MATH 4xx: Lie group analysis for mathematical modelling
Spring 2007

Class Hours: MWF 11:00-12:00, WEIR 133
Prerequisites: MATH 231 (Calculus and Analytic Geometry III),
MATH 335 (Ordinary Differential Equations)

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Literature:

[1] To be updated, *J.H. Spurk, Springer (1997),* ISBN 3-540-61651-9
[2] To be updated, *L.M. Milne-Thomson, Dover Publications, INC (1974),* ISBN 0-486-68970-0

1. Description of the subject

The formulation of fundamental natural laws and of technological problems in the form of rigorous mathematical models is given frequently, even prevalently, in terms of differential equations. The needs of new technologies require non-linear differential equations. An appropriate method for tackling these equations is provided by Lie group analysis.

Lie group analysis suggests a rigorous mathematical formulation of intuitive ideas of symmetry and provides constructive methods for solving non-linear differential equations analytically. Acquaintance with group analysis is important for constructing and investigating non-linear mathematical models of natural and engineering problems. Numerous physical phenomena can be investigated using Lie symmetries to unearth various group invariant solutions and conservation laws that provide significant physical insight into the problem. For example, at this point I have a project in mind related to stability and energy exchange of resonantly interacting nonlinear waves in the ocean. This problem can be considered from symmetry analysis point of vew and can be proposed as a Ph.D project for prospective graduate students.

For non-linear problems, Lie group analysis plays the same role as Fourier analysis for linear problems. Therefore, group analysis should be as familiar to the student as Fourier analysis, especially when so many real-world problems are strongly non-linear and are not tractable by means of mathematical methods taught within traditional university curricula.

2. Aim of the Programme

The programme is aimed at

- giving general knowledge of mathematics and deep understanding of Lie group analysis with applications in mathematical modelling,
- training skilled researchers capable to carry out their own research,
- investigating mathematical models of non-linear problems in physical, biological , economical and engineering sciences,
- incorporating the students in an international network in mathematical modelling, differential equations and Lie group analysis

3. About the course

This course provides audience with an easy to follow and comprehensive introduction to Lie's group analysis and develops simple practical methods for solving linear and nonlinear equations using their symmetries.

4. Course outline

1. **Linear partial differential equations of the first order**: First integrals of systems of ordinary differential equations, Integration of partial differential equations, Systems of homogeneous linear equations.

2. **One-parameter groups of transformations**: Definition of a local group, Canonical parameter, Lie equations, Exponential map, Invariant functions, Infinitesimal generator, Invariant equations.

3. **Notation from differential algebra:** Differential variables, total derivatives, the space of differential functions, Extension (prolongation) of the point transformations and group generators to derivatives.

4. **Symmetry of differential equations:** Two definitions of an admitted group, Determining equations, Solution of determining equations, Infinitesimal symmetries, Derivation of equations admitting a given group.

5. **Lie algebras:** Lie algebras of operators, Structure constants, Subalgebra and ideal, Derived algebras, Solvable Lie algebras, Construction of multi-parameter groups via Lie algebras, Classification of two-dimensional Lie algebras and integration of non-linear differential equations.

5. Sample problems

1. Solve the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + xu = x^2.$$

2. Find the one-parameter group with the generator

$$X = \frac{1}{2}x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

3. Find the general form of the first-order ordinary differential equations $y' = f(x, y)$ and the second-order equations $y'' = f(x, y, y')$ admitting

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

4. Find the symmetries of the following second-order equation:

$$y'' + e^{3y}(y')^4 + (y')^2 = 0.$$

5. Using Lie's integration algorithm, solve the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}.$$

6. Solve the equation

$$u_x u_{xx} + u_{yy} = 0.$$

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