Chapter 2 Derivatives

Section 2.1 An Intuitive Introduction to Derivatives

Consider a function:

Slope function:
- Derivative, $f'(x)$
- For each $x$, the slope of $f(x)$ is the height of $f'(x)$
- Where $f$ has a horizontal tangent line, the derivative $f'$ has a root.
- Where the graph of $f$ is increasing, the derivative $f'$ is above the $x$-axis.
- Where the graph of $f$ is decreasing, the derivative $f'$ is below the $x$-axis.
- Where $f$ has steep slope, the derivative $f'$ has large magnitude
- Where $f$ has shallow slope, the derivative $f'$ has small magnitude

Example (27, page 166) Sketch a graph of the associated slope function $f'$.  

![Graph of $f'$]
Example (32, page 166) The graph below is $f'$, sketch a possible graph of $f$.

![Graph](image)

Average Velocity:
For the function, $y = f(x)$, let’s look at the slope of the secant line from $x = a$ to $x = b$.

Now the slope of the secant line tells us the average rate change of the function from $x = a$ to $x = b = a + h$.

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h}$$

Suppose $s = f(t)$ where $s$ is the displacement (directed distance) from origin at time $t$ and $f$ is the position function (describes the motion) of the object. In the time interval from $t_1 = a$ to $t_2 = a + h$, the average rate of change would tell us the average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f(a + h) - f(a)}{h}$$
Consider the position function \( s(t) = -4.9t^2 + 30t + 20 \), what is the average rate of change for each of the following:

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>([2, 3])</th>
<th>([2, 2.5])</th>
<th>([2, 2.1])</th>
<th>([2, 2.01])</th>
<th>([2, 2.001])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Velocity</td>
<td></td>
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</table>

What happens to the slope of the secant line as \( h \to 0 \)?

The instantaneous rate of change of a function is the slope of \( f \) at \( x = c \). This is also the derivative of \( f \) at \( x = c \). Let's look at a graph of the function and the various secant lines from above. What would the slope of the tangent line be? What would you estimate the instantaneous velocity at \( t = 2 \)?
Example (#51, page 167)
Every morning Lynda takes a thirty-minute jog in Central Park. Suppose her distance \( s \) in feet from the oak tree on the north side of the park \( t \) minutes after she begins her jog is given by the function \( s(t) \), shown below on the left and suppose she jogs on a straight path leading into the park from the oak tree.

![Graph of distance from the oak tree and the post office](image)

a. What was the average rate of change of Lynda’s distance from the oak tree over the entire thirty-minute jog? What does that mean in the real world?

b. On which ten-minute interval was the average rate of change of Lynda’s distance from the oak tree the greatest: the first ten minutes, the second ten minutes the last ten minutes?

c. Use the graph of \( s(t) \) to estimate Lynda’s average velocity during the 5-minute interval from \( t = 5 \) to \( t = 10 \). What does the sign of this average velocity tell us?

d. Approximate the times at which Lynda’s (instantaneous) velocity was equal to zero. What is the physical significance of these times?

e. Approximate the time intervals during Lynda’s jog that her (instantaneous) velocity was negative. What does a negative velocity mean in terms of this physical exam?
Definition 2.1 The Derivative of a Function at a Point
The derivative \( x = c \) of a function \( f \) is the number
\[
 f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}
\]
or equivalently
\[
 f'(c) = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}
\]
provided the limit exists.

Example: Find the derivative of the function \( f(x) = \sqrt{x} \) at \( x = 4 \).

Definition 2.2 The Derivative of a Function
The derivative of a function \( f \) is the function \( f' \) defined by
\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
or equivalently
\[
 f'(c) = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}
\]
The domain of \( f' \) is the set of values \( x \) for which the defining limit of \( f' \) exists.

Example:
Find the derivative of \( f(x) = \frac{1}{x+2} \).

Definition 2.3 Differentiability at a Point
A function \( f \) is differentiable at \( x = c \) if
\[
 \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}
\]
exists.
Definition 2.4 One-Sided Differentiability at a Point

The left derivative and right derivative of a function $f$ at a point $x = c$ are respectively, equal to the following, if they exist:

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h}, \quad f'_+(c) = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h}$$

Theorem 2.5 Differentiability Implies Continuity

If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

Theorem (Alternative Version) Not Continuous Implies Not Differentiable

If $f$ is not continuous at $a$, then $f$ is not differentiable at $a$.

When is a Function Not Differentiable at a Point?

A function $f$ is not differentiable at $x = c$ if at least one of the following conditions holds:

a. $f$ is not continuous at $x = c$.

b. $f$ has a corner at $x = c$.

c. $f$ has a vertical tangent at $x = c$.

Example:
Show that the function $y = |x|$ has no derivative at $x = 0$. 
Note: If \( f'(x) \) does not exist

- At a corner \( \lim_{x \to c^-} f'(x) = l_1 \) and \( \lim_{x \to c^+} f'(x) = l_2 \) where \( l_1 \neq l_2 \).
- At a vertical tangent:
  - (Cusp) \( \lim_{x \to c^-} f'(x) = \infty \) and \( \lim_{x \to c^+} f'(x) = -\infty \) (or \( \lim_{x \to c^-} f'(x) = -\infty \) and \( \lim_{x \to c^+} f'(x) = \infty \))
  - \( \lim_{x \to c^-} f'(x) = \infty \) and \( \lim_{x \to c^+} f'(x) = \infty \) (or both equal \( -\infty \))

**Theorem 2.6 Equation of the Tangent Line to a Function at a Point**

The **tangent line** to the graph of a function \( f \) at a point \( x = c \) is defined to be the line passing through \((c, f(c))\) with slope \( f'(c) \), provided that the derivative \( f'(c) \) exists. This line has equation

\[
y = f(c) = f'(c)(x - c)
\]

**Definition 2.7 Local Linearity**

If \( f \) has a well-defined derivative \( f'(c) \) at a point \( x = c \), then for values of \( x \) near \( c \), the function \( f(x) \) can be approximated by the tangent line to \( f \) at \( x = c \) with the linearization of \( f \) around \( x = c \) by

\[
f(x) \approx f(c) + f'(c)(x - c)
\]

Other notation: \( f'(x) = y'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f(x)) \)
Section 2.3 Rules for Calculating Basic Derivatives

Theorem 2.8 Derivatives of Constant, Identity, and Linear Functions
For any real numbers \( k, m, \) and \( b \)

1. \( \frac{d}{dx}(k) = 0 \)

2. \( \frac{d}{dx}(x) = 1 \)

3. \( \frac{d}{dx}(mx + b) = m \)

Example: Find the derivative of the following:

1. \( \frac{d}{dx}(3) \)

2. \( f(x) = e^x \)

3. \( f(x) = 7x + 99 \)

Theorem 2.9 Power Rule
For any nonzero rational number \( k, \) \( \frac{d}{dx}(x^k) = kx^{k-1} \)

Proof: Let \( k \) be a positive integer. Then

\[
\frac{d}{dx}(x^k) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^k - x^k}{z - x} \\
= \lim_{z \to x} \left( z - x \right) \left( z^{k-1} + z^{k-2}x + \cdots + zx^{k-2} + x^{k-1} \right) \\
= \lim_{z \to x} \left( z^{k-1} + z^{k-2}x + \cdots + zx^{k-2} + x^{k-1} \right) \\
= kx^{k-1}
\]

Example: Find the derivative for each of the following:

1. \( f(x) = x^2 \)
2. \( f(x) = x^5 \)

3. \( f(x) = \sqrt{x} \)

4. \( f(x) = \frac{1}{x} \)

**Theorem 2.10 Constant Multiple Rule**

If \( f \) is a differentiable function of \( x \), and \( k \) is a constant, then

\[
\frac{d}{dx}[kf(x)] = k \frac{df}{dx} = kf'(x)
\]

**Example:** Find the derivative for each of the following:

1. \( f(x) = 7x^4 \)

2. \( f(x) = 2\sqrt{x} \)

3. \( f(x) = \sqrt{5x} \)

**Theorem 2.10 Sum Rule**

If \( f \) and \( g \) are differentiable functions of \( x \), then

\[
\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))
\]

**Example:** Find the derivative of

1. \( f(x) = 5x^2 + 6x + \pi \)
2. \( f(x) = 3\sqrt{x} + \frac{1}{2x^2} \)

3. \( y = \frac{12s^3 - 8s^2 + 12s}{4s} \)

**Theorem 2.11 Product Rule**
If \( f \) and \( g \) are differentiable at \( x \), then
\[
\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)
\]

Other notation: \( \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \)

**Example:** Find the derivative for each of the following:
1. \( f(x) = (2x^2 + 1)(x^3 + x) \)

2. \( f(x) = 10x^3(x^2 + 1) \)

**Theorem 2.11 The Quotient Rule**
If \( f \) and \( g \) are differentiable at \( x \), then the derivative of \( f/g \) at \( x \) exists provided \( g(x) \neq 0 \) and
\[
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\]
Other notation: \[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

One more way: \[ \frac{d}{dx} \left( \frac{\text{high}}{\text{low}} \right) = \frac{\text{low} \left( \frac{d \text{high}}{dx} \right) - \text{high} \left( \frac{d \text{low}}{dx} \right)}{(\text{low})^2} \]

**Example**  Find the derivative for each of the following:

1. \[ y = \frac{x^2 + 3x + 4}{x^2 + 2} \]

2. \[ f(x) = \frac{6x}{x^2 + 1} \]

Find \( \frac{dy}{dx} \) for each of the following:

1. \[ y = x \left( x^2 \right) \left( \sqrt{x} \right) \left( x^{2/3} \right) \]

2. \[ y = \frac{1 - 6x}{3} \]

3. \[ y = (3x + 2)^3 \]

4. \[ y = \frac{2x - 3}{5x + 4} \]

5. \[ y = |x^2 - 1| \]
Higher-Order Derivatives

Definition: Higher-Order Derivatives
Assuming \( f \) can be differentiated as often as necessary, the second derivative of \( f \) is

\[
f''(x) = f^{(2)}(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = y'' = D^2(f(x))
\]

For integers \( n \geq 1 \), the \( n \)th derivative is

\[
f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d}{dx} \left( f^{(n-1)}(x) \right)
\]

Example: Find the following: \( f(x) = 3x^2 + 6x + 4 \)

1. \( f'(x) = \)

2. \( f''(x) = \)

3. \( f'''(x) = \)

Example: Find the following if \( f(x) = 3\sqrt{x} \)

1. \( \frac{d}{dx}(3\sqrt{x}) \)

2. \( \frac{d^2}{dx^2}(3\sqrt{x}) \)

3. \( \frac{d^3}{dx^3}(3\sqrt{x}) \)
Theorem 2.12 The Chain Rule
Suppose \( f(u(x)) \) is a composition of functions. Then for all values of \( x \) at which \( u \) is differentiable at \( x \) and \( f \) is differentiable at \( u(x) \), the derivative of \( f \) with respect to \( x \) is equal to the product of the derivative of \( f \) with respect to \( u \) and the derivative of \( u \) with respect to \( x \).

In “prime” notation, we write it as
\[
\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}
\]

In Leibniz notation we write it at
\[
\frac{dy}{dx} = \frac{d}{dx}(f(u(x))) = f'(u(x))u'(x)
\]

Note we can translate the following in words; differentiate the “outside” function and evaluate it at the “inside” function, then multiply by the derivative of the “inside” function.

Example:
Find the derivatives of the following functions:

1. \( y = (x^2 + 1)^6 \)

2. \( y = \sqrt{5x^4 + 7x^3} \)

3. \( y = \frac{1}{x^{1/3} - 2x} \)

4. \( y = \left(5(3x^4 - 1)^3 + 3x - 1\right)^{100} \)
5. \( y = (1 - 4x)^2 \left( 3x^2 + 1 \right)^9 \)

6. \( f(x) = \left( x\sqrt{2x+1} \right)^{-2} \)

7. \( y = \frac{(x+1)(3x-4)}{\sqrt{x^3 - 27}} \)

8. \( y = x\sqrt{3x^2 + 1} + \sqrt[3]{2x+5} \)

Example: Determine an equation of the line tangent to the graph of \( y = x\sqrt{5-x^2} \) at the point (1,2).

**Implicit Differentiation**

In the previous sections, we have mostly dealt with equations of the form \( y = f(x) \) that expresses \( y \) explicitly in terms of the variable \( x \). (In other words, \( y \) is the dependent variable and is a function of only the independent variable \( x \).) However, what happens if we have an implicit relation between the variables \( y \) and \( x \). Look at the following examples:

\[
\begin{align*}
  x^3 + y^3 &= 18xy \\
  x^2 y - y^2 x &= x^2 + 3 \\
  \sqrt{3y-1} &= 5xy
\end{align*}
\]
Implicit Differentiation

1. Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.
2. Collect the terms with $dy/dx$ on one side of the equation.
3. Solve for $dy/dx$.

**Example:** Find $dy/dx$ for $x^2 + y^2 = r^2$.

**Example:** Find $dy/dx$ for $x^3 + y^3 = 18xy$.

**Example:** Find $dy/dx$ for $x^2y - y^2x = x^2 + 3$

**Example:** Find $dy/dx$ for $\sqrt{3y-1} = 5xy$

**Example:** Use implicit differentiation to find $dy/dx$ and then find $d^2y/dx^2$ for $2\sqrt{y} = x - y$

**Example:** For the equation $x^2y^2 = 9$, find the line that is tangent to the curve at $(-1,3)$
Section 2.5 Derivatives of Exponential and Logarithmic Functions

Theorem 2.13 Derivatives of Exponential Functions
For any constant \( k \), any constant \( b > 0 \) with \( b \neq 1 \), and all \( x \in \mathbb{R} \),

1. \( \frac{d}{dx}(e^x) = e^x \)
2. \( \frac{d}{dx}(b^x) = (\ln b) b^x \)
3. \( \frac{d}{dx}(e^{kx}) = ke^{kx} \)

Example: Find the derivatives of the following:
1. \( y = 5e^x \)
2. \( y = e^{6x} \)
3. \( y = e^{x^2} \)

Inverse Properties for \( e^x \) and \( \ln x \)
1. \( e^{\ln x} = x \) for \( x > 0 \), and \( \ln(e^x) = x \) for all \( x \).
2. \( y = \ln x \) if and only if \( x = e^y \)
3. For real numbers \( x \) and \( b > 0 \), \( b^x = e^{(\ln b)^x} = e^{x \ln b} \)
4. \( y = 2^x \)
5. \( y = 7^{x^2} \)

Theorem 2.14 Rates of Change and Exponential Functions
\( f'(x) = kf(x) \) for some constant \( k \) if and only if \( f \) is an exponential function of the form \( f(x) = Ae^{kx} \)
Example: Find the derivative of \( y = \ln x \).

Example: Find the derivative of

1. \( y = \ln 3x \)

2. \( y = \ln \left( x^{\sqrt{2}} \right) \).

3. \( y = \sqrt{\ln x^2} \)

4. \( y = \log_7 x \)

Theorem 2.15 Derivative of Logarithmic Functions

For any constant \( b > 0 \) with \( b \neq 1 \) and all appropriate values of \( x \),

1. \( \frac{d}{dx} (\ln x) = \frac{1}{x}, \quad x > 0 \)

2. \( \frac{d}{dx} (|x|) = \frac{1}{x}, \quad x \neq 0 \)

3. \( \frac{d}{dx} (\log_b x) = \frac{1}{x \ln b} \)

Suggestions:

\[
\frac{d}{dx} (e^u) = e^u \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx}
\]
Logarithmic Rules
For any base $b > 0 (b \neq 1)$ and positive real numbers $x$ and $y$ the following relations hold:

1. $\log_b (xy) = \log_b x + \log_b y$
2. $\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \quad \text{(includes } \log_b \frac{1}{x} = -\log_b x \text{)}$
3. $\log_b (x^y) = y \log_b x$
4. $\log_b b = 1$

Logarithmic Differentiation
1. Take the natural logarithm of both sides of the equation.
2. Simplify using the laws of logarithms.
3. Take the derivatives of both sides with respect to $x$.
4. Solve for $dy/dx$.

Example: Find $dy/dx$ for the following:

1. $y = \sqrt[6]{(x^2+1)(x-1)^2/(x^3+27)^6}$

2. $y = x^y$

Theorem 2.16 Derivatives of Inverse Function
If $f$ and $f^{-1}$ are inverse functions and both are differentiable, then for all appropriate values of $x$,

$$ (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} $$
Example: Given \( f(x) = (x+2)^2 \) for \( x \geq -2 \), find the slope of the line tangent to the graph of \( f^{-1} \) at the point \((36,4)\).

### Section 2.6 Derivatives of Trigonometric and Hyperbolic Functions

#### Theorem 2.17 Derivatives of Trigonometric Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>( \sec^2 x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( -\csc^2 x )</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \tan x \sec x )</td>
</tr>
<tr>
<td>( \csc x )</td>
<td>( -\csc x \cot x )</td>
</tr>
</tbody>
</table>

**Example:**

Show \( \frac{d}{dx} (\sin x) = \cos x \)

**Example:**

Show \( \frac{d}{dx} (\tan x) = \sec^2 x \)

**Example:**

Find the derivative for each of the following

1. \( y = -10x + 3\cos x \)

2. \( f(x) = (\sin x + \cos x) \sec x \)
3. \( y = x^2 \cos x - 2x \sin x - 2 \cos x \)

4. \( y = \sin(x^2) \)

5. \( y = \sin^2 x \)

**Example:**
Find \( y^{(4)} = d^4 y / dy^4 \) if \( y = 9 \cos x \).

**Example:**
Find the horizontal tangents for \( y = x + 2 \cos x \) on \( 0 \leq x \leq 2\pi \).

**Example:**
Find the derivative of \( y = \arcsin x \).

**Example:**
Find the derivative of \( y = \arctan x \).
Example:
Find the derivative of \( y = \arctan(\ln x) \).

Example: Find the derivative of \( y = \arcsin(6x) \)

**Theorem 2.18 Derivative of Inverse Trigonometric Functions**

1. \( \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1 \)
2. \( \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \)
3. \( \frac{d}{dx}(\text{arcsec} \, x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1 \)

Combining two exponential functions forms hyperbolic functions:

**Hyperbolic Functions:** (See page 228 for graphs)

- Hyperbolic sine: \( \sinh x = \frac{e^x - e^{-x}}{2} \)
- Hyperbolic cosecant: \( \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \)
- Hyperbolic cosine: \( \cosh x = \frac{e^x + e^{-x}}{2} \)
- Hyperbolic secant: \( \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \)
- Hyperbolic tangent: \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)
- Hyperbolic cotangent: \( \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \)

Trigonometric functions are sometimes referred to as circular functions. If we look at the unit circle, \( x^2 + y^2 = 1 \), then any point \( (x, y) = (\cos \theta, \sin \theta) \), where \( \theta \) is measured in radians. Also recall that \( \cos^2 x + \sin^2 x = 1 \).

Hyperbolic functions get their name because if we look at the right hand side of the hyperbola, \( x^2 - y^2 = 1 \), then any point \( (x, y) = (\cosh t, \sinh t) \). Note the first identity for hyperbolic functions is \( \cosh^2 t - \sinh^2 t = 1 \). Note in the diagram below, \( t \) does not represent the angle. However, it turns out \( t \) represents twice the area of the sector bounded by the hyperbola, \( x \)-axis,
and line. Hyperbolic functions are used for instance to model the wire hanging between two poles (assuming it is attached to both polls at the same height), then the shape of the curve can be modeled with $y = c + a \cosh(x/a)$ called a catenary.

\[ x^2 + y^2 = 1 \]

\[ x^2 - y^2 = 1 \]

**Identities for Hyperbolic Functions**

\[
\begin{align*}
\cosh^2 x - \sinh^2 x &= 1 \\
\sinh 2x &= 2 \sinh x \cosh x \\
\cosh^2 x &= \frac{\cosh 2x + 1}{2} \\
\tanh^2 x &= 1 - \sech^2 x \\
\cosh 2x &= \cosh^2 x + \sinh^2 x \\
\sinh^2 x &= \frac{\cosh 2x - 1}{2}
\end{align*}
\]

**Example:** Find the derivative $\frac{dy}{dx}$ for

1. $y = \sinh x$

2. $y = \tanh x$

3. $y = \frac{1}{2} \sinh(2x+1)$
**Theorem 2.20 Derivatives of Hyperbolic Functions**

For all real numbers \( x \),

1. \( \frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx} \)
2. \( \frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx} \)
3. \( \frac{d}{dx}(\tanh u) = \text{sech}^2 u \frac{du}{dx} \)

**Example:** Find the derivative of the following:

1. \( y = x \sinh^3 x \)

2. \( y = 3 \tanh^2 e^x \)

3. \( y = \frac{\tanh \sqrt{x}}{\sqrt{\cosh x}} \)

**Inverse Hyperbolic Functions**

| \( y = \sinh^{-1} x \) \( \iff \) \( \sinh y = x \) | \( \sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right) \) \( x \in \mathbb{R} \) |
|-----------------------------------------------|
| \( y = \cosh^{-1} x \) \( \iff \) \( \cosh y = x, \text{ and } y \geq 0 \) | \( \cosh^{-1} x = \ln \left( x + \sqrt{x^2 - 1} \right) \) \( x \geq 1 \) |
| \( y = \tanh^{-1} x \) \( \iff \) \( \tanh y = x \) | \( \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \) \( -1 < x < 1 \) |

**Example:** Find \( \frac{dy}{dx} \) for \( y = \sinh^{-1} x \).
Theorem 2.21 Derivatives of Inverse Hyperbolic Functions

For all $x$ for which the following are defined,

1. \[ \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}} \]

2. \[ \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1 \]

3. \[ \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}, \quad |x| < 1 \]

Examples: Find the derivative of each of the following:

1. \( y = \sinh^{-1}(x^3) \)

2. \( y = \frac{\sinh^{-1} x}{\cosh^{-1} x} \)