Chapter 1 Functions

Section 1.3 Inverse, Exponential and Logarithmic Functions

Exponential Functions

Properties of Exponential Function $f(x) = b^x$ for $b > 0, b \neq 1$

1. Because b^x is defined for all real numbers, the domain of *f* is $\{x : -\infty < x < \infty\}$. Because $b^x > 0$ for all values of *x*, the range of *f* is $\{y : 0 < y < \infty\}$.
2. For all $b > 0, b^0 = 1$ and thus $f(0) = 1$.
3. If $b > 1$, the *f* is an increasing function of *x*.
4. If $0 < b < 1$, the *f* is a decreasing function of *x*.

Graphs:

Exponential Rules
For any base $b > 0$ and real numbers *x* and *y*, we have

1. $b^x b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$ $\left(\text{which includes } \frac{1}{b^x} = b^{-x}\right)$
3. $(b^x)^y = b^{xy}$
4. $b^x > 0$ for all *x*.

Definition **The Natural Exponential Function**
The natural exponential function is $f(x) = e^x$, which has base $e = 2.718281828459\ldots$

Inverse Functions

Definition **Inverse Function**
Given a function *f*, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$.

Definition **One-to-One Functions and the Horizontal Line Test**
A function *f* is one-to-one on a domain *D* if each value of $f(x)$ corresponds to exactly one value of *x* in *D*. More precisely, *f* is one-to-one on *D* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for x_1 and x_2 in *D*. The horizontal line test says that every horizontal line intersects the graph of a one-to-one function at most once.

Theorem 1.1 **Existence of Inverse Function**
Let *f* be a one-to-one function on a domain *D* with range *R*. Then *f* has a unique inverse f^{-1} with domain *R* and range *D* such that

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

Where x is in *D* and *y* is in *R*.

Graphing the inverse function:

Procedure: **Finding an Inverse Function**
Suppose *f* is one-to-one on an interval *I*. To find f^{-1}

1. Solve $y = f(x)$ for *x*. If necessary, choose the function that corresponds to *I*.
2. Interchange *x* and *y* and write $y = f^{-1}(x)$.

Examples:

Definition **Logarithmic Function Base *b***

For any base $b > 0$, with $b \neq 1$, the logarithmic function base *b*, denoted $y = \log_b x$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the natural logarithmic function, denoted by $y = \ln x$.

Logarithmic Rules

For any base $b > 0 \left(b \neq 1\right)$ and positive real numbers *x* and *y* the following relations hold:

1. $\log_b\left(xy\right) = \log_b x + \log_b y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \quad \left(\text{includes } \log_b \frac{1}{x} = -\log_b x\right)$
3. $\log_b\left(x^y\right) = y \log_b x$
4. $\log_b b = 1$

Inverse Relations for Exponential and Logarithmic Functions

1. $y = \log_b x$ provided $b^y = x$
2. $b^{\log_b x} = x \quad \left(\text{for all } x > 0\right)$
3. $\log_b b^x = x \quad \left(\text{for all real values of } x\right)$

What do we know about the domain and range of the logarithmic function?

Change-of-Base Rules

Let *b* be a positive real number with $b \neq 1$. Then

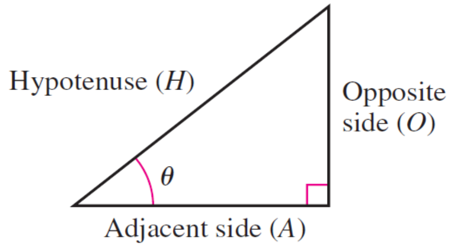
$$b^x = e^{x \ln b}, \text{ for all } x \text{ and } \log_b = \frac{\ln x}{\ln b} \text{ for } x > 0$$

More generally, if *c* is a positive real number with $c \neq 1$, then

$$b^x = c^{x \log_c b}, \text{ for all } x \text{ and } \log_b x = \frac{\log_c x}{\log_c b} \text{ for } x > 0.$$

Section 1.4 Trigonometric Functions and Their Inverses

Recall:



$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A} \qquad \cot\theta = \frac{A}{O}$$

$$\sec\theta = \frac{H}{A} \qquad \csc\theta = \frac{H}{O}$$

FIGURE 1.58

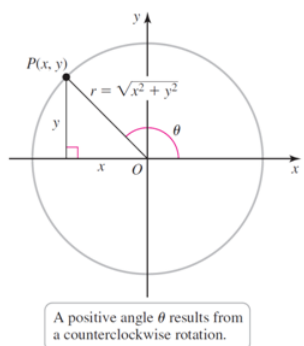


FIGURE 1.59

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Definition: **Trigonometric Functions**

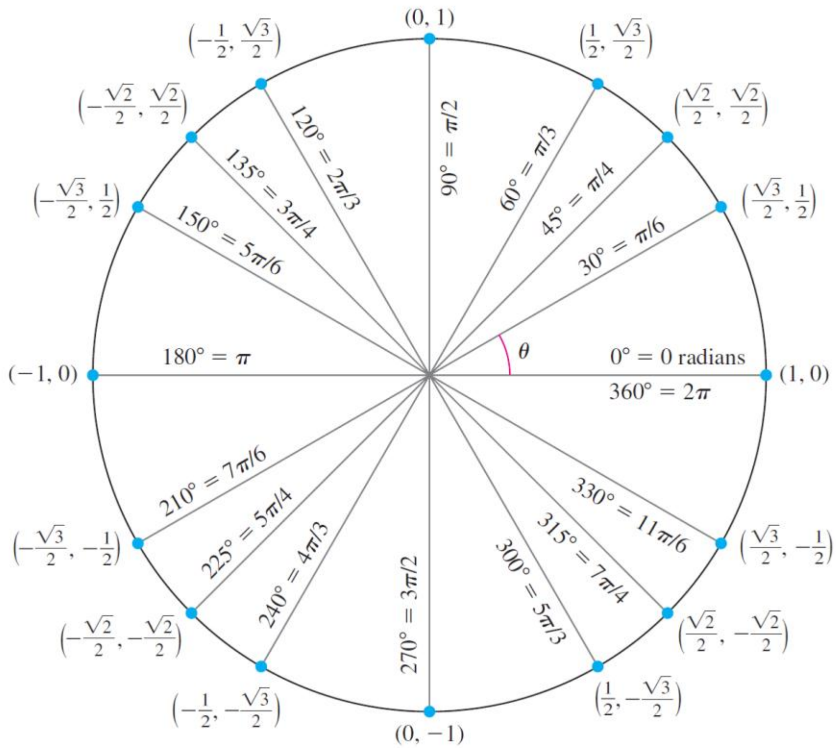
Let $P(x, y)$ be a point on a circle of radius r associated with the angle θ.

Then

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{r}$$



Or Standard Triangles:

Example: If $\cos\theta = \dfrac{5}{13}, 0 < \theta < 2\pi$, evaluate the other five trigonometric functions.

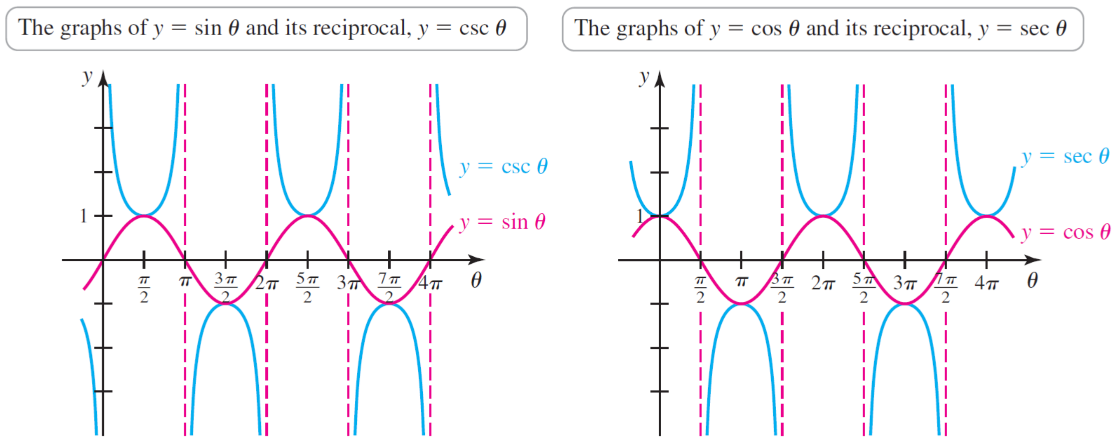
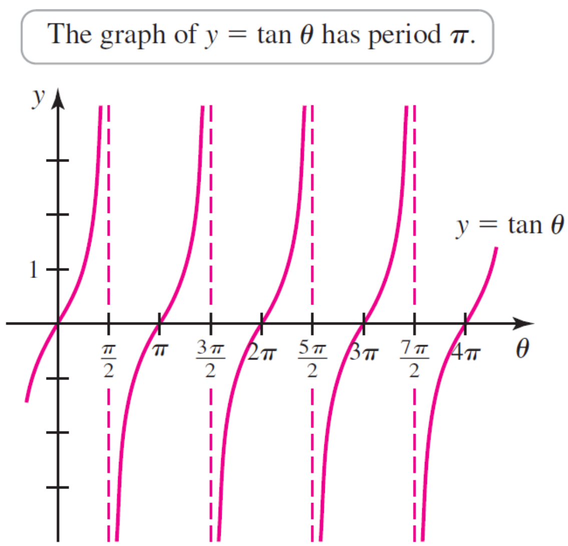
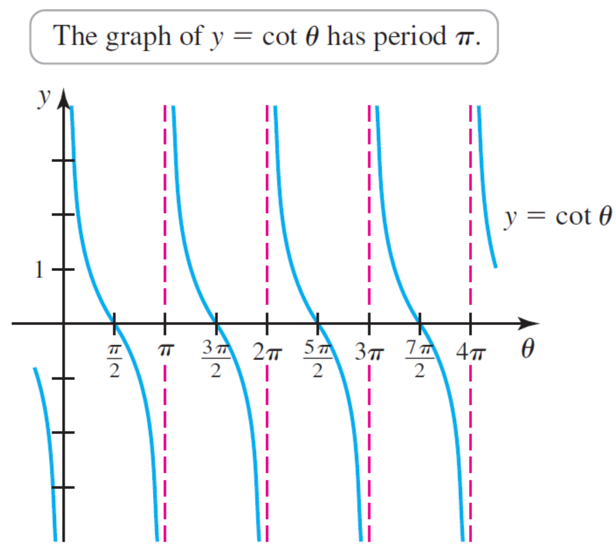


FIGURE 1.63 (a) (b)





Trigonometric Identities:

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Double and Half-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Example: Solve the following: $\sin 2\theta = \cos\theta$

Checkout page 39 and review transformed graphs of trig functions.

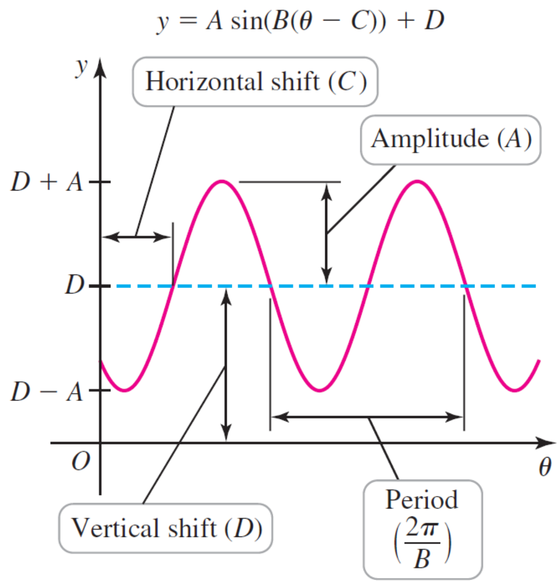


FIGURE 1.65

Inverse Trigonometric Functions:

Inverse Trigonometric Functions (see page 41 for graphs)
• $\sin^{-1} x = y \Leftrightarrow \sin y = x$ for $-\pi/2 \le y \le \pi/2$
• $\cos^{-1} x = y \Leftrightarrow \cos y = x$ for $0 \le y \le \pi$
• $\tan^{-1} x = y \Leftrightarrow \tan y = x$ for $-\pi/2 < y < \pi/2$
• $\cot^{-1} x = y \Leftrightarrow \cot y = x$ for $0 < y < \pi$
• $\sec^{-1} x = y \Leftrightarrow \sec y = x$ for $0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
• $\csc^{-1} x = y \Leftrightarrow \csc y = x$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

Example:

Find the angles in the following:

a. $\arccos\left(\frac{-1}{2}\right)$

b. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

c. $\arctan(-1)$

d. $\arctan\left(\tan\frac{\pi}{4}\right)$

e. $\arctan\left(\tan\frac{3\pi}{4}\right)$

f. $\tan(\arccos x)$