Section 4.8 Antiderivatives

If we know the function that gives us rate of change, what is the original function? For instance, if we know the velocity function, what is the position function?

Definition **Antiderivative**
A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example: Find the antiderivative for each of the following functions.

1. $f(x) = 4x^3$

2. $g(x) = \cos x$

3. $h(x) = \dfrac{1}{x}$

Theorem 4.16 The Family of Antiderivatives
If F be any antiderivative of f. Then all the antiderivatives of f have the form

$$F(x) + C$$

Where C is an arbitrary constant.

Definition
The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration.**

Example: Evaluate

1. $\int 2x\,dx$

2. $\int \dfrac{1}{2\sqrt{x}}\,dx$

3. $\int 5\cos 5x\,dx$

4. $\int \sin 3x\, dx$

Indefinite Integrals

1. $\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n \quad \rightarrow \quad \int x^n dx = \frac{1}{n+1}x^{n+1} + C \text{ for } n \neq -1$

2. $\frac{d}{dx}(\sin ax) = a\cos ax \quad \rightarrow \quad \int \cos ax dx = \frac{1}{a}\sin ax + C$

3. $\frac{d}{dx}(\cos ax) = -a\sin ax \quad \rightarrow \quad \int \sin ax dx = -\frac{1}{a}\cos ax + C$

4. $\frac{d}{dx}(\tan ax) = a\sec^2 ax \quad \rightarrow \quad \int \sec^2 ax dx = \frac{1}{a}\tan ax + C$

5. $\frac{d}{dx}(\sec ax) = a\sec ax \tan ax \quad \rightarrow \quad \int \sec ax \tan ax dx = \frac{1}{a}\sec ax + C$

6. $\frac{d}{dx}(\cot ax) = -a\csc^2 ax \quad \rightarrow \quad \int \csc^2 ax dx = -\frac{1}{a}\cot ax + C$

7. $\frac{d}{dx}(\csc ax) = a\csc ax \cot ax \quad \rightarrow \quad \int \csc ax \cot ax dx = -\frac{1}{a}\csc ax + C$

8. $\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \quad \rightarrow \quad \int e^{ax} dx = \frac{1}{a}e^{ax} + C$

9. $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \rightarrow \quad \int \frac{1}{x} dx = \ln|x| + C \text{ for } x \neq 0$

10. $\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \rightarrow \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C \text{ for } |x| \leq a,\ a > 0$

11. $\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{x^2 + a^2} \quad \rightarrow \quad \int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \text{ for all } x \text{ and } a \neq 0$

12. $\frac{d}{dx}\left(\operatorname{arcsec}\left(\frac{x}{a}\right)\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \rightarrow \quad \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C \text{ for } |x| \geq a > 0$

13. $\frac{d}{dx}\left(b^{ax}\right) = b^{ax} \cdot a\ln b \quad \rightarrow \quad \int b^{ax} dx = \frac{1}{a\ln b}b^{ax} + C$

Example: Evaluate the following:

1. $\int \sec 3x \tan 3x\, dx$

2. $\int \frac{1}{x^2 + 4} dx$

3. $\int e^{x/5} dx$

4. $\int 2^{-x} dx$

Table 4.18 Constant Multiple and Sum Rules	
Constant Multiple Rule:	$\int cf(x)\,dx = c\int f(x)\,dx$
Sum Rule:	$\int \left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx$

Example: Evaluate each of the following:

1. $\int \left(2x^3 - 5x + \frac{3}{x} + 7\right)dx$

2. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$

3. $\int 7\sin\left(\frac{\theta}{3}\right)d\theta$

4. $\int 3x^{\sqrt{3}} dx$

Initial Value Problems

Example:

1. Suppose $\frac{dy}{dx} = 10 - x$ and $y(0) = -1$, find *y*.

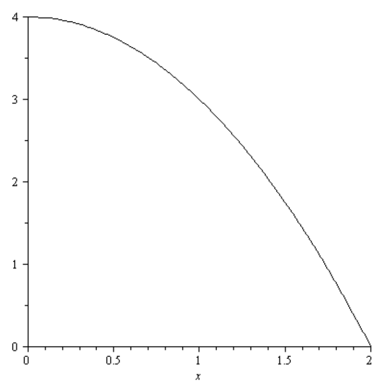
2. Suppose $\frac{d^2y}{dx^2} = 2 - 6x$, $y'(0) = 4$ and $y(1) = 1$, find *y*.

Chapter 5 Integration

Section 5.1 Approximating Area under Curves

Area

Find the area between the curve $y = 4 - x^2$, the *y*-axis $(x = 0)$ and the *x*-axis ($y = 0$).



We can estimate the area by summing the area of a collection of rectangles. Using more rectangles increases the accuracy of the estimation.

Estimating Area

1. Determine the number of rectangles you plan to use, *n*.

2. If the area is from $x = a$ to $x = b$, then find the width of the rectangles

$$\Delta x = \frac{b-a}{n}$$

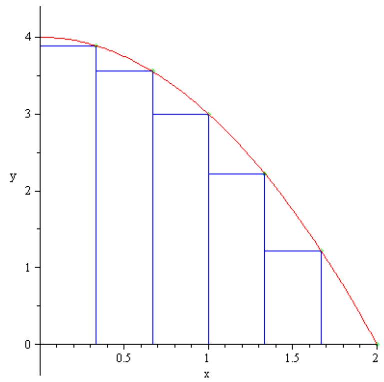
3. Create subinterval along the *x*-axis from $x = a$ to $x = b$.

4. The height of the rectangle, $f(c_i)$, is determined by which value of the subinterval you use, pick the right endpoint, the left endpoint or the midpoint of each subinterval.

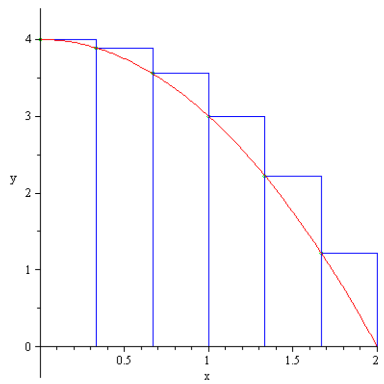
5. Sum the areas of the rectangles, $A \approx f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$.

Example: Estimate the area above

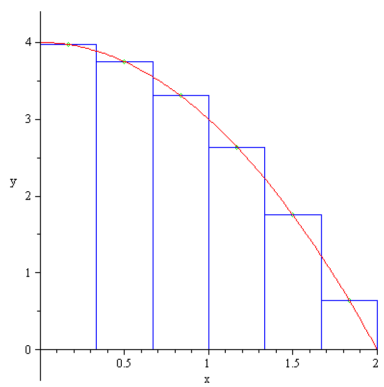
a. Using $n = 6$, and the right endpoint of the subinterval to determine height. Did we under estimate the area or over estimate the area?



b. Using $n = 6$, and the left endpoint of the subinterval to determine height. Did we under estimate the area or over estimate the area?



c. Using $n = 6$, and the midpoint of the subinterval to determine height.



d. What happens if we increase n?

Definition **Regular Partition**

Suppose $[a,b]$ is a closed interval containing n subintervals

$$[x_0,x_1],[x_1,x_2],\ldots[x_{n-1},x_n]$$

of equal lengths $\Delta x=\dfrac{b-a}{n}$ with $a=x_0$ and $b=x_n$. The endpoints $x_0,x_1,\ldots,x_{n-1},x_n$ of the subintervals are called **grid points** and they create a regular partition of the interval $[a,b]$. In general, the *k*th grid points is

$$x_k=a+k\Delta x \quad \text{for } k=0,1,2,\ldots n$$

Definition **Riemann Sum**

Suppose *f* is defined on a closed interval $[a,b]$, which is divided into n subintervals of equal length Δx. If $\overline{x}_k$ is any point in the *k*th subinterval $[x_{k-1},x_k]$, for $k=1,2,\ldots,n$ then

$$\sum_{k=1}^{n} f\left(\overline{x}_k\right)\Delta x = f\left(\overline{x}_1\right)\Delta x + f\left(\overline{x}_2\right)\Delta x+\cdots+f\left(\overline{x}_n\right)\Delta x$$

is called the Riemann Sum.

Algebra Rules for Finite Sums

Constant Multiple Rule:	$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, where c is a constant
Sum/Difference Rule	$\sum_{k=1}^{n}\left(a_k \pm b_k\right)=\sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Sums of Powers of Positive Integers

Sum of a constant c	$\sum_{k=1}^{n} c = cn$
Sum of first n integers:	$\sum_{k=1}^{n} k=\dfrac{n(n+1)}{2}$
Sum of squares of first n integers:	$\sum_{k=1}^{n} k^2=\dfrac{n(n+1)(2n+1)}{6}$
Sum of cubes of first n integers:	$\sum_{k=1}^{n} k^3=\left(\dfrac{n(n+1)}{2}\right)^2$

Example: Evaluate the sum $\sum_{k=1}^{6}\left(3-k^2\right)$

Example: Evaluate the sum $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k \right)^3$

Example:
Find a formula for the upper sum of $f(x) = x^2 + 1$ obtained by dividing the interval $[0,3]$ into n subintervals. Then take a limit of the sum as $n \to \infty$ to calculate the area under the curve over $[0,3]$.

Solution:

a. Width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$.

b. The partition P is $\{x_0, x_1, x_2, \ldots, x_n\}$ where $x_0 = a = 0$ and $x_k = a + k\Delta x = k\frac{3}{n}$. Thus our partition is $\left\{0, \frac{3}{n}, \frac{6}{n}, \ldots, 3\right\}$.

c. Now evaluate *f* at the right endpoint of each subinterval, $f\left(\frac{3k}{n}\right) = \left(\frac{3k}{n}\right)^2 + 1 = \frac{9}{n^2}k^2 + 1$.

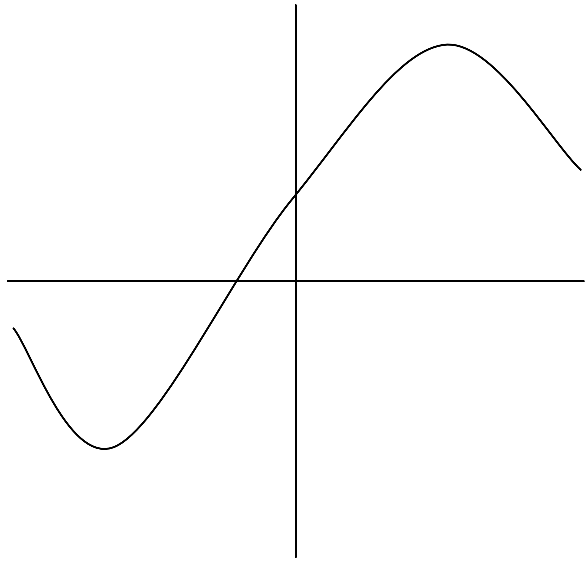
d. So the Riemann sum is:

$$
\begin{aligned}
S_p &= \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \left(\frac{9}{n^2}k^2 + 1\right)\frac{3}{n} \\
&= \sum_{k=1}^{n} \left(\frac{27}{n^3}k^2 + \frac{3}{n}\right) \\
&= \frac{27}{n^3}\sum_{k=1}^{n} k^2 + \frac{3}{n}\sum_{k=1}^{n} 1 \\
&= \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{3}{n}n \\
&= \frac{9}{2}\left(\frac{2n^3 + 3n^2 + n}{n^3}\right) + 3 \\
&= \frac{9}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 3 = 12 + \frac{27}{2n} + \frac{9}{2n^2}
\end{aligned}
$$

e. So what happens as n increases?

Section 5.2 Definite Integrals

Consider the following:



Riemann Sum

For a function, *f* on the closed interval $[a,b]$ Let's find the net area of the region between the *x*-axis and the curve.

Subdivide the interval $[a,b]$ into *n* subintervals, not necessarily of equal widths. So choose $n-1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between $a = x_0$ and $b = x_n$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The set $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is the set of grid points for the regular a partition of $[a,b]$. The partition *P* divides $[a,b]$ into *n* subintervals

$$[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$$

The **kth subinterval of *P*** is $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots n$.

In each subinterval, we select some point $x = \overline{x}_k$. The "height" of each rectangle is $f(\overline{x}_k)$. (Notice, for our problem some of the $f(\overline{x}_k)$ will be negative and some will be positive depending on whether *f* is above or below the *x*-axis.)

We form the product $f\left(\overline{x}_k\right)\Delta x$. Finally we the sum all of these products to get

$$S_p = \sum_{k=1}^{n} f\left(\overline{x}_k\right)\Delta x.$$

The sum S_p is called the **Riemann sum for *f* on the interval $[a,b]$.** (Notice the sum can be positive, negative or zero.) If the function $f(x) \geq 0, \ \forall x \in [a,b]$, then the Riemann sum is the approximate area between the curve and the *x*-axis from $x = a$ to $x = b$.

Definition:
Let $f(x)$ be a function defined on a closed interval $[a,b]$. Then the **definite integral of *f* over** $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\overline{x}_k\right)\Delta x$$

if the limit exists.

Comments:

- $\int_a^b f(x)\,dx$ is read the integral from *a* to *b* of *f* of *x* dee *x* or sometimes as the integral from *a* to *b* of *f* of *x* with respect to *x*.
- Recall from section 4.8, $\int$ is the integral sign, $f(x)$ is the integrand of the integral and *x* is the variable of integration.
- a is the lower limit of integration and *b* is the upper limit of integration.
- When the definition is satisfied, we say the Riemann sums of *f* on $[a,b]$ converge to the definite integral $I = \int_a^b f(x)\,dx$ and that *f* is integrable over $[a,b]$
- The value of the definite integral depends on the function not the letter we choose to represent the variable of integration. So if $\int_a^b f(x)\,dx = I$, then $\int_a^b f(t)\,dt = I$ or $\int_a^b f(u)\,du = I$. The variable of integration is called a dummy variable.

Theorem 5.2 **Integrable Function**
If *f* is continuous on $[a,b]$ or bounded on $[a,b]$, with a finite number of discontinuities, then f is integrable on $[a,b]$

Comments:

- Discontinuous functions may or may not be integrable.
- For instance, suppose a function is piecewise-continuous (discontinuous at a finite number of points), it is integrable.
- To fail, a function needs to be sufficiently discontinuous so that the region between the graph and the function cannot be approximated by increasingly thin rectangles.

Integrals with geometry:

Properties of Definite Integrals When *f* and *g* are integrable, then		
Order of Integration:	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	(definition)
Zero Width Interval	$\int_a^a f(x)\,dx = 0$	(definition)
Constant Multiple:	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, for any number *k*	
Sum and Difference:	$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
Additivity:	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
Max-Min Inequality:	If *f* has a maximum value *M* and a minimum value *m* on $[a,b]$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$	
Domination:	$f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ $f(x) \ge 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(special case)

Example: Suppose that $\int_{-3}^{0} g(t)\,dt = 2$ Find

a. $\int_{0}^{-3} g(t)\,dt$

b. $\int_{-3}^{0} g(u)\,du$

c. $\int_{-3}^{0} \frac{g(t)}{4}\,dt$

d. Also suppose $\int_{0}^{2} g(t)\,dt = 3$, find $\int_{-3}^{2} g(t)\,dt$.

e. Also suppose $\int_{-3}^{0} h(t)\,dt = 5$, find $\int_{-3}^{0} \left(g(t) - 3h(t)\right)dt$

Section 5.3 Fundamental Theorem of Calculus

Definition: **Area Function**
Let *f* be a continuous function for $t \geq a$. The **area function for *f* with left endpoint *a*** is

$$A(x) = \int_{a}^{x} f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the *t*-axis on the interval $[a, x]$.

Theorem 5.3 **Fundamental Theorem of Calculus, Part 1:**
If *f* is continuous on $[a,b]$ then the area function $A(x) = \int_{a}^{x} f(t)\,dt$ for $a \leq x \leq b$ is continuous on $[a,b]$ and differentiable on (a,b). The area function satisfies $A'(x) = f(x)$ or equivalently,

$$A'(x) = \frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x),$$

which means that the area function of *f* is an antiderivative of *f.*

Example: Find $\frac{dy}{dx}$

1. $y = \int_{0}^{x} \sqrt{1+t^2}\,dt$

2. $y = \int_0^{\sqrt{x}} \cos t\,dt$

3. $y = \int_{\tan x}^{0} \frac{1}{1+t^2}\,dt$

Theorem 5.3 (Continued) The Fundamental Theorem of Calculus, Part 2:
If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Comments:

- $\frac{d}{dx}\int_a^x f(t)\,dt = \frac{dF}{dt} = f(x)$, if you integrate *f* and then differentiate the results you get *f* back.
- $\int_a^x \frac{dF}{dt}\,dt = \int_a^x f(t)\,dt = F(x) - F(a)$, if you differentiate *F* and then integrate the result, you get back *F*.
- Find $\frac{d}{dx}\int_a^b f(t)\,dt$.

Example: Evaluate

1. $\int_{-3}^{4}\left(5 - \frac{x}{2}\right)dx$

2. $\int_0^{\pi/3} 2\sec^2 x\,dx$

3. $\int_{9}^{4} \frac{1-\sqrt{x}}{\sqrt{x}}\,dx$

4. $\int_{-4}^{4} |x|\,dx$

To find Total Area:

$$TA = \int_a^b \left| f(x) \right| dx$$

1. Find the zeros of $f(x)$ and subdivide the interval $[a,b]$ at the zeros.
2. Integrate *f* over each subinterval.
3. Add the absolute values of the integrals.

Example: Find the total area between $y = -x^2 - 2x$, $-3 \le x \le 2$ and the *x*-axis.

Section 5.4 Working with Integrals

Theorem 5.4 **Integrals of Even and Odd Functions**
Let *a* be a real number and let *f* be an integrable function on the interval $[-a,a]$

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

Example: Evaluate

1. $\int_{-1}^{1} \left(x^3 + 2x^2 - 3\right) dx$

2. $\int_{-2}^{2} \frac{x^3 - 4x}{x^2 + 1} dx$

3. $\int_{-\pi/4}^{\pi/4} \tan x dx$

Definition **Average Value of a Function**
The average value of an integrable function f on the interval $[a,b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example: Find the average value of the function $f(x) = \frac{1}{x}$ on $[1,e]$.

Theorem 5.5 **Mean Value Theorem for Integrals**
Let f be continuous on the interval $[a,b]$. There exists a point c in $[a,b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example Find or approximate the point at which the function $f(x) = \frac{\pi}{4}\sin x$ equals its average value on $[0,\pi]$.

Section 5.5 Substitution Rule

Recall from section 4.8, we know the indefinite integral of many functions; for instance,

$$\int x\,dx = \frac{x^2}{2} + C \text{ or } \int \left(\cos(4x) + e^{6x}\right)dx = \frac{1}{4}\sin(4x) + \frac{1}{6}e^{6x} + C.$$

Suppose we want to find $\int \cos x \sin x\, dx$ or $\int x\left(x^2+6\right)^5 dx$. We do not have any information from section 4.8 to help us solve these. However, we may recall that $\frac{d}{dx}\left(\sin^2 x\right) = 2\sin x \cos x$, so $\int \cos x \sin x\, dx = \frac{1}{2}\sin^2 x + C$. In this case, the integrand is the answer attained when we used the chain rule for $\frac{1}{2}\sin^2 x + C$.

> Theorem 5.6 **Substitution Rule**
> Let $u = g(x)$, where $g'(x)$ is continuous on an interval, and let f be continuous on the corresponding range of *g*. On that interval,
> $$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Recall from the chain rule that

$$\begin{aligned}\frac{d}{dx}\left(F(g(x))\right) &= F'(g(x))\,g'(x) \\ &= f(g(x))\,g'(x) \qquad \text{since } F' = f\end{aligned}$$

So

$$\begin{aligned}\int f(g(x))\,g'(x)\,dx = \int \frac{d}{dx}\left(F(g(x))\right)dx &= F(g(x)) + C \quad \text{by FTC} \\ &= F(u) + C \qquad \text{substitute } u \text{ for } g(x) \\ &= \int F'(u)\,du = \int f(u)\,du\end{aligned}$$

Examples: Evaluate

1. $\int x\left(x^2+6\right)^5 dx$

2. $\int \sqrt{3-2x}\,dx$

3. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

4. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

5. $\int \tan x\,dx$

6. $\int (\sin 2x) e^{\sin^2 x}\,dx$

7. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx$

8. $\int \frac{x}{\sqrt[3]{x+4}}\,dx$

Integrals of $\sin^2 x$ and $\cos^2 x$, to evaluate these we will need to use trigonometric identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2}$$

9. $\int \sin^2 x\,dx$

10. $\int \cos^2 x\,dx$

Theorem 5.7 **Substitution Rule for Definite Integrals**
Let $u = g(x)$, where g' is continuous on the interval $[a,b]$ and *f* is continuous on the range of *g*, then

$$\int_a^b f\left(g(x)\right)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Examples: Evaluate

1. $\int_0^3 \sqrt{y+1}dy$

2. $\int_0^{\pi} \frac{\sin t}{2-\cos t}dt$

3. $\int_1^2 \frac{2\ln x}{x}dx$

Areas Between Curves

Definition:
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$ then the **area of the region between the curves** $\boldsymbol{y = f(x)}$ **and** $\boldsymbol{y = g(x)}$ **from** $\boldsymbol{a}$ **to** $\boldsymbol{b}$ is the integral of $(f - g)$ from a to b:

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Example: Find the area enclosed by $y = x^2 - 2x$ and $y = x$.

Example: Find the area enclosed by $y = \sin\left(\frac{\pi x}{2}\right)$ and $y = x$.

Example: Find the area enclosed by $x = y^2$ and $x = y + 2$.