Chapter 13 Multiple Integration

Section 13.1Double Integrals over Rectangular Regions

Recall the Definite Integral from Chapter 5

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If $f(x) \ge 0$ then $\int_a^b f(x)\,dx$ is the area under the curve (and above the *x*-axis) between $x = a$ and $x = b$.

Volume and Double Integral

Suppose $z = f(x, y)$ defined on $R = \{(x, y) | a \le x \le b, c \le y \le d\}$. If $f(x, y) \ge 0$, then $z = f(x, y)$ is a surface above the *xy*-plane over *R*.

Let *S* be the solid that lies above *R* and under *z*.

$$S = \{(x, y, z) | 0 \le z \le f(x, y), (x, y) \in R\}$$

Subdivide the region *R* into rectangles, Δx, length and Δy, width, then area of each rectangle, R_{ij}, is $\Delta A = \Delta x \Delta y$ now the approximate height of the solid above R_{ij} is $f(x_{ij}^*, y_{ij}^*)$ where (x_{ij}^*, y_{ij}^*) is a point in R_{ij} and $\Delta V \approx f(x_{ij}^*, y_{ij}^*)\Delta A$.

So $V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$ and $V = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$.

Definition: The double integral of *f* over *R* is

$$\iint_R f(x, y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

If the limit exists.

If $f(x, y) \ge 0$ then the volume of the solid S is $V = \iint_R f(x, y)\,dA$.

To evaluate the double integral over *R*:

Recall from Chapter 6, to find the volume of a solid of revolution we made slices perpendicular to the *x*-axis,

$$V = \int_{x=a}^{x=b} A(x)\,dx$$

Where $A(x)$ is the cross-sectional area at *x*. Now the area of the cross-section is:

$$A(x) = \int_{y=c}^{y=d} f(x,y)\,dy$$

To evaluate hold *x* fixed and integrate with respect to *y* (partial integration with respect to *y*).

So $V = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_a^b \int_c^d f(x,y)\,dydx$

Example: Find the volume of the solid under $f(x,y) = x^2 + y^2$ and over the rectangle $R = \{(x,y) | 2 \le x \le 4, -1 \le y \le 1\}$.

Theorem 13.1: (Fubini) Double Integrals on Rectangular Regions
Let $f(x,y)$ be continuous on the rectangular region *R*: $a \le x \le b, c \le y \le d$. The double integral of *f* over *R* may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx.$$

Example: Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = (x+y)^{-2}$ and $R = \{(x,y) | 0 \le x \le 1, 1 \le y \le 2\}$

Example: Evaluate $\iint\limits_R \frac{x}{(1+xy)^2}\,dA$. Where $R=\{(x,y)\,|\,0\le x\le 4, 1\le y\le 2\}$

Note: $\iint\limits_R f(x,y)\,dA$ will be positive if $f(x,y)$ is above the region *R* on the *xy*-plane; it is negative if $f(x,y)$ is below the region *R* on the *xy*-plane; it can also be zero.

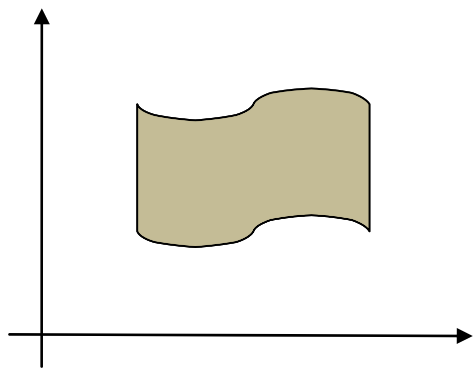
Definition: **Average Value of a Function Over a Plane Region**
The average value of an integrable function *f* over a region *R* is

$$\overline{f}=\frac{1}{\text{area of } R}\iint\limits_R f(x,y)\,dA$$

Example: Find the average value of the function $f(x,y)=\sin x\sin y$ over the region $R=\{(x,y)\,|\,0\le x\le \pi, 0\le y\le \pi\}$.

Section 13.2 Double Integrals over General Regions

Suppose $f(x, y)$ is defined over the bounded region *D* (i.e. *D* can be enclosed in a rectangular region *R*.



Let $F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\in R \text{ but not } D \end{cases}$ then

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

- Any *R* will do as long as it contains *D*.
- If $f(x,y)\geq 0$ then the $V=\iint_D f(x,y)\,dA$
- Note *F* may have discontinuities at the boundary of *D*. Nonetheless, if *f* is continuous on *D* and the boundary curve is well behaved then $\iint_R F(x,y)\,dA$ exist and therefore $\iint_D f(x,y)\,dA$ exists.

Type I:

Suppose $D=\{(x,y)\,|\,a\leq x\leq b, g_1(x)\leq y\leq g_2(x)\}$, then $\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$.

Example: Evaluate $\iint_D f(x,y)\,dA$ where $f(x,y)=x^2-y$ where *D* is the region bounded by $x=0$, $y=x$ and $y=2-x$.

Type II:

Suppose $D=\{(x,y)|c\le y\le d, h_1(y)\le x\le h_2(y)\}$ then $\iint\limits_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$

Example: Find the volume of the solid bounded above by $f(x,y)=x+y$ over the region bounded by $y=2x$, $x=1$ in the first quadrant.

Example: Evaluate $\iint\limits_D f(x,y)\,dA$ where $f(x,y)=2y^2+x$ where D is the region bounded by $y=2$, $y=-x$ and $y=x-2$.

Example: For $\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}\,dydx$, sketch the region of integration, reverse the order of integration and evaluate the integral.

Example: Find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 8 - 2x - 4y$.

Example: Find the volume of the wedge sliced from the cylinder $x^2 + y^2 = 1$ by the planes $z = 1 - x$ and $z = x - 1$.

Area by Double Integral

Definition:
The area of a closed bounded plane region R is

$$A = \iint_R dA$$

Example: Find the area of the region bounded by the curves $x = y^2 - 1$ and $x = 2y^2 - 2$.

Example: $\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$ gives the area of a region in the xy-plane. Sketch the region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Find the area.

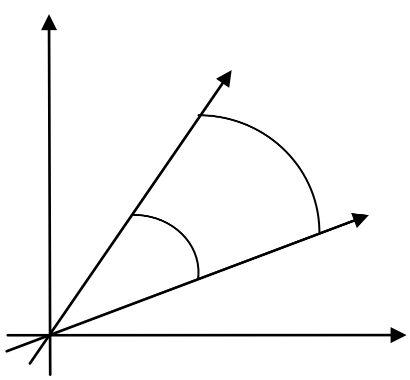
Section 13.3 Double Integrals in Polar Coordinates

Recall Polar Coordinates from Chapter 10

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \; y = r\sin\theta$$

Polar Rectangle

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$$



Recall the area of sector of a circle is

$$A = \frac{1}{2} r^2 \theta$$

What is the area of the polar rectangle?

$$\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f\left(r_i^* \cos\left(\theta_j^*\right), r_i^* \sin\left(\theta_j^*\right)\right) dA$$

$$= \lim_{n,m\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} g\left(r_i^*, \theta_j^*\right) r^* \Delta r \Delta\theta = \int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\, r\,dr\,d\theta$$

$$= \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta, \text{ where } \beta - \alpha \le 2\pi$$

Area in Polar Coordinates

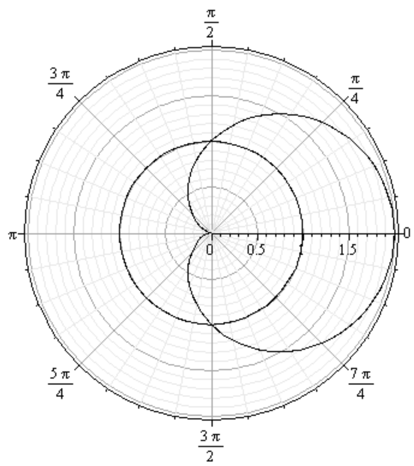
The area of a closed bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta$$

Example: Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dx\,dy$

Example: Evaluate $\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^{0} xy^2\,dx\,dy$

Example: Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$



Example: Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \le a^2$ in the *xy*-plane.

Section 13.4 Triple Integrals in Rectangular Coordinates

Triple Integral

$$\iiint_T f(x,y,z)\,dV = \lim_{|P|\to 0} \sum f(x,y,z)\,\Delta V$$

- *T* is a solid body with density function *f*.
- Limit exists as norm $|P|$ (diagonal of block) approaches zero provided *f* is continuous on *T* and that the boundary region of *T* is reasonably well behaved.

Example: Evaluate $\iiint_T f(x,y,z)\,dV$ for $f(x,y,z) = x + y + z$ and *T* is a rectangular box where $0 \le x \le 2, 0 \le y \le 3, 0 \le z \le 1$.

Suppose *T* is described by $z_1(x,y) \le z \le z_2(x,y),\ g_1(x) \le y \le g_2(x),\ a \le x \le b$ then

$$\iiint_T f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dzdydx$$

Example: Evaluate $\iiint_T f(x,y,z)\,dV$ for $f(x,y,z) = x + y$ and *T* is the region bounded between the surfaces $z = 2 - x^2$ $z = x^2$ and $0 \le y \le 3$.

Definition:
The volume of a closed, bounded region *D* in space is

$$V = \iiint_D dV$$

Example: Find the volume bounded by the surfaces $z = x^2$, $y + z = 4$, $y = 0$, and $z = 0$

Example: Find the volume bounded by the planes $z=0$, $x=0$, $y=2$, and $z=y-2x$.
Note: $dV = dzdydx = dzdxdy = dydzdx = dydxdz = dxdzdy = dxdydz$

Definition:
The average value of a function *F* over a region *D* in space is

$$f_{avg} = \frac{1}{V(D)} \iiint_D f(x,y,z)\,dV$$

Where $V(D)$ is the volume of the region *D*.

Section 13.5 Triple Integrals in Cylindrical and Spherical Coordinates

Definition:
Cylindrical coordinates represent a point P in space by ordered triples (r,θ,z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane.
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x,y,z) and Cylindrical (r,θ,z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

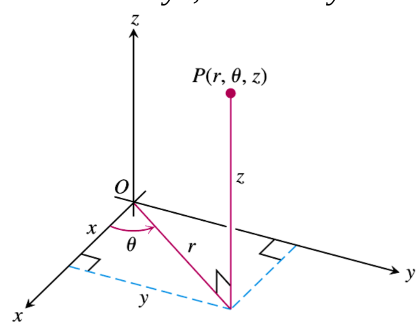


FIGURE 13.35 The cylindrical coordinates of a point in space are r, θ, and z.

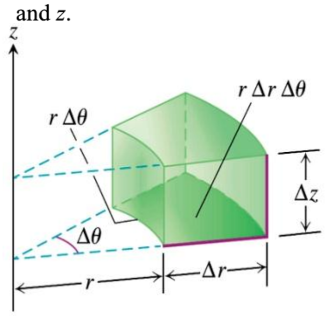


FIGURE 13.37 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

To integrate over $T = \{(x,y,z) | (x,y) \in D, h_1(x,y) \le z \le h_2(x,y)\}$ where D is given in polar coordinated by

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, g_1(\theta) \le r \le g_2(\theta)\}$$

Then $\iiint_T f(x,y,z)\,dV = \int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$

Example: Evaluate $\iiint_T x\,dV$, where T is enclosed by the planes $z=0$ and $z=x+y+3$ and by the cylinders $x^2+y^2=4$ and $x^2+y^2=9$

Spherical Coordinates

A point P in space is (ρ,ϕ,θ)

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0\le\phi\le\pi)$.
3. θ is the angle from cylindrical coordinates.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r=\rho\sin\phi,\quad x=r\cos\theta=\rho\sin\phi\cos\theta$$
$$z=\rho\cos\phi,\quad y=r\sin\theta=\rho\sin\phi\sin\theta$$
$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}$$

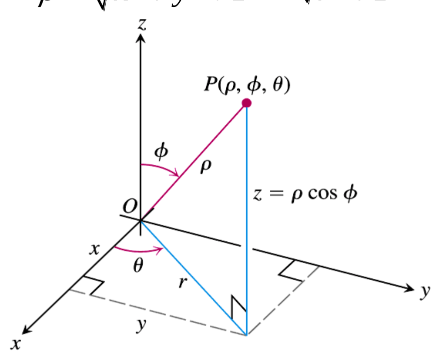


FIGURE 13.40 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

Note:

- $\rho=c$ is a sphere centered at the origin with radius c.
- $\theta=c$ is a half plane
- $\phi=c$ is either the top half of a cone if $0\le\phi\le\pi/2$ and the bottom half if $\pi/2\le\phi\le\pi$

Example: Plot the point $(2,\pi/3,\pi/4)$ then find the rectangular coordinates.