Find the volume of the solid generated when the region bounded by \( x = \sqrt{y} \), \( x = 0 \), and \( y = 1 \) is rotated about the \( x \)-axis.

Example 3:
Find the volume of the solid generated when the region bounded by \( y = \sqrt{x - 1} \), \( y = 0 \), and \( x = 5 \) is revolved about the line \( y = 3 \).

Example 4: Which method should you use?
Find the volume of the solid generated when the region bounded by \( y = x^2 \) and \( x = y^2 \) is rotated about the \( y \)-axis
Section 6.5 Lengths of Curves

To find the length of a curve from \( (a, f(a)) \) to \( (b, f(b)) \):
- Create small arcs
- Length is the sum of the small arcs
- What is the approximate length of a segment?

\[
\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x
\]

So
\[
= \sqrt{1 + \left(f'(x)\right)^2} \Delta x \quad \text{by Mean Value Theorem}
\]

**Arc Length is** \( L = \lim_{n \to \infty} \sum_{i=1}^{n} ds_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x)\right)^2} \Delta x = \int_{a}^{b} \sqrt{1 + \left(f'(x)\right)^2} \, dx \)

**Example:** Find the length of the curve \( y = \frac{x^2}{2} - \frac{\ln x}{4} \) for \( 2 \leq x \leq 4 \)

**Example:** Find the length of the curve \( x = \frac{y^{3/2}}{3} - y^{1/2} \) for \( 1 \leq y \leq 9 \)
Section 6.6 Physical Applications

From physics, force is the influence that tends to cause motion in a body.

Work Done by Constant Force
If a body moves a distance, \(d\), in the direction of an applied constant force, \(F\), then the work done is

\[ W = F \cdot d \]

For example, the work to lift a 90 lb bag of concrete 3 feet is

\[ W = Fd = (90\text{ lb})(3\text{ ft}) = 270\text{ ft} \cdot \text{lb} \]

Common Units of Work and Force

<table>
<thead>
<tr>
<th>Mass</th>
<th>Distance</th>
<th>Force</th>
<th>Work</th>
</tr>
</thead>
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<tr>
<td>kg</td>
<td>M</td>
<td>newton</td>
<td>Joules</td>
</tr>
<tr>
<td>g</td>
<td>cm</td>
<td>dyne</td>
<td>Erg</td>
</tr>
<tr>
<td>slug</td>
<td>ft</td>
<td>pound</td>
<td>Ft-lb</td>
</tr>
</tbody>
</table>

Work Done with Variable Force

Suppose \(F\) is a continuous function and \(F(x)\) is a variable force that acts on an object moving along the \(x\)-axis from \(x = a\) to \(x = b\).

To determine work:
Partition the interval \([a, b]\) into \(n\) subintervals of length \(\Delta x\). If \(\Delta x\) is sufficiently small, then the force is essentially constant on the subinterval. So \(\Delta W \approx F(x_i)\Delta x\) for each subinterval and

\[ W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i) \Delta x = \int_{a}^{b} F(x) \, dx \]

Example: An object located \(x\) ft from a fixed starting position is moved along a straight road by force \(F(x) = (3x^2 + 5)\text{ lb}\). What work is done by the force to move the object from \(x = 1\) to \(x = 4\) feet?
**Hooke’s Law**

When a spring is pulled $x$ units past its equilibrium (rest) position, there is a restoring force $F(x) = kx$ that pulls the spring back toward equilibrium.

**Example:** The natural length of a certain spring is 10 cm. If it requires 2 joules of ergs of work to stretch the spring to 18 cm, how much work will be needed to stretch the spring to 20 cm?

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**Example:** A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/sec, but the water leaks from a hole in the bucket at a rate of 0.2 lbs/sec. Find the work to pull the bucket to the top of the well.

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**Example:** A cylindrical tank of radius 3 feet and height of 10 feet is filled to a depth of 2 feet with a liquid of density $\rho = 40$ lb/ft$^3$. Find the work done in pumping all the liquid to a height of 2 feet above the top of the tank.
Example: You are in charge of the evacuation and repair of a hemispherical storage tank (with base down, see picture in text). The tank has radius 10 ft and is full of benzene weighing 56 lb/ft³. A firm you contacted says it can empty the tank for ½ cent per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 feet above the top of the tank. If you have $5000 budgeted for the job, can you afford to hire the firm?

Force and Pressure

\[ \Delta F = \text{(area of strip)} \times \text{(pressure)} = w(y) \Delta y \rho g (a - y) \]

So force is

\[ F = \int_{a}^{b} \rho g (a - y) w(y) dy \]

Example: The lower edge of a dam is defined by the parabola \( y = x^2 / 16 \), with the top at \( y = 25 \). Determine the force on the dam. Lengths are measured in meters.
Section 6.7 Logarithmic and Exponential Functions Revisited.

Definition: The Natural Logarithm
The natural logarithm of a number \( x > 0 \) denoted \( \ln(x) \) is defined as
\[
\ln x = \int_1^x \frac{1}{t} \, dt
\]

Theorem 6.4 Properties of Natural Logarithms
1. The domain and range of \( y = \ln x \) are \((0, \infty)\) and \((-\infty, \infty)\), respectively.
2. \( \frac{d}{dx} \left( \ln(u(x)) \right) = \frac{u'(x)}{u(x)} , \ u(x) \neq 0 \)
3. \( \ln(x + y) = \ln x + \ln y , \ (x > 0, y > 0) \)
4. \( \ln \left( \frac{x}{y} \right) = \ln x - \ln y , \ (x > 0, y > 0) \)
5. \( \ln x^p = p \ln x \ (x > 0, p \text{ are real number}) \)
6. \( \int \frac{1}{x} \, dx = \ln |x| + C \)

Definition: Base of the Natural Logarithm
The natural logarithm is the logarithm with a base of \( e = \lim_{h \to 0} \left( 1 + h \right)^{1/h} \approx 2.781828 \). It follows that \( \ln e = 1 \).

Theorem 6.5 Properties of \( e^x \)
The exponential function \( f(x) = e^x \) satisfies the following properties, all of which follow from the integral definition of \( \ln x \). Let \( x \) and \( y \) be real numbers
1. \( e^{x+y} = e^x e^y \)
2. \( e^{x-y} = \frac{e^x}{e^y} \)
3. \( (e^x)^y = e^{xy} \)
4. \( \ln (e^x) = x \)
5. \( e^{\ln x} = x , \ x > 0 \)
Theorem 6.6 **Derivative and Integral of the Exponential Function**
For real number $x$,
\[
\frac{d}{dx}(e^{u(x)}) = e^{u(x)}u'(x) \quad \text{and} \quad \int e^x \, dx = e^x + C
\]

General Logarithmic and Exponential Function

**Change of Base Rules (page 31)**
Let $b$ be a positive real number with $b \neq 1$. Then
\[
b^x = e^{\ln b}, \quad \text{for all } x \quad \text{and} \quad \log_b x = \frac{\ln x}{\ln b}, \quad \text{for } x > 0
\]