Heat Kernel Approach in Quantum Field Theory

Ivan G. Avramidi

Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, NM 87801, USA

We give a short overview of the effective action approach in quantum field theory and quantum gravity and describe various methods for calculation of the asymptotic expansion of the heat kernel for second-order elliptic partial differential operators acting on sections of vector bundles over a compact Riemannian manifold. We consider both Laplace type operators and non-Laplace type operators on manifolds without boundary as well as Laplace type operators on manifolds with boundary with oblique and non-smooth boundary conditions.

1. Effective Action in Gauge Field Theories and Quantum Gravity

In this lecture we briefly describe the standard formal construction of the generating functional and the effective action in gauge theories following the covariant spacetime approach to quantum field theory developed mainly by DeWitt [1]. The basic object of any physical theory is the spacetime manifold, which is assumed to be a \( n \)-dimensional manifold with the topological structure of a cylinder \( M = I \times \Sigma \), where \( I \) is an open interval of the real line (or the whole real line) and \( \Sigma \) is some \((m-1)\)-dimensional manifold. The spacetime manifold is here assumed to be globally hyperbolic and equipped with a (pseudo)-Riemannian metric \( g \) of signature \((-+\cdots+)\); thus, a foliation of spacetime exists into spacelike sections identical to \( \Sigma \). Usually one also assumes the existence of a spin structure on \( M \). A point \( x = (x^\mu) \) in the spacetime is described locally by the time \( x^0 \) and the space coordinates \((x^1, \ldots, x^{m-1})\). We label the spacetime coordinates by Greek indices, which run from 0 to \( m-1 \), and sum over repeated indices.

Let us consider a vector bundle \( V \) over the spacetime \( M \) each fiber of which is isomorphic to a vector space, on which the spin group \( \text{Spin}(1, m-1) \), i.e. the covering group of Lorentz group, acts. The vector bundle \( V \) can also have an additional structure on which a gauge group acts. The sections of the vector bundle \( V \) are called fields. The tensor fields describe the particles with integer spin (bosons) while the spin-tensor fields describe particles with half-odd spin (fermions). Although the whole scheme can be developed for the superfields (a combination of boson and fermion fields), we restrict ourselves in the present lecture to boson fields. A field \( \varphi \) is represented locally by a set of functions \( \varphi = \varphi^A(x) \), where \( A = 1, \ldots, \dim V \). Capital Latin indices will be used to label the local components of the fields. We will also use the condensed DeWitt notation, where the discrete index \( A \) and the spacetime point \( x \) are combined in one lower case Latin index \( i \equiv (A, x) \). Then the components of a field \( \varphi \) are \((\varphi^i) \equiv (\varphi^A(x))\). As usual, we will also assume that a summation over repeated lower case Latin indices, i.e. a combined summation-integration, is performed, viz. \( J \cdot \varphi \equiv J_i \varphi^i \equiv \int_M d\text{vol}(x) J_A(x) \varphi^A(x) \), where \( d\text{vol}(x) = dx |g|^{1/2} \), \( |g| = \text{det} g_{\mu\nu} \), is the natural Riemannian volume element defined by some background metric \( g \). We will often omit the volume element when this does not cause any misunderstanding.

In quantum field theory (QFT) the vector bundle \( V \) is called the configuration space. One assumes that the configuration space is an infinite-dimensional manifold \( \mathcal{M} \). The fields \( \varphi^i \) are the coordinates on this manifold, the variational derivative \( \delta \varphi^i / \delta \varphi \) is a tangent vector, a small disturbance \( \delta \varphi \) is a one-form and so on. If \( S(\varphi) \) is a scalar field on the configuration space, then its variational derivative \( \delta S / \delta \varphi \) is a one-form on \( \mathcal{M} \) with the components that we denote by \( S_i = \delta S / \delta \varphi^i \). By using the functional differentiation one can define formally the concept of tan-
gent space, the tangent vectors, Lie derivative, one-forms, metric, connection, geodesics and so on.

The dynamics of quantum field theory is determined by an action functional $S(\phi)$, which is a differentiable real-valued scalar field on the configuration space. The dynamical field configurations are defined as the field configurations satisfying the stationary action principle, i.e. they must satisfy the dynamical equations of motion $\delta S/\delta \phi = 0$ with given boundary conditions. The set of all dynamical field configurations, i.e. those that satisfy the dynamical equations of motion, $\mathcal{M}_0$, is a subspace of the configuration space called the dynamical subspace.

Quantum field theory is basically a theory of small disturbances on the dynamical subspace. Most of the problems of standard QFT deal with scattering processes, which are described by the transition amplitudes between some well defined initial and final states in the remote past and the future. The collection of all these amplitudes is called the scattering matrix, or shortly $S$-matrix.

Let us single out in the space-time two causally connected in– and out– regions, that lie in the past and in the future respectively relative to the region $\Omega$, which is of interest from the dynamical standpoint. Let $|\text{in}\rangle$ and $|\text{out}\rangle$ be some initial and final states of the quantum field system in these regions. Let us consider the transition amplitude $\langle \text{out}|\text{in}\rangle$ and ask the question: how does this amplitude change under a variation of the initial and final states of the quantum field system in $\Omega$, i.e. $\delta S = J \cdot \phi$. The amplitude $\langle \text{out}|\text{in}\rangle$ becomes a functional of the sources that we denote by $Z(J)$. By using the Schwinger variational principle one can obtain the chronological mean values in terms of the derivatives of the functional $Z(J)$:

$$\langle \varphi_{i_1} \cdots \varphi_{i_n} \rangle = \left( \frac{\hbar}{i} \right)^n Z^{-1} \frac{\delta^n}{\delta J_{i_n} \cdots \delta J_{i_1}} Z,$$

where $T$ denotes the operator of chronological ordering that orders the (non-commuting) operators in order of their time variables from right to left. In other words, the functional $Z(J)$ is the generating functional for chronological amplitudes. Let us now define another functional $W(J)$ by $Z = \exp \left( \frac{1}{\hbar} W \right)$. We have obviously

$$\langle \varphi_{i_1} \cdots \varphi_{i_n} \rangle = \left( \frac{\hbar}{i} \right)^n e^{-\frac{i}{\hbar} W} \frac{\delta^n}{\delta J_{i_n} \cdots \delta J_{i_1}} e^{\frac{i}{\hbar} W},$$

in particular,

$$\langle \varphi_i \rangle = \phi^i, \quad \langle \varphi^i \varphi^k \rangle = \phi^i \phi^k + \frac{\hbar}{i} G^{ik},$$

where

$$\phi^i = \frac{\delta W}{\delta J_i} \quad \text{and} \quad G^{i_1 \cdots i_n} = \frac{\delta^n}{\delta J_{i_n} \cdots \delta J_{i_1}} W.$$
There are many different ways to show that there is a functional $\Gamma(\phi)$ such that $\langle \delta S(\hat{\phi}) / \delta \hat{\phi} \rangle = \delta \Gamma(\phi) / \delta \phi$. This functional is defined by

$$\langle \text{out} | \text{in} \rangle = \exp \left\{ \frac{i}{\hbar} (\Gamma + J \cdot \phi) \right\},$$

or by the functional Legendre transform

$$\Gamma(\phi) = W(J(\phi)) - J(\phi) \cdot \frac{\delta}{\delta J} W(J(\phi)).$$

This is the most important object in quantum field theory. It contains all the information about quantized fields. First of all, the first variation of $\Gamma$ gives the effective equations for the background fields

$$\delta \Gamma / \delta \phi = -J.$$

These equations replace the classical equations of motion and describe the effective dynamics of the background field with regard to all quantum corrections. That is why $\Gamma$ is called the effective action. Furthermore, the second derivative of $\Gamma(\phi)$ determines the full propagator

$$G = \left( -\frac{\delta^2 \Gamma}{\delta \phi^2} \right)^{-1}.$$

The higher derivatives determine the so-called full vertex functions $\Gamma_{i_1 \ldots i_k}$, which are also called strongly connected, or one-particle irreducible functions. In other words, $\Gamma(\phi)$ is the generating functional for the full vertex functions. The full vertex functions together with the full propagator determine the full connected Green functions and, therefore, all chronological amplitudes and, hence, the $S$-matrix. Thus, the entire quantum field theory is summed up in the functional structure of the effective action.

One can obtain a very useful formal representation for the effective action in terms of functional integrals (called also path integrals, or Feynman integrals). A functional integral is an integral over the (infinite-dimensional) configuration space $\mathcal{M}$. Although a rigorous mathematical definition for the functional integrals is absent, they can be used in perturbation theory of QFT as an effective tool, especially in gauge theories, for manipulating the whole series of perturbation theory. The point is that in perturbation theory one encounters only the functional integrals of Gaussian type, which can be well defined effectively in terms of the classical propagators and vertex functions. The Gaussian integrals do not depend much on the dimension and, therefore, many formulas from the finite-dimensional case, like the Fourier transform, integration by parts, delta-function, change of variables etc. are valid in the infinite-dimensional case as well. One has to note that the functional integrals are formally divergent — if one tries to evaluate the integrals, one encounters meaningless divergent expressions. This difficulty can be overcome in the framework of the renormalization theory in so-called renormalizable field theories, but we will not discuss this problem in the present lectures.

Integrating the Schwinger variational principle one can obtain the following functional integral:

$$\langle \text{out} | \text{in} \rangle = \int_{\mathcal{M}} D\varphi \exp \left\{ \frac{i}{\hbar} \left[ S(\varphi) + J \cdot \varphi \right] \right\}.$$  

Correspondingly, for the effective action one obtains a functional equation

$$\exp \left\{ \frac{i}{\hbar} \Gamma(\phi) \right\} = \int_{\mathcal{M}} D\varphi \exp \left\{ \frac{i}{\hbar} \left[ S(\varphi) - \frac{\delta \Gamma(\phi)}{\delta \phi} \cdot (\varphi - \phi) \right] \right\}.$$  

The only way to get numbers from this formal expression is to take advantage of the semiclassical approximation within a formal expansion in powers of Planck constant $\hbar$:

$$\Gamma = S + \sum_{k=1}^{\infty} \hbar^k \Gamma_{(k)}.$$  

Substituting this expansion in the functional equation for the effective action, shifting the integration variable in the functional integral $\varphi = \phi + \sqrt{\hbar} \chi$, expanding the action $S(\varphi)$ in functional Taylor series in quantum fields $\hbar$, expanding both sides of the equation in powers on $\hbar$ and equating the coefficients of equal powers of $\hbar$, one gets the recurrence relations that uniquely define all coefficients $\Gamma_{(k)}$. All functional integrals appearing in
this expansion have the form
\[ \int_\mathcal{M} Dh \exp \left( -\frac{i}{2} h \cdot \Delta h \right) h^{i_1} \cdots h^{i_n}, \]
where \( \Delta \) is a partial differential operator defined by the second variation of the action
\[ \Delta = -\frac{\delta^2 S}{\delta \varphi^2}. \]

These integrals are Gaussian and can be calculated in terms of the functional determinant of the operator \( \Delta \) and the bare propagator \( G = \Delta^{-1} \), i.e., the Green function of the operator \( \Delta \) with Feynman boundary conditions, and the local classical vertex functions \( S_{i_1 \cdots i_n} \).

In particular, the one-loop effective action is determined by the functional determinant of the operator \( \Delta \)
\[ \Gamma_{(1)} = \frac{1}{2i} \log \text{Det} \Delta. \]

Strictly speaking, the Gaussian integrals are well defined for elliptic differential operators in terms of the functional determinants and their Green functions. Although the Gaussian integrals of quantum field theory are determined by hyperbolic differential operators with Feynman boundary conditions they can be well defined by means of the analytic continuation from the Euclidean sector of the theory where the operators become elliptic. This is done by so-called Wick rotation—one replaces the real time coordinate by a purely imaginary one \( x^0 \to i\tau \) and singles out the imaginary factor also from the action \( S \to iS \) and the effective action \( \Gamma \to i\Gamma \). Then the metric of the spacetime manifold becomes positive definite and the classical action in all ‘nice’ field theories becomes a positive-definite functional. Then the fast oscillating Gaussian functional integrals become exponentially decreasing and can be given a rigorous mathematical meaning.

Let us try to apply the formalism described above to a gauge field theory. A characteristic feature of a gauge field theory is the fact that the dynamical equations \( \delta S/\delta \varphi = 0 \) are not independent — there are certain identities, called Noether identities, between them. This means that there are some nowhere vanishing vector fields \( R_\alpha = R^\mu_{\alpha\nu} \delta/\delta \varphi^\nu \) on the configuration space \( \mathcal{M} \) that annihilate the action \( R_\alpha S = 0 \), and, hence, define invariance flows on \( \mathcal{M} \). The transformations of the fields \( \delta \xi^\alpha = R^\nu_{\alpha\mu} \xi^\mu \) are called the invariance transformations and \( R_\alpha \) are called the generators of invariance transformations. The infinitesimal parameters of these transformations \( \xi \) are sections of another vector bundle (usually the tangent bundle \( TG \) of a compact Lie group \( G \)) that are represented locally by a set of functions \( \xi^\alpha = (\xi^\alpha(x)), a = 1, \ldots, \text{dim} \ G \) over spacetime with compact support. To distinguish between the components of the gauge fields and the components of the gauge parameters we introduce lower case Latin indices from the beginning of the alphabet; the Greek indices from the beginning of the alphabet are used as condensed labels \( \alpha = (a, x) \) that include the spacetime point. We assume that the vector fields \( R_\alpha \) are linearly independent and complete, which means that they form a complete basis in the tangent space of the invariant subspace of configuration space. The vector fields \( R_\alpha \) form the gauge algebra. We restrict ourselves to the simplest case when the gauge algebra is the Lie algebra of an infinite-dimensional gauge Lie group \( \mathcal{G} \), which is the case in Yang-Mills theory and gravity. Then the flow vectors \( R_\alpha \) decompose the configuration space into the invariants subspaces of \( \mathcal{M} \) (called the orbits) consisting of the points connected by the gauge transformations. The space of orbits is then \( \mathcal{M}/\mathcal{G} \). The linear independence of the vectors \( R_\alpha \) at each point implies that each orbit is a copy of the group manifold. One can show that the vector fields \( R_\alpha \) are tangent to the dynamical subspace \( \mathcal{M}_0 \) which means that the orbits do not intersect \( \mathcal{M}_0 \) and the invariance flow maps the dynamical subspace \( \mathcal{M}_0 \) into itself. Since all field configurations connected by a gauge transformation, i.e., the points on an orbit, are physically equivalent, the physical dynamical variables are the classes of gauge equivalent field configurations, i.e., the orbits. The physical configuration space is, hence, the space of orbits \( \mathcal{M}/\mathcal{G} \). In other words the physical observables must be the invariants of the gauge group.

To quantize a gauge theory by means of the
functional integral, we consider the in- and out-regions, define some |in⟩ and |out⟩ states in these regions and study the amplitude ⟨out|in⟩. Since all field configurations along an orbit are physically equivalent we have to integrate over the orbit space M/G. To deal with such situations one has to choose a representative field in each orbit. This can be done by choosing special coordinates (Iα, χα(φ)) on the configuration space M, where Iα label the orbits and χα the points in the orbit. Computing the Jacobian of the field transformation and introducing a delta functional δ(χ − ζ) we can fix the coordinates on the orbits and obtain the measure on the orbit space M/G

\[ D\varphi \text{Det} F(\varphi) \delta(\chi(\varphi) - \zeta), \]

where \( F^{\alpha}_\beta = R^a_{\alpha} \chi^{\beta}_a \) is a non-degenerate operator. Thus we obtain a functional integral for the transition amplitude

\[ \langle \text{out}|\text{in} \rangle = \int_M D\varphi \text{Det} F(\varphi) \delta(\chi(\varphi) - \zeta) \times \exp \left\{ \frac{i}{\hbar} S(\varphi) \right\}. \]

Now one can go further and integrate this equation over parameters ζ with a Gaussian measure determined by a nondegenerate matrix γ = (γαβ), which most naturally can be chosen as the metric on the orbit (gauge group metric). As a result we get

\[ \langle \text{out}|\text{in} \rangle = \int_M D\varphi \text{Det}^{1/2} \gamma \text{Det} F(\varphi) \times \exp \left\{ \frac{i}{\hbar} \left[ S(\varphi) + \frac{1}{2} \chi(\varphi) \cdot \gamma \chi(\varphi) \right] \right\}. \]

The functional equation for the effective action takes the form

\[ \exp \left\{ \frac{i}{\hbar} \Gamma(\phi) \right\} = \int_M D\varphi \text{Det}^{1/2} \gamma \text{Det} F(\varphi) \times \exp \left\{ \frac{i}{\hbar} \left[ S(\varphi) + \frac{1}{2} \chi(\varphi) \cdot \gamma \chi(\varphi) - \frac{\delta \Gamma(\phi)}{\delta \phi} \cdot (\varphi - \phi) \right] \right\}. \]

This equation can be used to construct the semi-classical perturbation theory in powers of the Planck constant (loop expansion), which gives the effective action in terms of the bare propagators and the vertex functions. The new feature is though that the bare propagator and the vertex functions are determined by the action

\[ S_{\text{eff}}(\varphi) = S(\varphi) + \frac{1}{2} \chi(\varphi) \cdot \gamma \chi(\varphi) + \frac{i}{\hbar} \log \text{Det} F(\varphi) + \frac{\hbar}{2i} \log \text{Det} \gamma \]

In particular, one finds the one-loop effective action

\[ \Gamma_{(1)} = -\frac{1}{2i} \log \text{Det} \Delta + \frac{i}{\hbar} \log \text{Det} F + \frac{1}{2i} \log \text{Det} \gamma, \]

where

\[ \Delta = -\frac{\delta^2 S}{\delta \varphi^2} - \frac{\delta \chi}{\delta \varphi} \cdot \gamma \frac{\delta \chi}{\delta \varphi}. \]

2. Heat Kernel Asymptotic Expansion

As we have seen in the previous lecture the effective action in quantum field theory can be computed within the semi-classical perturbation theory—the one-loop effective action is determined by the functional determinants of second-order hyperbolic partial differential operators with Feynman boundary conditions and the higher-loop approximations are determined in terms of the Feynman propagators and the classical vertex functions. As we noted above these expressions are purely formal and need to be regularized and renormalized, which can be done in a consistent way in renormalizable field theories.

One should stress, of course, that many physically interesting theories (including Einstein’s general relativity) are perturbatively non-renormalizable. Since we only need Feynman propagators we can do the Wick rotation and consider instead of hyperbolic operators the elliptic ones. The Green functions of elliptic operators and their functional determinants can be expressed in terms of the heat kernel. That is why we concentrate in the subsequent lectures on the calculation of the heat kernel.
The gauge invariance (or covariance) in quantum gauge field theory and quantum gravity is of fundamental importance. That is why, a calculational scheme that is manifestly covariant is an inestimable advantage. A manifestly covariant calculus is such that every step is expressed in terms of geometric objects; it does not have some intermediate “non-covariant” steps that lead to an “invariant” result. Below we describe a manifestly covariant method for calculation of the heat kernel following mainly our papers [2, 3, 4, 5].

Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $m$ without boundary, equipped with a positive definite Riemannian metric $g$. Let $V$ be a vector bundle over $M$, $V^*$ be its dual, and $\text{End} (V) \cong V \otimes V^*$ be the corresponding bundle of endomorphisms. Given any vector bundle $V$, we denote by $C^\infty (V)$ its space of smooth sections. We assume that the vector bundle $V$ is equipped with a Hermitian metric. This naturally identifies the dual vector bundle $V^*$ with $V$, and defines a natural $L^2$ inner product and the $L^2$-trace $\text{Tr}_{L^2}$ using the invariant Riemannian measure on the manifold $M$. The completion of $C^\infty (V)$ in this norm defines the Hilbert space $L^2 (V)$ of square integrable sections. We denote by $TM$ and $T^* M$ the tangent and cotangent bundles of $M$. Let a connection, $\nabla : C^\infty (V) \to C^\infty (T^* M \otimes V)$, on the vector bundle $V$ be given, which we assume to be compatible with the Hermitian metric on the vector bundle $V$. The connection is given its unique natural extension to bundles in the tensor algebra over $V$ and $V^*$. In fact, using the Levi-Civita connection $\nabla^L C$ of the metric $g$ together with $\nabla$, we naturally obtain connections on all bundles in the tensor algebra over $V, V^*, TM$ and $T^* M$; the resulting connection will usually be denoted just by $\nabla$. It is usually clear which bundle’s connection is being referred to, from the nature of the section being acted upon. Let $\nabla^*$ be the formal adjoint to $\nabla$ defined using the Riemannian metric and the Hermitian structure on $V$ and let $Q \in C^\infty (\text{End} (V))$ be a smooth Hermitian section of the endomorphism bundle $\text{End} (V)$.

A Laplace type operator $F : C^\infty (V) \to C^\infty (V)$ is a partial differential operator of the form

$$ F = \nabla^* \nabla + Q = -g^{\mu \nu} \nabla_\mu \nabla_\nu + Q. \quad (1) $$

It is obviously symmetric, i.e. $(F \varphi, \psi) = (\varphi, F \psi)$, elliptic, and can be made essentially self-adjoint, i.e. its closure is self-adjoint, which implies that it has a unique self-adjoint extension. We will not be very careful about this and will simply say that $F$ is elliptic and self-adjoint. It is well known [6] that: i) the operator $F$ has a discrete real spectrum, $\{\lambda_n\}^\infty_{n=1}$, bounded from below:

$$ \Lambda < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots $$

with some real constant $\Lambda$, ii) all eigenspaces of the operator $F$ are finite-dimensional, and iii) the eigenvectors, $\{ \varphi_n \}^\infty_{n=1}$, of the operator $F$, are smooth sections of the vector bundle $V$ that form a complete orthonormal basis in $L^2 (V)$.

For $t > 0$ the operators $U(t) = \exp (-tF)$ form a semi-group of bounded operators on $L^2 (V)$, so called heat semi-group. The kernel of this operator is defined by

$$ U(t|x, x') = \sum_n e^{-t\lambda_n} \varphi_n(x) \otimes \varphi^*_n(x'), $$

where each eigenvalue is counted with multiplicities. It is a section of the external tensor product of vector bundles $V \boxtimes V^*$ over $M \times M$, which can also be regarded as an endomorphism from the fiber of $V$ over $x'$ to the fiber of $V$ over $x$. This kernel satisfies the heat equation

$$ (\partial_t + F) U(t) = 0 \quad (2) $$

with the initial condition

$$ U(0^+|x, x') = \delta(x, x') \quad (3) $$

and is called the heat kernel.

Moreover, the heat semigroup $U(t)$ is a trace-class operator with a well defined $L^2$-trace

$$ \text{Tr}_{L^2} \exp (-tF) = \int_M \text{tr}_V U^{\text{diag}}(t). \quad (4) $$

Hereafter $\text{tr}_V$ denotes the fiber trace and the label ‘diag’ means the diagonal value of a two-point quantity, e.g.

$$ U^{\text{diag}}(t|x) = U(t|x, x') \big|_{x = x'}. $$
The trace of the heat kernel is obviously a spectral invariant of the operator $F$. It determines other spectral functions by integral transforms. Of particular importance is the so-called zeta function, which enables one to define, in particular, the regularized functional determinant of an elliptic operator.

In these lectures we will study the heat kernel only locally, i.e. in the neighbourhood of the diagonal of $M \times M$, when the points $x$ and $x'$ are close to each other. We fix a point $x'$ of the manifold and consider a small geodesic ball with a radius smaller than the injectivity radius of the manifold, so that each point $x$ of the ball can be connected by a unique geodesic with the point $x'$. Such geodesic ball can be covered by a single coordinate patch with normal local coordinates centered at $x'$. Let $\sigma(x,x')$ be the geodesic interval, defined as one half the square of the length of the geodesic connecting the points $x$ and $x'$, i.e. $\sigma(x,x') = (1/2)[\text{dist}(x,x')]^2$. The first derivatives of this function with respect to $x$ and $x'$ define tangent vector fields to the geodesic at the points $x$ and $x'$

$$w^\mu(x,x') = g^{\mu\nu} \partial_\nu \sigma, \quad u^\mu(x,x') = g^{\mu\nu'} \partial_{\nu'} \sigma,$$  
(5)

and the determinant of the mixed second derivatives defines the so-called Van Vleck–Morette determinant

$$\Delta(x,x') = |g(x)|^{-\frac{1}{2}} |g(x')|^{-\frac{1}{2}} \det(-\partial_\mu \partial_{\nu'} \sigma).$$  
(6)

Let, finally, $P(x,x')$ denote the parallel transport operator of sections of the vector bundle $V$ along the geodesic from the point $x'$ to the point $x$. It is an endomorphism from the fiber of $V$ over $x'$ to the fiber of $V$ over $x$ (or a section of the external tensor product $V \boxtimes V^*$ over $M \times M$). Near the diagonal of $M \times M$ all these two-point functions are smooth single-valued functions of the coordinates of the points $x$ and $x'$. We should point out from the beginning that we will construct all two-point geometric quantities (in particular, the coefficients of the asymptotic expansion of the heat kernel as $t \to 0$) in form of covariant Taylor series. The Taylor series do not necessarily converge in smooth case; they do, however, converge in the analytic case in a sufficiently small neighborhood of the diagonal.

Further, one can easily prove that the function

$$U_0(t) = (4\pi t)^{-n/2} \Delta^{1/2} \exp \left(-\frac{\sigma}{2t}\right) P$$

satisfies the initial condition (3). Moreover, locally it also satisfies the heat equation (2) in the free case, when the Riemannian curvature $\text{Riem}$ of the manifold, the curvature $\mathcal{R}$ of the connection $\nabla V$, and the endomorphism $Q$ vanish: $\text{Riem} = \mathcal{R} = Q = 0$. Therefore, $U_0(t)$ is the exact heat kernel for the Laplacian in flat Euclidean space with a flat trivial bundle connection. This function gives a good framework for the approximate solution in the general case. Namely, by factorizing out the free factor we get an ansatz

$$U(t) = (4\pi t)^{-n/2} \Delta^{1/2} \exp \left(-\frac{\sigma}{2t}\right) P \Omega(t).$$  
(7)

The function $\Omega(t|x,x')$, called the transport function, is a section of the endomorphism bundle $\text{End}(V)$ over the point $x'$. It satisfies the transport equation

$$\left(\partial_t + \frac{1}{t} D + L\right) \Omega(t) = 0,$$

with the initial condition

$$\Omega(0|x,x') = I,$$

where $I$ is the identity endomorphism of the bundle $V$ (we will often omit it) and $D$ and $L$ are differential operators defined by

$$D = u^\mu \nabla_\mu,$$

$$L = P^{-1} \Delta^{-1/2} F \Delta^{1/2} P.$$

(8)
(9)

Now, let us fix a sufficiently large negative parameter $\lambda$, viz. $\lambda < \Lambda$, so that $(F - \lambda I)$ is a positive operator. Since $\exp[-t(F - \lambda I)] = e^{t\lambda} \exp[-tF]$, the transport function for the operator $(F - \lambda I)$ is $e^{t\lambda} \Omega(t)$. Clearly, for sufficiently large negative $\lambda$, $\lambda << 0$, the function $e^{t\lambda} \Omega(t)$ with all its derivatives decreases faster than any power of $t$ as $t \to \infty$. Let us consider a slightly modified version of the Mellin transform of the function $e^{t\lambda} \Omega(t)$

$$b_q(\lambda) = \frac{1}{\Gamma(-q)} \int_0^\infty dt t^{-q-1} e^{t\lambda} \Omega(t).$$

(10)
Note that for fixed $\lambda$ this is a Mellin transform of $e^{\lambda t} \Omega(t)$ and for a fixed $q$ this is a Laplace transform of the function $e^{t \gamma^{-1}} \Omega(t)$. The integral (10) converges for $\text{Re} \ q < 0$. By integrating by parts and analytical continuation one can prove that the function $b_q(\lambda)$ is an entire function of $q$. The values of the function $b_q(\lambda)$ at the integer positive points $q = k$ are given by

$$b_k(\lambda) = \sum_{n=0}^{k} \binom{k}{n} (-\lambda)^{k-n} a_n,$$

where

$$a_k = (-\partial_t)^k \Omega(t) \bigg|_{t=0}.$$

These coefficients $a_k = a_k(x, x')$ are called Hadamard–Minakshisundaram–DeWitt–Seeley (HADS) coefficients.

By inverting the Mellin transform we obtain a new ansatz for the transport function and, hence, for the heat kernel

$$\Omega(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \ e^{-t\lambda q} \Gamma(-q) b_q(\lambda),$$

where $c < 0$ and $\text{Re} \ \lambda < \Lambda$. Clearly, since the left-hand side of this equation does not depend on $\lambda$, neither does the right hand side. Thus, $\lambda$ serves as an auxiliary parameter that regularizes the behavior at $t \to \infty$.

Substituting this ansatz into the transport equation we get a functional-differential equation for the function $b_q$

$$\left(1 + \frac{1}{q} D\right) b_q(\lambda) = (L - \lambda I) b_{q-1}(\lambda)$$

with the initial condition

$$b_0(\lambda) = I.$$

Note that for integer $q = k$ and $\lambda = 0$ this becomes a differential recursion system for the coefficients $a_k$

$$a_0 = I, \quad \left(1 + \frac{1}{k} D\right) a_k = L a_{k-1}.$$

It is interesting to note that there is an asymptotic expansion of $b_q(\lambda)$ as $\lambda \to -\infty$

$$b_q(\lambda) \sim \sum_{n=0}^{\infty} \frac{\Gamma(q + 1)}{n! \Gamma(q - n + 1)} (-\lambda)^{q-n} a_n,$$

that coincides with (11) for integer $q$.

By computing the inverse Mellin transform we obtain the asymptotic expansion of the transport function as $t \to 0$ in terms of the coefficients $a_k$

$$\Omega(t) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k.$$

Using our ansatz (7) we also find the trace of the heat kernel in form of an inverse Mellin transform

$$\text{Tr}_{L^2} \exp(-tF) = (4\pi t)^{-m/2} e^{-t\lambda}$$

$$\times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \ t^q \Gamma(-q) B_q(\lambda),$$

where

$$B_q(\lambda) = \frac{1}{M} \int_{M} \text{tr}_V b_q^{\text{diag}}(\lambda).$$

Noting that $B_q$ is an entire function of $q$, this gives the standard asymptotic expansion as $t \to 0$

$$\text{Tr}_{L^2} \exp(-tF) \sim \sum_{k=0}^{\infty} t^{(k-m)/2} A_k,$$

where

$$A_{2k+1} = 0$$

and

$$A_{2k} = (4\pi)^{-m/2} \frac{(-1)^k}{k!} \frac{1}{M} \int_{M} \text{tr}_V a_k^{\text{diag}}.$$

This is the famous Minakshisundaram–Pleijel asymptotic expansion, which is called Schwinger–DeWitt expansion in the physics literature. This expansion is of great importance in differential geometry, spectral geometry, quantum field theory and other areas of mathematical physics, such as theory of Huygens’ principle, heat kernel proofs of the index theorems, Korteveg–De Vries hierarchy, Brownian motion etc..

The (off-diagonal) HADS coefficients $a_k$ are determined by the recursion system (13). The formal solution of this recursion system is

$$a_k = D_k^{-1} L D_{k-1}^{-1} L \ldots D_1^{-1} L I,$$
where
\[
D_k = 1 + \frac{1}{k} D .
\] (20)

To give a precise meaning to this formal operator solution we need to define the inverse operator \(D_k^{-1}\). This can be done in terms of the covariant Taylor series. We will need the following notions from the theory of symmetric tensors. Let \(S^n_m\) be the bundle of symmetric tensors of type \((m, n)\).

First of all, we define the exterior symmetric tensor product \(\vee\) : \(S^n_m \times S^n_j \rightarrow S^{n+r}_{m+j}\) of symmetric tensors by
\[
(A \vee B)^{\alpha_1 \cdots \alpha_m + \beta_1 \cdots \beta_n} = A^{\alpha_1 \cdots \alpha_m} B^{\beta_1 \cdots \beta_n} .
\] (21)

This naturally leads to the following definition of the exterior symmetric power of a symmetric tensor \(\vee^k\) : \(S^n_m \rightarrow S^{nk}_m\)
\[
\vee^k A = A \vee \cdots \vee A .
\] (22)

Next, we define the inner product \(\ast\) : \(S^n_m \times S^n_i \rightarrow S^n_{m+i}\)
\[
(A \ast B)^{\alpha_1 \cdots \alpha_m \gamma_1 \cdots \gamma_i} = A^{\alpha_1 \cdots \alpha_m} B^{\gamma_1 \cdots \gamma_i} .
\] (23)

We also define the exterior symmetric covariant derivative \(\nabla^S\) : \(S^n_m \rightarrow S^n_{m+1}\)
\[
(\nabla^S A)^{\alpha_1 \cdots \alpha_m + \beta} = \nabla^{(\alpha_1} A^{\beta_2 \cdots \beta_m)} .
\] (24)

These definitions are naturally extended to End \((V)\)-valued symmetric tensors, i.e. to the sections of the bundle \(S^n_m \otimes \text{End}(V)\).

Let us consider the space of smooth two-point functions in a small neighborhood of the diagonal \(x = x'\) that we will denote by \(|f\rangle\). Let us define a special set of such functions \(|n\rangle\), labeled by a non-negative integer \(n\), by \(|0\rangle = 1\), \(|n\rangle = \frac{(-1)^n}{n!} \vee^n u'\), where \(u'\) is the tangent vector field to the geodesic connecting the points \(x\) and \(x'\) at the point \(x'\) defined by (5).

It is easy to show that these functions satisfy the equation
\[
D|n\rangle = n|n\rangle
\] (25)
and, hence, are the eigenfunctions of the operator \(D\) with positive integer eigenvalues.

Let \(|n\rangle\) denote the dual functions defined by
\[
\langle n| f \rangle = \langle \nabla^S f \rangle_{x=x'} ,
\] (26)

so that
\[
\langle n|m \rangle = \delta_{mn} I_{(n)} ,
\] (27)

where \(I_{(n)}\) is the identity endomorphism on the space of symmetric \(n\)-tensors. Using this notation the covariant Taylor series for an analytic function \(|f\rangle\) can be written in the form
\[
|f\rangle = \sum_{n=0}^{\infty} |n\rangle \ast \langle n| f \rangle ,
\] (28)

and, therefore, the functions \(|n\rangle\) form a complete orthonormal basis in the subspace of analytic functions.

The complete set of eigenfunctions \(|n\rangle\) can be employed to present the action of the operator \(L\) on a function \(|f\rangle\) in the form
\[
L|f\rangle = \sum_{m,n \geq 0} |m\rangle \ast \langle m| L|n\rangle \ast \langle n| f \rangle ,
\] (29)

where \(|m| L|n\rangle\) are the ‘matrix elements’ of the operator \(L\) that are just End \((V)\)-valued symmetric tensors, i.e. sections of the vector bundle \(S^n_m(M) \otimes \text{End}(V)\). Now it should be clear that the inverse of the operator \(D_k\) in (20) can be defined by
\[
D_k^{-1}|f\rangle = \sum_{n=0}^{\infty} \frac{k}{k+n} |n\rangle \ast \langle n| f \rangle .
\] (30)

Using such representations for the operators \(D_k^{-1}\) and \(L\) we obtain a covariant Taylor series for the coefficients \(a_k\)
\[
a_k = \sum_{n=0}^{\infty} |n\rangle \ast \langle n| a_k \rangle
\] (31)

where
\[
\langle n| a_k \rangle = \sum_{n_1, \ldots, n_{k-1} \geq 0} \left( \prod_{j=1}^{k} \frac{j}{j+n_j} \right) \langle n| L|n_{k-1}\rangle \ast \langle n_{k-1}| L|n_{k-2}\rangle \ast \cdots \ast \langle n_1| L|0\rangle ,
\] (32)
with $n_k \equiv n$.

Thus, we have reduced the problem of computation of the HMDS-coefficients $a_k$ to the evaluation of the matrix elements $(m|L|n)$ of the operator $L$. For a differential operator $L$ of second order, the matrix elements $(m|L|n)$ vanish for $n > m + 2$. Therefore, the summation over $n_i$ in (32) is limited from above: $n_1 \geq 0$, and $n_i \leq n_{i+1} + 2$, for $i = 1, 2, \ldots, k - 1$, and, hence, the sum (32) always contains only a finite number of terms. We will not present here explicit formulas, (they have been computed explicitly for arbitrary $m, n$ in [3]), but note that all these quantities are expressed polynomially in terms of three sorts of geometric data: i) symmetric tensors of type $(2, n)$, i.e. sections of the bundle $S^2_n$ obtained by symmetric derivatives $K(n) = (\nabla^S)^{n-2} \text{Riem}$ of the symmetrized Riemann tensor $\text{Riem}$ taken as a section of the bundle $S^2_1$; ii) sections $R(n)$ of the form $\nabla^S$ obtained by symmetrized derivatives of the curvature $\nabla$ of the connection $\nabla^V$ taken as a section of the bundle $S^1_1 \otimes \text{End}(V)$ obtained by symmetrized derivatives of the vector bundle $S^1_1 \otimes \text{End}(V)$; iii) $\nabla^V$-valued symmetric forms, i.e. sections of the vector bundle $S^0_2 \otimes \text{End}(V)$, constructed from the symmetrized covariant derivatives $Q(n) = (\nabla^S)^n Q$ of the endomorphism $Q$.

3. Approximation Schemes for Calculation of the Heat Kernel

In this lecture we are going to investigate the general structure of the heat kernel coefficients $A_k$. We will follow mainly our papers [3, 7, 8, 9, 10] (see also our review papers [11, 12, 13]). Our analysis will be again purely local. Since locally one can always expand the metric, the connection and the endomorphism $Q$ in the covariant Taylor series, they are completely characterized by their Taylor coefficients, i.e. the covariant derivatives of the curvatures, more precisely by the objects $K(n)$, $R(n)$ and $Q(n)$ defined in the previous lecture. We introduce the following notation for all of them $R(n) = \{ K(n+2), R(n+1), Q(n) \}$, and call these objects covariant jets; $n$ will be called the order of a jet $R(n)$. It is worth noting that the jets are defined by symmetrized covariant derivatives. This makes them well defined as ordering of the covariant derivatives becomes not important—it is only the number of derivatives that plays a role.

The coefficients $A_k$ are integrals of local invariants $\text{tr}_V a^{\text{diag}}$ which are polynomial in the jets. The first two coefficients have the well known form

$$a_0^{\text{diag}} = I, \quad a_1^{\text{diag}} = Q - \frac{1}{6} R,$$

where $R$ is the scalar curvature. For $k \geq 2$ one can classify the terms in $A_{2k}$ according to the number of the jets and their order

$$A_{2k} = \sum_{n=2}^k A_{2k,(n)},$$

where $A_{2k,(n)}$ can be presented symbolically in the form

$$A_{2k,(2)} = \int_M \text{tr}_V \sum_i R(0) R(2k-4), \quad (33)$$

$$A_{2k,(3)} = \int_M \text{tr}_V \sum_{i=0}^{2k-6} \sum R(0) R(i) R(2k-6-i),$$

$$A_{2k,(k-1)} = \int_M \text{tr}_V \sum_{i=0}^{k-3} \sum R(0) R(1) R(2k-3-i) R(1),$$

$$A_{2k,(k)} = \int_M \text{tr}_V \sum R^k(0). \quad (34)$$

More precisely, all quadratic terms can be reduced to a finite number of invariant structures, viz. [3]

$$A_{2k,(2)} = (4\pi)^{-m/2}\left(\frac{1}{2(2k-3)}\right)^k \sum_{i=0}^{2k-6} \sum R(0) R(i) R(2k-6-i),$$

$$\times \int_M \text{tr}_V \left\{ f^{(1)}_k Q \Box^{k-2} Q + 2 f^{(2)}_k R^\beta \gamma \Box^{k-3} \nabla^\beta R^\alpha \gamma + f^{(3)}_k Q \Box^{k-2} R + f^{(4)}_k R^\beta \gamma \Box^{k-2} R^\alpha + f^{(5)}_k R^\beta \gamma \Box^{k-2} R \right\},$$

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and $f^{(i)}_k$ are some numerical coefficients. These numerical coefficients can
be computed by the technique developed in the previous section. The result reads

\[ f^{(1)}_k = 1, \quad f^{(2)}_k = \frac{1}{2(2k - 1)}, \quad f^{(3)}_k = \frac{k - 1}{2(2k - 1)}, \]

\[ f^{(4)}_k = \frac{1}{2(4k^2 - 1)}, \quad f^{(5)}_k = \frac{k^2 - k - 1}{4(4k^2 - 1)}. \]

One should note that the same results were obtained by a completely different method in [14].

Let us consider the situation when the curvatures are small but rapidly varying, i.e. the derivatives of the curvatures are more important than the powers of them. Then the leading derivative terms in the heat kernel are the largest ones and, therefore, the trace of the heat kernel has the form

\[ \text{Tr}_{L^4} \exp(-tF) = t^{-m/2}A_0 + t^{1-m/2}A_2 + t^{2-m/2}H(t) + O(\mathcal{H}^3), \]

where \( H(t) \) is some complicated nonlocal functional that has the following asymptotic expansion as \( t \to 0 \):

\[ H(t) \sim \sum_{k=2}^{\infty} t^{k-2}A_{2k,(2)}. \]

Using the results for \( A_{2k,(2)} \) one can easily construct such a functional \( H \) just by a formal summation of leading derivatives

\[ H(t) = (4\pi)^{-m/2} \frac{1}{2} \int_M \{ Q\gamma^{(1)}(-t\Box)Q + 2R^\alpha_\beta \nabla_\sigma \nabla_\alpha \gamma^{(2)}(-t\Box) \nabla_\beta \gamma^{(3)}(-t\Box)R + R^\alpha_\beta \nabla_\sigma \gamma^{(3)}(-t\Box)R + R\gamma^{(4)}(-t\Box)R \}, \]

where \( \nabla \text{Riem} = 0, \quad \nabla \mathcal{R} = 0, \quad \nabla \mathcal{Q} = 0. \)

The trace of the heat kernel has then the form

\[ \text{Tr}_{L^4} \exp(-tF) = t^{-m/2}\Theta(t) + O(\mathcal{H}^3), \]

where \( \Theta(t) \) is a functional that has the following asymptotic expansion as \( t \to 0 \):

\[ \Theta(t) \sim \sum_{k=0}^{\infty} t^k A_{2k,(k)}, \]

where \( A_{2k,(k)} \) are the terms without covariant derivatives (highest order terms in the jets) in the coefficients \( A_{2k} \) and \( O(\mathcal{H}^3) \) denotes the terms with at least one derivative that vanish in the covariantly constant background curvatures.
polynomials in the curvatures and the endomorphism $Q$. Therefore, the functional $\Theta(t)$ is a generating functional for all heat kernel coefficients $A_k$ for a covariantly constant background, in particular, for all symmetric spaces.

There is a very elegant indirect way to construct the heat kernel without solving the heat equation but using only the commutation relations of differential operators $[7, 8, 9, 10]$. The main idea is in a generalization of the usual Fourier transform to the case of operators and consists in the following. Let us consider for a moment a trivial case, where the curvatures vanish but the potential term does not:

\[
\text{Riem} = 0, \quad \mathcal{R} = 0, \quad \nabla Q = 0.
\]

In this case the operators of covariant derivatives obviously commute and form an Abelian Lie algebra, i.e. $[\nabla_{\mu}, \nabla_{\nu}] = 0$. It is easy to show that the heat semigroup operator can be presented in the form

\[
\exp(-tF) = (4\pi t)^{-m/2} \exp(-tQ)
\]

\[
\times \int_{\mathbb{R}^m} dk |g|^{1/2} \exp \left( -\frac{\langle k, k \rangle}{4t} + k \cdot \nabla \right),
\]

where $\langle k, k \rangle = k^\mu g_{\mu\nu} k^\nu$ and $k \cdot \nabla = k^\mu \nabla_\mu$.

Here, of course, it is assumed that the covariant derivatives also commute with the metric, i.e. $[\nabla, g] = 0$. Acting with this operator on the Dirac distribution and using the obvious relation

\[
\exp(k \cdot \nabla)\delta(x, x') \bigg|_{x = x'} = \delta(k), \tag{35}
\]

one integrates easily over $k$ and obtains the diagonal of the heat kernel and the trace

\[
\text{Tr}_{L^2} \exp(-tF) = (4\pi t)^{-m/2} \int_{\mathbb{R}^m} \text{tr}_V \exp(-tQ).
\]

Let us consider now a more complicated case when there is a nontrivial covariantly constant curvature $\mathcal{R} \neq 0$ in flat space:

\[
\text{Riem} = 0, \quad \nabla \mathcal{R} = 0, \quad \nabla Q = 0.
\]

In this case the covariant derivatives form a nilpotent Lie algebra

\[
[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu},
\]

\[
[\nabla_\mu, \mathcal{R}_{\alpha\beta}] = [\nabla_\mu, Q] = 0,
\]

\[
[\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] = [\mathcal{R}_{\mu\nu}, Q] = 0.
\]

For this algebra one can prove a theorem expressing the heat semigroup operator in terms of an average over the corresponding Lie group $[7]$

\[
\exp(-tF) = (4\pi t)^{-m/2} \exp(-tQ)
\]

\[
\times \det_{\text{End}(TM)}^{1/2} \left( \frac{t\mathcal{R}}{\sinh (t\mathcal{R})} \right)
\]

\[
\times \int_{\mathbb{R}^m} dk |g|^{1/2} \exp \left[ -\frac{1}{4t} \langle k, t\mathcal{R} \coth (t\mathcal{R})k \rangle \right]
\]

\[
\times \exp (k \cdot \nabla).
\]

Here functions of the curvature $\mathcal{R}$ are understood as functions of sections of the bundle $\text{End}(TM) \otimes \text{End}(V)$, and the determinant $\det_{\text{End}(TM)}$ is taken with respect to End $(TM)$ indices, End $(V)$ indices being intact.

It is not difficult to show that in this case the equation (35) is still valid, so that the integral over $k^\mu$ becomes trivial and we obtain immediately the trace of the heat kernel $[7]$

\[
\text{Tr}_{L^2} \exp(-tF) = (4\pi t)^{-m/2} \int_{\mathbb{R}^m} \text{tr}_V \exp(-tQ)
\]

\[
\times \det_{\text{End}(TM)}^{1/2} \left( \frac{t\mathcal{R}}{\sinh (t\mathcal{R})} \right).
\]

Expanding this in a power series in $t$ one can find all covariantly constant terms $A_{k,k}$ in all heat kernel coefficients $A_k$.

Let us now generalize the algebraic approach to the case of curved manifolds with covariantly constant Riemann curvature and trivial connection $\nabla^V$

\[
\nabla \text{Riem} = 0, \quad \mathcal{R} = 0, \quad \nabla Q = 0.
\]

First of all, we give some definitions. The condition $\nabla \text{Riem} = 0$ defines the geometry of locally symmetric spaces. A Riemannian locally symmetric space which is simply connected and complete is a globally symmetric space (or, simply, symmetric space). A symmetric space
is said to be of compact, noncompact or Euclidean type if all sectional curvatures $K(u, v) = R_{\alpha\beta\gamma\delta} u^\alpha v^\beta u^\gamma v^\delta$ are positive, negative or zero. A direct product of symmetric spaces of compact and noncompact types is called semisimple symmetric space. A generic complete simply connected Riemannian symmetric space is a direct product of a flat space and a semisimple symmetric space.

It should be noted that our analysis is purely local. We are looking for a universal (in the category of locally symmetric spaces) local generating function of the curvature invariants, that reproduces adequately the asymptotic expansion of the function of the curvature tensor of a symmetric space, i.e. in other words all heat kernel coefficients reproduce the heat kernel coefficients whose Taylor coefficients satisfy

$$D_i = (D_{ab})$$

are known to be the generators of the holonomy algebra $\mathcal{H}$

$$[D_i, D_k] = F^j_{ik} D_j,$$

where $F^j_{ik}$ are the structure constants.

In symmetric spaces a much richer algebraic structure exists. Indeed, let us define the quantities $C^A_{BC} = -C^A_{CB}, A = 1, \ldots, D$, where $D = m + p$, by

$$C^a_{ab} = E^i_{ab}, \quad C^a_{ib} = D^a_{ib}, \quad C^a_{ikl} = F^a_{ikl},$$

and the matrices $C_A = (C^B_{AC}) = (C_a, C_i)$:

$$C_a = \begin{pmatrix} 0 & D^b_{ai} \\ E^j_{ac} & 0 \end{pmatrix},$$

$$C_i = \begin{pmatrix} D^b_{ia} & 0 \\ 0 & F^j_{ik} \end{pmatrix}.$$  

One can show that they satisfy the Jacobi identities [9, 10]

$$[C_A, C_B] = C^C_{AB} C_C$$

and, hence, define a Lie algebra $\mathcal{G}$ of dimension $D$ with the structure constants $C^A_{BC}$, the matrices $C_A$ being generators of the adjoint representation.

In symmetric spaces one can find explicitly the generators of the infinitesimal isometries, i.e. the Killing vector fields $\xi_A$, and show that they form a Lie algebra of isometries that is (in case of semisimple symmetric space) isomorphic to the Lie algebra $\mathcal{G}$, viz.

$$[\xi_A, \xi_B] = C^C_{AB} \xi_C.$$  

Moreover, introducing a symmetric nondegenerate $D \times D$ matrix

$$\gamma_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ij} \end{pmatrix},$$

that plays the role of the metric on the algebra $\mathcal{G}$, one can express the operator $F$ in semisimple
symmetric spaces in terms of the generators of isometries

\[ F = -\gamma^{AB} \xi_A \xi_B + Q, \]

where \( \gamma^{AB} = (\gamma_{AB})^{-1} \).

Using this representation one can prove that the heat semigroup operator can be presented in terms of an average over the group of isometries \( G \).

\[ \exp(-tF) = (4\pi t)^{-D/2} \exp \left[ -t \left( Q - \frac{1}{6} R_G \right) \right] \]

\[ \times \int_{R^D} dk |\gamma|^{1/2} \det_{\text{Ad}(G)}^{1/2} \left( \frac{\sinh(k \cdot C/2)}{k \cdot C/2} \right) \]

\[ \times \exp \left[ -\frac{1}{4t} (k, k) + k \cdot \xi \right] \]

where \( |\gamma| = \det \gamma_{AB}, \langle k, k \rangle = k^A \gamma^{AB} k^B, \]

\[ k \cdot C = k^A C_A, k \cdot \xi = k^A \xi_A, \quad \text{and} \quad R_G = \text{the scalar curvature of the group of isometries } G \]

\[ R_G = -\frac{1}{4} \gamma^{AB} C^C_{AD} C^D_{BC}. \]

Acting with this operator on the Dirac distribution \( \delta(x, x') \) one can, in principle, evaluate the off-diagonal heat kernel \( \exp(-tF) \delta(x, x') \), i.e. for non-coinciding points \( x \neq x' \) (see [10]). To calculate the trace of the heat kernel, it is sufficient to compute only the coincidence limit \( x = x' \). Splitting the integration variables \( k^A = (q^a, \omega^i) \) and solving the equations of characteristics one can obtain the action of the isometries on the Dirac distribution [9, 10]

\[ \exp(k \cdot \xi) \delta(x, x') \bigg|_{x=x'} = \det_{\text{End}(TM)}^{-1} \left( \frac{\sinh(\omega \cdot D/2)}{\omega \cdot D/2} \right) \delta(q), \]

where \( \omega \cdot D = \omega^i D_i \).

Using this result one can easily integrate over \( q \) to get the heat kernel diagonal. After changing the integration variables \( \omega \rightarrow \sqrt{t} \omega \) it takes the form [9, 10]

\[ U^{\text{diag}}(t) = (4\pi t)^{-m/2} \exp \left[ -t \left( Q - \frac{1}{8} R - \frac{1}{6} R_H \right) \right] \]

\[ \times (4\pi)^{-n/2} \int_{R^n} d\omega \beta^{1/2} \exp \left( -\frac{1}{4}(\omega, \omega) \right) \]

\[ \times \det_{\text{Ad}(\mathcal{H})}^{1/2} \left( \frac{\sinh(\sqrt{t} \omega \cdot F/2)}{\sqrt{t} \omega \cdot F/2} \right) \]

\[ \times \det_{\text{End}(TM)}^{-1/2} \left( \frac{\sinh(\sqrt{t} \omega \cdot D/2)}{\sqrt{t} \omega \cdot D/2} \right), \]

where \( \beta = \det \beta_{ij}, \quad \langle \omega, \omega \rangle = \omega^i \beta_{ij} \omega^j, \quad \omega \cdot F = \omega^i F_i, \quad F_i = (F^j)_{ik} \) are the generators of the holonomy algebra \( \mathcal{H} \) in adjoint representation and

\[ R_H = -\frac{1}{4} \beta^k F^m_{il} F^l_{km} \]

is the scalar curvature of the holonomy group.

The remaining integration over \( \omega \) in (36) can be done in a rather formal way [12, 13]. Let \( a^i_k \) and \( a_k \) be some operators acting on a Hilbert space that form the following \( p \)-dimensional Lie algebra

\[ [a^i_j, a^k_l] = \delta^k_j, \]

\[ [a^i_k, a^k_j] = 0. \]

Let \( |0\rangle \) be the ‘vacuum vector’ in the Hilbert space, i.e.

\[ \langle 0|0 \rangle = 1, \quad \langle 0|a^k \rangle = 0. \]

Then the heat kernel (36) can be presented in an algebraic form without any integration, i.e.

\[ U^{\text{diag}}(t) = (4\pi t)^{-m/2} \exp \left[ -t \left( Q - \frac{1}{8} R - \frac{1}{6} R_H \right) \right] \]

\[ \times \langle 0|\det_{\text{Ad}(\mathcal{H})}^{1/2} \left( \frac{\sinh(\sqrt{t} a \cdot F/2)}{\sqrt{t} a \cdot F/2} \right) \]

\[ \times \det_{\text{End}(TM)}^{-1/2} \left( \frac{\sinh(\sqrt{t} a \cdot D/2)}{\sqrt{t} a \cdot D/2} \right) \]

\[ \times \exp \left( \langle a^k, \beta^{-1} a^k \rangle \right) |0\rangle, \]

where \( a \cdot F = a^k F_k \) and \( a \cdot D = a^k D_k \). This formal solution should be understood as a power series.
in the operators $a^k$ and $a^*_k$: it determines a well defined asymptotic expansion as $t \to 0$.

By expanding these formulas in an asymptotic power series as $t \to 0$ one obtains all HMDS-coefficients $a^\text{diag}_{k, \ell}$ for any locally symmetric space. Thereby one finds all covariantly constant terms $A_{2k,(k)}$ in all heat kernel coefficients.

4. Heat-kernel Asymptotics for Non-Laplace Type Operators

In this lecture, we study a general class of second-order non-Laplace type elliptic partial differential operators, acting on sections of a vector bundle $V$ over a Riemannian manifold $M$ without boundary following our papers [15, 16]. In general, the study of spin-tensor quantum gauge fields in a general gauge necessarily leads to non-Laplace type operators acting on sections of general spin-tensor bundles described in the first lecture. It is precisely these operators that are of prime interest in the present lecture. The study of non-Laplace operators is quite new, and the available methods are still underdeveloped in comparison with the Laplace type theory. The only exception to this is the case of anti-symmetric forms, which is pretty simple and, therefore, is well understood now [17, 18, 19, 20, 21].

So, we will restrict our attention to operators acting on tensor-spinor bundles. These bundles may be characterized as those appearing as direct summands of iterated tensor products of the cotangent and spinor bundles, i.e. $V = TM \oplus \cdots \oplus TM \otimes T^*M \otimes \cdots T^*M \otimes S$, with $S$ being the spinor bundle. Alternatively, they may be described abstractly as bundles associated to representations of the spin group Spin$(m)$. These are extremely interesting and important bundles, as they describe the fields in Euclidean quantum field theory. The connection on the tensor-spinor bundles is built in a canonical way from the Levi-Civita connection. The generators are determined by the representation of Spin$(m)$ which induces the bundle $V$: they are tensor-spinors constructed purely from Kronecker symbols, together with the fundamental tensor-spinor if spin structure is involved. More general bundles appearing in field theory are actually tensor products of these with auxiliary bundles, usually carrying another (gauge) group structure.

Let $Q$ be a smooth Hermitian section of the bundle $\text{End}(V)$, i.e. $Q = Q$, and $a$ be a parallel symmetric Hermitian $\text{End}(V)$-valued tensor, more precisely, a smooth section of the vector bundle $TM \otimes TM \otimes \text{End}(V)$ satisfying the following conditions

$$a^{\mu\nu} = a^{\nu\mu}, \quad (a^{\mu\nu})^* = a^{\nu\mu}, \quad \nabla a = 0. \quad (37)$$

The operator of our primary interest in this lecture has the form

$$F = \nabla^* a \nabla + Q = -a^{\mu\nu} \nabla_\mu \nabla_\nu + Q. \quad (38)$$

Non-Laplace type operators appear naturally in the context of Stein-Weiss operators [21]. Let $T^*M \otimes V = W_1 \oplus \cdots \oplus W_n$ be the decomposition of the bundle $T^*M \otimes V$ in irreducible components $W_j$ and $\text{Pr}_i : T^*M \otimes V \to W_i$ be the corresponding projections. Stein-Weiss operators $G_i : C^\infty(V) \to C^\infty(W_i)$ are first order partial differential operators (called the gradients) defined by $G_i = \text{Pr}_i \nabla$. Then the operator $F : C^\infty(V) \to C^\infty(V)$ defined by

$$F = \sum_{i=1}^n c_i G_i^* G_i,$$

with some constants $c_i$, is a second-order non-Laplace type operator of the form (38) with

$$a = \sum_{i=1}^n c_i (\text{Pr}_i)^* \text{Pr}_i.$$

Now it is obvious that the structure of the coefficient $a$ depends solely on the representation of the spin group with which the bundle $V$ is associated.

We will restrict ourselves to a special class when the coefficient $a$ is built in a universal, polynomial way, using tensor product and contraction from the metric $g$ and its inverse $g^*$, together with (if applicable) the volume form $\epsilon$ and/or the fundamental tensor-spinor $\gamma$. Such a tensor-endomorphism $a$ is obviously parallel. Here we do not assume that $a^{\mu\nu}$ has the form $g^{\mu\nu} \text{Id}_V$ or $g^{\mu\nu} B$ with some automorphism $B \in \text{Aut}(V)$. We do not set any conditions on the endomorphism $Q$, except that it should be Hermitian.
In the following we will denote the leading symbol of the operator $F$, $\sigma_L(F; x, \xi)$, with $\xi$ a cotangent vector, just by $A(x, \xi)$. For the non-Laplace type operator $F$ in (38) it has the form

$$A(x, \xi) = a^{\mu\nu}(x)\xi_\mu \xi_\nu.$$  

We require that the leading symbol should be positive definite, i.e. $A(x, \xi)$ is a Hermitian and positive definite endomorphism for any $(x, \xi)$, with $\xi \neq 0$. In particular, $F$ is elliptic. Positive definiteness implies that the roots of the characteristic polynomial

$$\chi(x, \xi, \lambda) := \det_V[A(x, \xi) - \lambda]$$

are positive functions on $M$.

A very important point is that the structure of the spectrum, i.e. the number $s$ of eigenvalues, $\lambda_1, \ldots, \lambda_s$, and their multiplicities $d_1, \ldots, d_s$ are constant on $M$. Moreover, one can show that $\text{tr}_V A^n(x, \xi)$, and, therefore, the characteristic polynomial $\chi(x, \xi, \lambda)$ depends on $(x, \xi)$ only through $|\xi|^2 = g^{\mu\nu}(x)\xi_\mu \xi_\nu$. As a result, the dependence of the eigenvalues $\lambda_i(x, \xi)$ on $(x, \xi)$ is only through $|\xi|^2$ as well. Since $A(x, \xi)$ is 2-homogeneous in $\xi$, the $\lambda_i$ must be also:

$$\lambda_i(x, \xi) = \mu_i |\xi|^2,$$

for some positive real numbers $\mu_1, \ldots, \mu_s$, which are independent of the point $(x, \xi) \in T^*M$, and, in fact, independent of the specific Riemannian manifold $(M, g)$.

Let $\Pi_i$ be the orthogonal projection onto the $\lambda_i$-eigenspace. The $\Pi_i$ satisfy the conditions

$$\Pi_i^2 = \Pi_i, \quad \Pi_i \Pi_k = 0, \quad (i \neq k),$$

$$\sum_{i=1}^s \Pi_i = I_V, \quad \text{tr}_V \Pi_i = d_i.$$  

In contrast to the eigenvalues, the projections depend on the direction $\xi/|\xi|$ of $\xi$, rather than on the magnitude $|\xi|$. In other words, they are 0-homogeneous in $\xi$. Furthermore, they are even polynomials in $\xi/|\xi|:

$$\Pi_i(x, \xi) = \sum_{n=0}^p \frac{1}{n!2^n} \xi_{\mu_1} \cdots \xi_{\mu_{2n}} \Pi_{i(2n)}^{\mu_1 \cdots \mu_{2n}}(x),$$

where $p$ is some positive integer. Here the $\Pi_{i(2n)}$ are some $\text{End}(V)$-valued trace-free symmetric $2n$-tensors that do not depend on $\xi$. Clearly, the leading symbol can be written in terms of eigenvalues and projections

$$A(x, \xi) = |\xi|^2 \sum_{i=1}^s \mu_i \Pi_i(x, \xi).$$  

There is also a converse formula for the projections in terms of powers of the leading symbol. Note that the highest degree of projections, $p$, is also a constant that depends only on the representation to which $V$ is associated; both $s$ and $p$ can be computed explicitly in representation-theoretic terms [16].

The non-Laplace type operator $F$ with a positive definite leading symbol is an elliptic self-adjoint operator of second order. Therefore, there is a well defined heat kernel $U(t|x, x')$. Moreover, there is a well defined heat kernel diagonal $U_{\text{diag}}(t)$ and the trace of the heat kernel $\text{Tr}_{L_2} \exp(-tF)$ that have the asymptotic expansion of the standard form (16). Since the global heat kernel coefficients $A_k$ are determined by the integrals of the fiber trace of the local ones $a_k^{\text{diag}}$, it is sufficient to compute the local heat kernel coefficients, more precisely, their fiber traces, $\text{tr}_V a_k^{\text{diag}}$. By invariance theory, these coefficients are linear combinations of the local invariants built from the geometric objects (curvatures, the potential $Q$, and their derivatives) with universal numerical constants. It is these universal constants that we want to compute. Therefore, this can be done at any fixed point of the manifold.

Let us stress here that our purpose is not to provide a rigorous construction of the heat kernel with estimates; for this we rely on the standard references [6]. Rather, given that the existence of heat kernel asymptotic expansion is known, our aim is to compute its coefficients.

Our analysis will be again purely local. We fix a point $x'$ in the manifold $M$ and consider a small geodesic ball with the radius smaller than the injectivity radius of the manifold. Then any point in this ball can be connected with the fixed point $x'$ by a unique geodesic. Further, we represent
the heat kernel in the form
\[ U(t|x, y) = \Delta^{1/2}(x, x') P(x, x') U(t|x, x') \]
\[ \times P^{-1}(y, x') \Delta^{1/2}(y, x'), \quad (40) \]
where \( \Delta \) is the Van Vleck-Morette determinant and \( P \) is the parallel transport operator defined in lecture 2. Then the modified heat kernel \( U \) is a section of the bundle \( \text{End}(V) \) at \( x' \) but is scalar at \( x \) and \( y \). It satisfies the modified heat equation
\[ (\partial_t + L) U(t) = 0 \]
\[ (41) \]
where \( L = P^{-1} \Delta^{-1/2} F \Delta^{1/2} P \) is the operator defined by (9), with the initial condition
\[ U(0^+|x, y; x') = \Delta^{-1}(x, x') \delta(x, y). \]
Here and everywhere below, as usual, the differential operators act on the first space argument of the heat kernel (recall that \( x' \) is being fixed).

We shall employ the standard scaling device for the heat kernel \( U(t|x, y; x') \) when \( x \to x', \ y \to x', \) and \( t \to 0 \). We introduce a small expansion parameter \( \varepsilon \), choose the normal coordinates at \( x' \), and scale the coordinates according to
\[ x \to x' + \varepsilon(x - x'), \quad y \to x' + \varepsilon(y - x'), \]
\[ t \to \varepsilon^2 t, \quad (42) \]
Note that this also means that the derivatives scale according to
\[ \partial_t \to \frac{1}{\varepsilon^2} \partial_t, \quad \partial_{\mu} \to \frac{1}{\varepsilon} \partial_{\mu}. \]
Note also that in normal coordinates \( \Delta(x, x') = |g(x)|^{-1/2} \), so that \( \Delta^{-1}(x, x') \delta(x, y) = \delta(x - y) \), and \( P(x, x') = 1 \) (however, \( \nabla P \neq 0 \)).

Next, we expand the operator \( L \) and the heat kernel \( U \) in a formal asymptotic series in \( \varepsilon \)
\[ L \sim \sum_{n=0}^\infty \varepsilon^n L_n, \]
and
\[ U(t) \sim \sum_{n=0}^\infty \varepsilon^n U_n(t), \]
The zeroth order heat kernel is determined by the equation
\[ (\partial_t + F_0) U_0(t) = 0 \]
\[ (43) \]
with the initial condition
\[ U_0(0^+|x, y; x') = \delta(x - y). \]
The higher order approximations are determined by the following differential recurrence relations
\[ (\partial_t + F_0) U_k(t) = - \sum_{n=0}^{k-1} F_{k-n} U_n(t) \]
with the initial conditions
\[ U_k(0^+|x, y; x') = 0. \]
By construction the coefficients \( U_n \) are homogeneous functions, i.e.
\[ U_n(t|x, y; x') = t^{(n-m)/2} \]
\[ \times U_n \left( 1 \left| x' + \frac{x - x'}{\sqrt{t}}, y - \frac{x - x'}{\sqrt{t}} \right| x' \right). \]
Therefore, on the diagonal one obtains the asymptotic expansion
\[ U_{\text{diag}}(t) \sim \sum_{n=0}^\infty t^{(n-m)/2} U_n^{\text{diag}}(1), \]
\[ (46) \]
where
\[ U_n^{\text{diag}}(t|x) = U_n(t|x, x). \]
Comparing this with the standard heat kernel asymptotic expansion (16) we see that the diagonal odd-order coefficients vanish
\[ U_{2k+1}^{\text{diag}}(t) = 0, \]
and the even-order ones give the heat kernel coefficients
\[ A_{2k} = \int_M \text{tr}_{U_{2k}^{\text{diag}}(x)}. \]
Using the form of the operator \( L \) in the leading order
\[ F_0 = -g^{\mu\nu}(x') \partial_{\mu} \partial_{\nu}. \]
we easily find the leading order heat kernel by Fourier transform

\[ U_0(t|x,y;x') = \int_{\mathbb{R}^m} \frac{d\xi}{(2\pi)^m} e^{i\xi \cdot (x-y)} \exp[-tA(x',\xi)], \]

where \( \xi \cdot (x-y) = \xi_{\mu}(x^\mu - y^\mu) \). Here and everywhere below all integrals over \( \xi \) will be over \( \mathbb{R}^m \). Writing the leading symbol in terms of the projections, we get

\[ U_0(t|x,y;x') = \sum_{i=1}^s \int \frac{d\xi}{(2\pi)^m} e^{i\xi \cdot (x-y) - t\mu_i|\xi|^2} \times \Pi_i(x',\xi). \]

The trace of the diagonal can now be easily computed

\[ \text{tr} V U_0^{\text{diag}}(t) = \sum_{i=1}^s d_i (4\pi t\mu_i)^{-m/2}. \quad (48) \]

It gives the diagonal value of the lowest order heat kernel coefficient \( a_0 \):

\[ \text{tr} V a_0^{\text{diag}} = \sum_{i=1}^s d_i \mu_i^{m/2}, \]

and, therefore,

\[ A_0 = (4\pi)^{-m/2} \sum_{i=1}^s \frac{d_i}{\mu_i^{m/2}} \text{vol}(M). \]

These formula points out a new feature of non-Laplace type operators; one which complicates life somewhat. Whereas the dimension dependence of the heat coefficients of Laplace type operators is isolated in the overall factor \( (4\pi)^{-m/2} \), the dimension dependence for non-Laplace type operators is more complicated.

The calculation of higher-order coefficients is a challenging task. We will indicate how the coefficient \( A_1 \) is computed. By the invariance theory we have,

\[ A_2 = \int M \text{tr} V (HQ + \mathbf{1}\beta R), \]

where \( \beta \) is a universal constant and \( H \) is some endomorphism. Both \( \beta \) and \( H \) depend only on the leading symbol of the operator \( F \).

To compute these quantities we need, first of all, the Taylor expansion of the metric and connection in normal coordinates

\[ g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\nu\alpha\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta) + O[(x-x')^3], \]

\[ A_\mu(x) = -\frac{1}{2} R_{\mu\nu}(x^\alpha - x'^\alpha) + O[(x-x')^2], \]

\[ Q(x) = Q + O[(x-x')^3]. \]

Here and below all coefficients are computed at the fixed point \( x' \); we do not indicate this explicitly. Similarly, the Taylor expansion of the tensor-endomorphism \( a^{\mu\nu}(x) \) is determined by the equation \( \nabla_\mu a^{\alpha\beta} = 0 \), which gives

\[ a^{\mu\nu}(x) = a^{\mu\nu} + \frac{1}{3} a^{\lambda(\mu R^\nu)}_{\alpha\lambda\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta) + O[(x-x')^3]. \]

Using these formulas we obtain

\[ F_1 = 0, \]

\[ F_2 = X^{\mu\nu}_{\alpha\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta) \partial_\mu \partial_\nu + Y^{\mu}_{\alpha}(x^\alpha - x'^\alpha) \partial_\mu + Q, \]

where

\[ X^{\mu\nu}_{\alpha\beta} = -\frac{1}{3} a^{\lambda(\mu R^\nu)}_{(\alpha|\lambda|\beta)}, \]

\[ Y^{\mu}_{\alpha} = \frac{2}{3} a^{\mu\lambda} R_{\lambda\alpha} - \frac{1}{2} [R_{\alpha\mu}, a^{\mu\nu}]_+, \]

and \([A,B]_+ = AB + BA\) denotes the anticommutator.

From the recurrence relations (and the initial conditions) we find

\[ U_1 = 0 \]

and

\[ U_2(t) = -\int_0^t dt U_0(t - \tau) F_2 U_0(\tau). \]
By using the Fourier representation for $U_0$ we obtain for the diagonal

$$U_2^{\text{diag}}(t) = -\int \frac{d\xi}{(2\pi)^m} \int_0^t d\tau e^{-(t-\tau)A(x,\xi)} \hat{F}_2 e^{-\tau A(x,\xi)},$$

where

$$\hat{F}_2 = X^{\mu\nu} \partial_\mu \partial_\nu \xi^\alpha \xi^\beta - Y^\mu \partial_\mu \xi^\nu + Q.$$  

This gives finally the heat kernel coefficient $A_2$

$$A_2 = -(4\pi)^{-m/2} \sum_{i=1}^{s} \mu_i^{-m/2} \int \frac{d\xi}{\pi^{m/2}} e^{-|\xi|^2} \Omega_i(x, \xi).$$

The calculation of the coefficient $\beta$ is a much more complicated problem. After a long calculation one obtains a complicated expression in terms of the constant $\mu_i$ (the eigenvalues of the leading symbol) and the leading symbol of the operator $F$ (see, [15]).

An interesting feature of the non-Laplace type operators is the semi-classical polarization. This means that, unlike Laplace type operators, the asymptotic expansion of the heat kernel for non-Laplace type operators has the form

$$U(t|x, x') = \sum_{i=1}^{s} (4\pi \mu_i)^{-m/2} \Delta^{1/2}(x, x') \times \exp \left[ -\frac{\sigma(x, x')}{2t\mu_i} \right] \Omega_i(t|x, x'),$$

where each transport function $\Omega_i$ satisfies a different transport equation. This also implies that the differential recursion system for the coefficients of the asymptotic expansion of the transport functions is much more complicated.

5. Heat-Kernel Asymptotics of Oblique Boundary-Value Problem

In this lecture we study the heat kernel asymptotics for a Laplace type partial differential operator acting on sections of a vector bundle over a compact Riemannian manifold with boundary. In this case one has to impose some boundary conditions in order to make a (formally self-adjoint) differential operator self-adjoint (at least symmetric) and elliptic. There are many admissible boundary conditions that guarantee the self-adjointness and ellipticity of the problem. The simplest boundary conditions are the classical Dirichlet and the Neumann ones. There exist also slight modifications of the Neumann boundary conditions (called Robin boundary conditions in physical literature) when the normal derivative of the field at the boundary is proportional to the value of the field at the boundary [22, 23]. In an even more general scheme, called mixed boundary conditions, the Dirichlet and Robin boundary conditions are mixed by using some projectors. In this lecture we study a more general setup, called the oblique boundary-value problem, which includes both normal and tangential derivatives of the fields at the boundary [24, 25]. We will follow mainly our papers [26, 27].

Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $m$ with smooth boundary $\partial M$ with a positive-definite Riemannian metric $g$ on $M$ and induced metric $\hat{g}$ on $\partial M$. In this lecture the Greek indices range from 1 through $m$ and lower case Latin indices range from 2 through $m$. Let $\{\hat{e}_i\} = \{\hat{e}_2, \ldots, \hat{e}_m\}$ be the local frame for the tangent bundle $T\hat{M}$ and $\hat{x} = (\hat{x}^2, \ldots, \hat{x}^m)$, be the local coordinates on $\partial M$. Let $r$ be the normal geodesic distance to $\partial M$, and $\hat{N} = \partial_r|_{\partial M}$ be the inward pointing unit normal to $\partial M$. Let $V$ be a (smooth) vector bundle over the manifold $M$, $\nabla$ be a connection on $V$ and $\hat{\nabla}$ be the induced connection on the boundary. Further let $Q$ be a smooth endomorphism of $V$ and $F$ be a Laplace type operator $F = -g^{\mu\nu} \nabla_\mu \nabla_\nu + Q$.

Let $W = V|_{\partial M}$ be the restriction of the vector bundle $V$ to the boundary $\partial M$. We define the boundary data map $\psi : \text{C}^\infty(V) \to \text{C}^\infty(W \oplus W)$
by
\[
\psi(\varphi) = \left( \varphi|_{\partial M}, \nabla_N \varphi|_{\partial M} \right).
\]

Let \( \Pi \) be an orthogonal Hermitian projector acting on \( W \) and \( \Gamma \) be an anti-Hermitian endomorphism of the bundle \( W \) orthogonal to \( \Pi \), i.e., \( S^\ast = S, \Pi S = S \Pi = 0 \). We use these to define a first-order self-adjoint tangential differential operator \( \Lambda : C^\infty(W) \to C^\infty(W) \) by
\[
\Lambda = (\mathbf{I} - \Pi) \left\{ \frac{1}{2} \left( \Gamma^\ast \nabla_i + \nabla_i \Gamma^\ast \right) + S \right\} (\mathbf{I} - \Pi).
\]

We study the oblique boundary-value problem for the Laplace type operator \( F \). The oblique boundary conditions now read
\[
B \psi(\varphi) = 0,
\]
where \( B : C^\infty(W \oplus W) \to C^\infty(W \oplus W) \) is the boundary operator defined by
\[
B = \left( \begin{array}{cc} \Pi & 0 \\ \Lambda & \mathbf{I} - \Pi \end{array} \right).
\]

This is equivalent to the following boundary conditions
\[
\Pi \varphi|_{\partial M} = \Lambda \varphi|_{\partial M} = 0,
\]
\[
(\mathbf{I} - \Pi) \nabla_N \varphi|_{\partial M} + \Lambda \varphi|_{\partial M} = 0.
\]

The boundary operator \( B \) (49) incorporates all standard types of boundary conditions. Indeed, by choosing \( \Pi = \mathbf{I} \) and \( \Lambda = 0 \) one gets the Dirichlet boundary conditions, by choosing \( \Pi = 0, \Lambda = \mathbf{I} \) one gets the Neumann boundary conditions. More generally, the choice \( \Gamma = 0 \) corresponds to the mixed boundary conditions.

Integration by parts shows that the Laplace-type operator \( F \) endowed with oblique boundary conditions is symmetric, i.e., \( \langle \varphi, F \psi \rangle = \langle F \varphi, \psi \rangle \). However, it is not necessarily elliptic. To be elliptic the boundary-value problem \((F, B)\) has to satisfy two conditions. First of all, the leading symbol of the operator \( F \) should be elliptic in the interior of \( M \). Second, the so-called strong ellipticity condition should be satisfied. This question was studied in [26] where it has been shown that the oblique boundary-value problem for a Laplace type operator is strongly elliptic with respect to the cone \( \mathbf{C} \setminus \mathbf{R}^+ \) if and only if for any nonvanishing cotangent vector \( \zeta \) on the boundary the endomorphism \( \langle \zeta | I - i \Gamma \cdot \zeta \rangle \) is positive-definite, i.e., \( \langle \zeta | I - i \Gamma \cdot \zeta \rangle > 0 \). A sufficient condition for strong ellipticity is: \( \langle \zeta | 2 I + (\Gamma \cdot \zeta)^2 \rangle > 0 \).

The heat kernel \( U(t|x, y) \) is now defined by the heat equation
\[
(\partial_t + F) U(t) = 0,
\]
the initial condition
\[
U(0^+ | x, y) = \delta(x, y),
\]
and the boundary conditions
\[
B \psi[U(t|x, y)] = 0. \quad (50)
\]
Hereafter the boundary data map (as well as the boundary operator) acts on the first argument of the heat kernel.

The heat kernel of a smooth boundary-value problem on a manifold with boundary has the following asymptotic expansion as \( t \to 0^+ \) [6]
\[
\text{Tr}_L \exp(-tF) \sim \sum_{k=0}^{\infty} t^{(k-m)/2} A_k. \quad (51)
\]

In the case of manifolds without boundary only even order terms were present (see 16). Now, in contrast, all \( A_k \) are non-zero. They have the following general form:
\[
A_{2k} = \int_{\partial M} a_{2k}^{(0)} + \int_{\partial M} a_{2k}^{(1)} \quad (52)
\]
\[
A_{2k+1} = \int_{\partial M} a_{2k+1}^{(1)} \quad (53)
\]
Hereafter the integration over the boundary is defined with the help of the usual Riemannian volume element \( d\text{vol}_\hat{g} \) on \( \partial M \) with the help of the induced metric \( \hat{g} \).

Here \( a_k^{(0)} \) are the (local) interior heat-kernel coefficients and \( a_k^{(1)} \) are the boundary ones. The local interior coefficients \( a_k^{(0)} \) are determined by the
same local invariants as in the manifolds without boundary, i.e. by the HMDG coefficients \( \text{tr}_V a^{\text{diag}}_k \), and therefore, do not depend on the boundary conditions. The boundary coefficients \( a^{(1)}_k \) are far more complicated because in addition to the geometry of the manifold \( M \) they depend essentially on the geometry of the boundary \( \partial M \) and on the boundary conditions. For Laplace-type operators they are known for the usual boundary conditions (Dirichlet, Neumann, or mixed version of them) up to \( a^{(1)}_5 \) [22, 23]. For oblique boundary conditions including tangential derivatives some coefficients were recently computed in [26, 27, 28, 29].

In this lecture we will evaluate the coefficient \( A_1 \), following our recent work [26].

Let us fix a positive number \( \delta > 0 \). We split the whole manifold in a disjoint union of two different parts: \( M = M^{\text{int}} \cup M^{\text{bound}} \), where \( M^{\text{bound}} \) is a narrow geodesic strip near the boundary \( \partial M \) of the width \( \delta \) and \( M^{\text{int}} \) is the interior of the manifold \( M \) (without the thin strip), i.e. \( M^{\text{int}} = M \setminus M^{\text{bound}} \).

We will construct the parametrix on \( M \) by using different approximations in \( M^{\text{bound}} \) and \( M^{\text{int}} \). Strictly speaking, to glue them together in a smooth way one should use ‘smooth characteristic functions’ of different domains (partition of unity) and carry out all necessary estimates. What one has to control is the order of the remainder terms in the limit \( t \to 0 \) and their dependence on \( \delta \). Since our task here is not to prove the form of the asymptotic expansion, which is known, but rather to compute explicitly the coefficients of the asymptotic expansion, we will not worry about such subtle details. We will compute the asymptotic expansion as \( t \to 0 \) in each domain and then take the limit \( \delta \to 0 \).

We will use different local coordinates in different domains. In \( M^{\text{int}} \) we can, for example, choose normal coordinates centered at a fixed point \( x_0 \). In fact, this is not necessary—we can use a manifestly covariant technique described in lecture 2. In \( M^{\text{bound}} \) we choose the local coordinates as follows. By using the geodesic flow we get the local frame \( \{ N, e_i \} \) for the tangent bundle \( TM \) and the local coordinates \( x = (x, \tilde{x}) \) on \( M^{\text{bound}} \), which identifies \( M^{\text{bound}} \) with \( \partial M \times (0, \delta) \).

The construction of the parametrix in the interior \( U^{\text{int}}(t|x, y) \) goes along the same lines as for manifolds without boundary described in the previous lectures. The idea is always to separate the basic case (when the coefficients of the operator \( F \) are frozen at a fixed point \( x_0 \)). In the case of manifolds without boundary the basic case is, in fact, zero-dimensional, i.e. algebraic. The interior parametrix is defined by the heat equation (2), the initial condition (3) and by an asymptotic condition at infinity (instead of the boundary conditions). This means that effectively one introduces a small expansion parameter \( \varepsilon \) reflecting the fact that the points \( x \) and \( y \) are close to each other and the parameter \( t \) is small. This can be done by fixing a point \( x_0 = x' \) in \( M^{\text{int}} \), choosing the normal coordinates at this point (with \( g_{\mu\nu}(x') = \delta_{\mu\nu} \)), scaling like in (42) and expanding in a power series in \( \varepsilon \). This construction is standard and we do not repeat it here (see lectures 2 and 4). Locally, at any point in \( M^{\text{int}} \), the resulting interior parametrix is given by the same formulas as the heat kernel for a manifold without boundary, i.e. by the same formulas as in the lecture 2. For a fixed finite \( \delta > 0 \) the error of this approximation is exponentially small as \( t \to 0 \). Thus, the interior parametrix has the same asymptotic expansion with \( t \to 0 \) as the heat kernel for a manifold without boundary. In other words, as \( t \to 0 \)

\[
\int_{M^{\text{int}}} \text{tr}_V U^{\text{int}}_{\text{diag}}(t) \sim \sum_{k=0}^{\infty} t^{k-m/2} (4\pi)^{-m/2} \frac{(-1)^k}{k!} \int_{M^{\text{int}}} \text{tr}_V a^{\text{diag}}_k(t),
\]

where \( a^{\text{diag}}_k \) are the standard local HMDG coefficients for manifolds without boundary computed in lecture 2. By taking the limit \( \delta \to 0 \) of this equation we obtain

\[
a^{(0)}_{2k+1} = 0, \tag{55}
\]

\[
a^{(0)}_{2k} = (4\pi)^{-m/2} \frac{(-1)^k}{k!} \text{tr}_V a^{\text{diag}}_k. \tag{56}
\]

For an elliptic boundary-value problem the diagonal of the parametrix \( T^{\text{bound}}_{\text{diag}}(t) \) in \( M^{\text{bound}} \)
has exponentially small terms, i.e. \( \sim \exp(-r^2/t) \), as \( t \to 0^+ \) and \( 0 < r < \delta \). These terms behave like distributions near the boundary, and, therefore, the integrals over \( M^{\text{bnd}} \), more precisely, the integrals \( \lim_{\delta \to 0} \int_{\partial M} \int_{0}^{\delta} d\tau \ldots \), do contribute to the asymptotic expansion with coefficients being the integrals over \( \partial M \). It is this phenomenon that leads to the boundary terms in the heat kernel coefficients. Thus, such terms determine the local boundary contributions \( a_k^{(1)} \) to the global heat-kernel coefficients \( A_k \).

The boundary parametrix \( U^{\text{bnd}}(t|x, x') \) in \( M^{\text{bnd}} \) is constructed as follows. Now we want to find the fundamental solution of the heat equation near diagonal, i.e. for \( x \to x' \) and for small \( t \to 0 \) in the region \( M^{\text{bnd}} \) close to the boundary, i.e. for small \( r \) and \( r' \), that satisfies the boundary conditions on \( \partial M \) and an asymptotic condition at infinity. We fix a point on the boundary \( x_0 \in \partial M \) and choose normal coordinates on \( \partial M \) at this point (with \( g_{ij}(0, x_0) = \delta_{ij} \)).

To construct the boundary parametrix, we again scale the coordinates. But now we include the coordinates \( r \) and \( r' \) in the scaling

\[
\begin{align*}
\hat{x} &= \hat{x}_0 + \varepsilon (\hat{x} - \hat{x}_0), \\
\hat{x}' &= \hat{x}_0 + \varepsilon (\hat{x}' - \hat{x}_0), \\
r &= \varepsilon r, \\
r' &= \varepsilon r', \\
t &= \varepsilon^2 t.
\end{align*}
\]

The corresponding differential operators are scaled by

\[
\hat{\partial} \to \frac{1}{\varepsilon} \hat{\partial}, \quad \hat{\partial}_r \to \frac{1}{\varepsilon} \hat{\partial}_r, \quad \hat{\partial}_t \to \frac{1}{\varepsilon^2} \hat{\partial}_t.
\]

Then, we expand the scaled operator \( F \) in a power series in \( \varepsilon \), i.e.

\[
F \sim \sum_{n=0}^{\infty} \varepsilon^{n-2} F_n,
\]

where \( F_n \) are second-order differential operators with homogeneous symbols. Next, we expand the scaled boundary operator (with an extra factor \( \varepsilon \) at the operator \( \Lambda \))

\[
B \sim \sum_{n=0}^{\infty} \varepsilon^n B_n,
\]

where \( B_n \) are first-order tangential operators with homogeneous symbols. At zeroth order we have

\[
\begin{align*}
F_0 &= -\hat{\partial}_r^2 - \hat{\partial}_t^2, \\
B_0 &= \begin{pmatrix} \Pi_0 & 0 \\ \Lambda_0 & I - \Pi_0 \end{pmatrix},
\end{align*}
\]

where \( \Pi_0 = \Pi(\hat{x}') \) and

\[
\hat{\partial}_t^2 = \hat{g}_{ik}(\hat{x}')\hat{\partial}_i\hat{\partial}_k, \quad \Lambda_0 = \Gamma^j(\hat{x}')\hat{\partial}_j.
\]

Note that all leading-order operators \( F_0, B_0 \) and \( \Lambda_0 \) have constant coefficients and, therefore, are very easy to handle.

The subsequent strategy is rather simple. We expand the scaled heat kernel in \( \varepsilon \)

\[
U^{\text{bnd}} \sim \sum_{n=0}^{\infty} \varepsilon^{2-m+n} U_n^{\text{bnd}},
\]

and substitute into the scaled version of the heat equation and the boundary condition. Then, by equating the like powers in \( \varepsilon \) one gets an infinite set of recursive differential equations for \( U_n \)

\[
(\hat{\partial}_r + F_0)U_k^{\text{bnd}} = - \sum_{n=1}^{k} F_n U_{k-n}^{\text{bnd}},
\]

with the boundary conditions

\[
B_0 \psi[U_k^{\text{bnd}}] = - \sum_{n=1}^{k} B_n \psi[U_{k-n}^{\text{bnd}}],
\]

and the asymptotic condition at infinity

\[
\lim_{r \to \infty} U_k^{\text{bnd}}(t|r, \hat{x}; r', \hat{x}') = 0.
\]

In other words, we decompose the parametrix into the homogeneous parts with respect to \((\hat{x} - \hat{x}_0), (\hat{x}' - \hat{x}_0), \hat{r}, \hat{r}' \) and \( t \). By using this homogeneity we obtain finally the asymptotic expansion of the diagonal of the boundary parametrix

\[
U_{\text{diag}}^{\text{bnd}}(t) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} U_k^{\text{bnd}} \left( \frac{r}{\sqrt{t}}, \frac{\hat{r}}{\sqrt{t}}, \frac{\hat{r}'}{\sqrt{t}}, \frac{\hat{x}}{\sqrt{t}}, \frac{\hat{x}'}{\sqrt{t}} \right).
\]

Now we have to integrate the diagonal \( U_{\text{diag}}^{\text{bnd}} \) over \( M^{\text{bnd}} \), expand it in an asymptotic series as \( t \to 0 \), and then take the limit \( \delta \to 0 \). One should
stress that the volume element $d\text{vol}(x) = \sqrt{|g|}dx$ should also be scaled, i.e.
\[ d\text{vol}(\varepsilon r, \hat{x}) \sim d\text{vol}(0, \hat{x}) \cdot \sum_{k=0}^{\infty} \varepsilon^k k! g_k(\hat{x}), \]
where
\[ g_k(\hat{x}) = \frac{\partial^k}{\partial r^k} \left[ \frac{d\text{vol}(r, \hat{x})}{d\text{vol}(0, \hat{x})} \right] \bigg|_{r=0}. \]
The coefficients $g_k$ will contribute directly to the coefficients of the asymptotic expansion.

We have
\[
\int_{M^{\text{bnd}}} \text{tr} V U_{\text{diag}}^{\text{bnd}}(t) \sim \int_{\partial M} \int_0^\delta \frac{d\xi t^{(k-m)/2}}{dt} \frac{d\text{vol}(r, \hat{x})}{d\text{vol}(0, \hat{x})} \cdot \text{tr} V U_{\text{bnd}}^{\text{bnd}} \left( 1 \frac{r}{\sqrt{t}}, \hat{x} ; \frac{r}{\sqrt{t}}, \hat{x} \right).
\]
Since as $\delta \to 0$ the volume of $M^{\text{bnd}}$ vanishes, i.e. $\lim_{\delta \to 0} \text{vol}(M^{\text{bnd}}) = 0$, the contribution of all regular terms will vanish in the limit $\delta \to 0$. In contrary, the singular terms, which behave like distributions near $\partial M$, will give the $\partial M$ contributions to the boundary heat kernel coefficients $a_k^{(1)}$. By changing the integration variable $r = \sqrt{\delta} \xi$ the integral $\int_{\partial M} \frac{d\xi}{\sqrt{\delta}} \frac{d\text{vol}(r, \hat{x})}{d\text{vol}(0, \hat{x})}$ becomes $\int_0^\infty d\xi t^{1/2}(\ldots)$ and in the limit $t \to 0$ becomes the improper integral $\int_0^\infty d\xi t^{1/2}(\ldots)$ plus an exponentially small remainder term. Then in the limit $\delta \to 0$ we obtain integrals over $\partial M$ up to an exponentially small function that we are not interested in. More precisely, as the result we get the coefficients $a_k^{(1)}$ in the form
\[
a_k^{(1)} = \sum_{n=0}^{k-1} \frac{1}{n!} g_n \lim_{\delta \to 0} \int_0^{\delta/\sqrt{t}} d\xi \xi^n \times \text{tr} V U_{k-n-1}^{\text{bnd}}(1 | \xi, \hat{x} ; \xi, \hat{x}).
\]

In this lecture we will find the boundary parametrices of the heat equation to leading order, i.e. $U_{0}^{\text{bnd}}$. We fix a point $\hat{x} \in \partial M$ on the boundary and the normal coordinates at this point (with $\hat{g}_{ik}(\hat{x}') = \delta_{ik}$), take the tangent space $T\partial M$ and replace the manifold $M$ by $M_0 \equiv T\partial M \times \mathbb{R}^+$. By using the explicit form of the zeroth-order operators $F_0$, $B_0$ and $A_0$ we obtain the equation
\[
\left( \partial_t - \partial_r^2 - \partial_{\hat{x}}^2 \right) U_0^{\text{bnd}}(t | x, y) = 0,
\]
and the boundary conditions
\[
\Pi_0 U_0^{\text{bnd}}(t | x, y) \bigg|_{r(x)=0} = 0,
\]
\[
(I - \Pi_0) \left( \partial_r + i \Gamma_0 \partial_{\hat{x}} \right) U_0^{\text{bnd}}(t | x, y) \bigg|_{r(x)=0} = 0,(62)
\]
where $\Pi_0 = \Pi(\hat{x}')$, $\Gamma_0 = \Gamma^j(\hat{x}')$. As usual the differential operators always act on the first argument of a kernel. Moreover, for simplicity of notation, we will denote $\Pi_0$ and $\Gamma_0$ just by $\Pi$ and $\Gamma$ and omit the dependence of all geometric objects on $\hat{x}'$. To leading order this does not cause any misunderstanding. Furthermore, the heat kernel should be symmetric and vanish at infinity.

By using the Laplace transform in $t$ and Fourier transform in $(\hat{x} - \hat{y})$ this equation reduces to an ordinary differential equation of second order in $r$ on $\mathbb{R}^+$, which can be easily solved taking into account the boundary conditions at $r = 0$ and $r \to \infty$. Omitting simple but lengthy calculations we obtain
\[
U_0^{\text{bnd}}(t | x, y) = \int_{\mathbb{R}^+} \frac{d\xi}{(2\pi)^{n-r}} \int_{w-i\infty}^{u+i\infty} \frac{d\lambda}{2\pi i}
\times e^{-t\lambda + ic |\hat{x} - \hat{y}|} G(\lambda | \xi, r(x), r(y)),
\]
where $w$ is a negative constant and $G$ is the leading-order resolvent kernel in momentum representation. It reads
\[
G(\lambda | \xi, u, v) = \frac{1}{2\sqrt{\xi^2 - \lambda}}
\times \exp \left\{ -|u - v| \sqrt{\xi^2 - \lambda} \right\}
+ \left[ I - 2\Pi + 2i\Gamma \cdot \xi \left( I \sqrt{\xi^2 - \lambda} - i\Gamma \cdot \xi \right)^{-1} \right] \times \exp \left\{ -(u + v) \sqrt{\xi^2 - \lambda} \right\},
\]
where Re $\sqrt{|\xi|^2 - \lambda} > 0$.

By changing the integration variables, deforming the contour of integration and computing certain Gaussian integrals, we obtain the heat kernel diagonal

$$U_0^{\text{bdy}}(t|x, x) = (4\pi t)^{-n/2} \left\{ I + \exp \left( -\frac{r^2}{t} \right) (I - 2\Pi) + \Phi \left( \frac{r}{\sqrt{t}} \right) \right\},$$

where

$$\Phi(z) = -2 \int_{\mathbb{R}^{n-1}} \frac{d\xi}{\pi^{(m-1)/2}} \int_C \frac{d\omega}{\sqrt{\pi}} \times \exp \left[ -|\xi|^2 - \omega^2 + 2i\omega z \right] \Gamma \cdot \zeta (\omega I + \Gamma \cdot \zeta)^{-1}.$$

Here the contour of integration $C$ comes from $-\infty + i\varepsilon$, encircles the point $\omega = i|\xi|$ in the clockwise direction and goes to $+\infty + i\varepsilon$, with $\varepsilon > 0$ a positive infinitesimal parameter; the contour $C$ does not cross the interval $\text{Re} \omega = 0$, $0 < |\omega| < |\xi|$, on the imaginary axis and is above all singularities of the resolvent $G$.

Now by using eq. (59) we obtain the coefficient $a_1^{(1)}$

$$a_1^{(1)} = (4\pi)^{-(m-1)/2} \int_0^\infty d\xi \times \Gamma \cdot \{ e^{-\xi^2} (I - 2\Pi) + \Phi(\xi) \},$$

and finally, by computing the integral over $\xi$ we get

$$a_1^{(1)} = (4\pi)^{-(m-1)/2} \left\{ -I - 2\Pi + 2 \int_{\mathbb{R}^{n-1}} \frac{d\xi}{\pi^{(m-1)/2}} \exp \left[ -|\xi|^2 - (\Gamma \cdot \zeta)^2 \right] \right\}. \quad (64)$$

We now consider two particular cases. First of all, if the matrices $\Gamma$ form an Abelian algebra, i.e. $[\Gamma^i, \Gamma^j] = 0$, then the integral (64) is Gaussian and can be easily evaluated explicitly:

$$a_1^{(1)} = (4\pi)^{-(m-1)/2} \left\{ -I - 2\Pi + 2(I + \Gamma^2)^{-1/2} \right\}.$$

Another very important case is when the operator $\Lambda$ is a natural Dirac type operator when the matrices $\Gamma^j$ form a Clifford like algebra

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2 \hat{g}^{ij} \frac{1}{(m-1)} \Gamma^2,$$

where $\Gamma^2 = \hat{g}_{ij} \Gamma^i \Gamma^j$. In this case one obtains

$$a_1^{(1)} = (4\pi)^{-(m-1)/2} \left\{ -I - 2\Pi + 2 \left( I + \frac{1}{(m-1)} \Gamma^2 \right)^{-(m-1)/2} \right\}.$$

Note that the integral (64) diverges when the strong ellipticity condition, $|\zeta|^2 I + (\Gamma \cdot \zeta)^2 > 0$, is violated. This leads to singularities in the heat kernel coefficients. This is a general feature of the oblique boundary-value problem.

6. Heat-Kernel Asymptotics for Non-Smooth Boundary Conditions

The boundary-value problem studied in the previous lecture was in the smooth category. A more general (and much more complicated) setting, so called singular boundary-value problem, arises when either the symbol of the differential operator or the symbol of the boundary operator (or the boundary itself) are not smooth. In this lecture we study a singular boundary-value problem for a second order partial differential operator of Laplace type when the operator itself has smooth coefficients but the boundary operator is not smooth. The case when the manifold as well as the boundary are smooth, but the boundary operator jumps from Dirichlet to Neumann on the boundary, is known in the literature as the Zaremba problem. Zaremba problem belongs to a much wider class of singular boundary-value problems, i.e. manifolds with singularities (corners, edges, cones etc.). There is a large body of literature on this subject where the problem is studied from a very abstract function-analytical point of view (see [30] and the references therein.) However, the study of heat kernel asymptotics of Zaremba type problems is quite new, and there are only some preliminary results in this area ([31, 32]. Moreover, compared to the smooth category the
needed machinery is still underdeveloped. In this lecture we will closely follow our papers [33, 34]. Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(m\) with smooth boundary \(\partial M\) with a positive-definite Riemannian metric \(g\) on \(M\) and induced metric \(\hat{g}\) on \(\partial M\). In this lecture we will be dealing with submanifolds of Riemannian manifolds of codimension one and two. Therefore, we need to fix notation, first of all. With our notation, Greek indices, \(\mu, \nu, \ldots\), label the local coordinates on \(M\) and range from 1 through \(m\), lower case Latin indices from the middle of the alphabet, \(i, j, k, l, \ldots\), label the local coordinates on \(\partial M\) (codimension one manifold) and range from 2 through \(m\), and lower case Latin indices from the beginning of the alphabet, \(a, b, c, d, \ldots\), label the local coordinates on a codimension two manifold \(\Sigma_0 \subset \partial M\) that will be described later and range over \(3, \ldots, m\). Further, we will denote by \(\hat{g}\) the induced metric on the submanifolds (of the codimension one or two). We should stress from the beginning that we slightly abuse the notation by using the same symbols for all submanifolds (of codimension one or two). This should not cause any misunderstanding since it is always clear from the context what is meant.

Let \(V\) be a vector bundle over the manifold \(M\), \(\nabla\) be a connection on \(V\) and \(\nabla\) be the induced connection on the boundary. Further let \(Q\) be a smooth endomorphism of \(V\) and \(F\) be a Laplace type operator \(F = -g^{\mu\nu} \nabla_\mu \nabla_\nu + Q\). Let \(W = V|_{\partial M}\) be the restriction of the vector bundle \(V\) to the boundary \(\partial M\) and \(N\) be the inward pointing unit normal to the boundary. We use these to define the boundary data map \(\psi : C^\infty(V) \to C^\infty(W \oplus W)\) and the boundary operator \(B : C^\infty(W \oplus W) \to C^\infty(W \oplus W)\): the boundary conditions then are \(B\psi(\varphi) = 0\).

We always assume the manifold \(M\) itself and the coefficients of the operator \(F\) to be smooth in the interior of \(M\). If, in addition, the boundary \(\partial M\) is smooth, and the boundary operator \(B\) is a differential operator with smooth coefficients, then \((F, B)\) is a smooth local boundary-value problem. If the boundary \(\partial M\) consists of a finite number of disjoint connected parts, \(\partial M = \cup_{i=1}^n \Sigma_i\), with each \(\Sigma_i\) being compact connected manifold without boundary, \(\partial \Sigma_i = \emptyset\) and \(\Sigma_i \cap \Sigma_j = \emptyset\) if \(i \neq j\), then one can impose different boundary conditions on different connected parts of the boundary \(\Sigma_i\). This means that the full boundary operator decomposes \(B = B_1 \oplus \cdots \oplus B_n\), with \(B_i\) being different boundary operators acting on different bundles. If the boundary operators are smooth (even if different), then such a boundary-value problem is still smooth.

In this lecture we are interested in a different class of boundary conditions. Namely, we do not assume the boundary operator to be smooth. Instead, we will study the case when it has discontinuous coefficients. Such problems are often also called mixed boundary conditions; to avoid misunderstanding we will not use this terminology. We impose different boundary conditions on connected parts of the boundary, which makes the boundary-value problem discontinuous. Roughly speaking, one has a decomposition of a smooth boundary in some parts where different types of the boundary conditions are imposed, i.e. say Dirichlet or Neumann. The boundary operator is then discontinuous at the intersection of these parts. We consider the simplest case when there are just two components. We assume that the boundary of the manifold \(\partial M\) is decomposed as the disjoint union

\[ \partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0, \quad (65) \]

where \(\Sigma_1\) and \(\Sigma_2\) are smooth compact submanifolds of dimension \((m-1)\) (codimension 1 submanifolds), with the same boundary \(\Sigma_0 = \partial \Sigma_1 = \partial \Sigma_2\), that is a smooth compact submanifold of dimension \((m-2)\) (codimension 2 submanifold) without boundary, i.e. \(\partial \Sigma_0 = \emptyset\). Let us stress here that when viewed as sets both \(\Sigma_1\) and \(\Sigma_2\) are considered to be disjoint open sets, i.e. \(\Sigma_1 \cap \Sigma_2 = \emptyset\).

Let \(\pi_1\) and \(\pi_2\) be the trivial projections of sections, \(\psi\), of a vector bundle \(W\) to \(\Sigma_i\) defined by \((\pi_i\psi)(\hat{x}) = \psi(\hat{x})\) if \(\hat{x} \in \Sigma_i\) and \((\pi_i\psi)(\hat{x}) = 0\) if \(\hat{x} \notin \Sigma_i\). In other words \(\pi_1\) maps smooth sections of the bundle \(W\) to their restriction to \(\Sigma_1\), extending them by zero on \(\Sigma_2\), and similarly for \(\pi_2\). Let \(S \in C^\infty(\text{End}(W))\) be a smooth Hermitian endomorphism of the vector bundle \(W\).

We study the Zaremba type boundary-value
problem for the Laplace type operator $F$. The Zaremba boundary conditions are
\[ B\psi(\varphi) = 0, \]
where $B : C^\infty(W \oplus W) \to C^\infty(W \oplus W)$ is a Zaremba type boundary operator defined by
\[ B = \begin{pmatrix} \pi_1 & 0 \\ \pi_2 & \pi_2 \end{pmatrix}. \] (66)
In other words, we have Dirichlet boundary conditions on $\Sigma_1$ and Neumann (Robin) boundary conditions on $\Sigma_2$:
\[ \varphi|_{\Sigma_1} = 0, \]
\[ (\nabla_N + S)\varphi|_{\Sigma_2} = 0. \] (67) (68)
In the following, for simplicity, we restrict ourselves to the case $S = 0$. The projectors $\pi_1$ and $\pi_2$ as well as the boundary operator $B$ are clearly non-smooth (discontinuous) on $\Sigma_0$. Note that the boundary conditions are set only on open subsets $\Sigma_1$ and $\Sigma_2$; the boundary conditions do not say anything about the boundary data on $\Sigma_0$.

By integrating by parts on $\partial M$, it is not difficult to check that the Zaremba type boundary-value problem $(F,B)$ for a Laplace-type operator with the boundary operator $B$ (66) is symmetric. One can also show that it is elliptic with respect to $C \setminus \mathbb{R}_p$.

The heat kernel is defined by the equation
\[ (\partial_t + F)U(t|x,x') = 0, \] (69)
with the initial condition
\[ U(0^+|x,x') = \delta(x,x'), \] (70)
and the boundary condition
\[ B\psi[U(t|x,x')] = 0. \] (71)

Since coefficients of the boundary operator $B$ are discontinuous on $\Sigma_0$, a Zaremba type boundary-value problem is essentially singular. For such problems the asymptotic expansion of the trace of the heat kernel has additional nontrivial logarithmic terms [30], i.e.
\[ \text{Tr}_{L^2} \exp(-tF_B) \sim \sum_{k=0}^{\infty} t^{(k-m)/2} B_k + \log t \sum_{k=0}^{\infty} t^{k/2} H_k. \] (72)
Whereas there are some results concerning the coefficients $B_k$, almost nothing is known about the coefficients $H_k$. Since the Zaremba problem is local, or better to say 'pseudo-local', all these coefficients have the form
\[ B_{2k} = \int_M b_{2k}^{(0)} + \int_{\Sigma_1} b_{2k}^{(1),1} + \int_{\Sigma_2} b_{2k}^{(1),2} + \int_{\Sigma_0} b_{2k}^{(2)}, \] (73)
\[ B_{2k+1} = \int_{\Sigma_1} b_{2k+1}^{(1),1} + \int_{\Sigma_2} b_{2k+1}^{(1),2} + \int_{\Sigma_0} b_{2k+1}^{(2)}, \] (74)
\[ H_k = \int_{\Sigma_0} h_k. \] (75)
Here the new feature is the appearance of the integrals over $\Sigma_0$, which complicates the problem even more, since the coefficients now depend on the geometry of the imbedding of the codimension 2 submanifold $\Sigma_0$ in $M$ that could be pretty complicated, even if smooth.

Let us stress here that we are not going to provide a rigorous construction of the parametrix with all the estimates, which, for a singular boundary-value problem, is a task that would require a separate paper. For a complete and mathematically rigorous treatment the reader is referred to [30] and references therein. Here we keep instead to a pragmatic approach and will describe the construction of the parametrix that can be used to calculate explicitly the heat kernel coefficients $B_k$ as well as $H_k$.

First of all, we need to properly describe the geometry of the problem. Let us fix two small positive numbers $\varepsilon_1, \varepsilon_2 > 0$. We split the whole manifold in a disjoint union of four different parts:
\[ M = M^{\text{int}} \cup M^{\text{band}} \]
\[ = M^{\text{int}} \cup M_1^{\text{band}} \cup M_2^{\text{band}} \cup M_0^{\text{band}}. \]
Here $M_0^{\text{band}}$ is defined as the set of points in the narrow strip $M^{\text{band}}$ of the manifold $M$ near the boundary $\partial M$ of the width $\varepsilon_1$ that are at the same time in a narrow strip of the width $\varepsilon_2$ near $\Sigma_0$
\[ M_0^{\text{band}} = \{ x \in M | \text{dist}(x,\partial M) < \varepsilon_1, \text{dist}(x,\Sigma_0) < \varepsilon_2 \}. \]
Further, \(M_1^{\text{bnd}}\) is the part of the thin strip \(M^{\text{bnd}}\) of the manifold \(M\) (of the width \(\varepsilon_1\)) near the boundary \(\partial M\) that is near \(\Sigma_1\) but at a finite distance from \(\Sigma_0\), i.e.

\[
M_1^{\text{bnd}} = \{ x \in M \mid \text{dist}(x, \Sigma_1) < \varepsilon_1, \quad \text{dist}(x, \Sigma_0) > \varepsilon_2 \}.
\]

Similarly,

\[
M_2^{\text{bnd}} = \{ x \in M \mid \text{dist}(x, \Sigma_2) < \varepsilon_1, \quad \text{dist}(x, \Sigma_0) > \varepsilon_2 \}.
\]

Finally, \(M^{\text{int}}\) is the interior of the manifold \(M\) without a thin strip at the boundary \(\partial M\), i.e.

\[
M^{\text{int}} = M \setminus (M_1^{\text{bnd}} \cup M_2^{\text{bnd}} \cup M_0^{\text{bnd}}) = \{ x \in M \mid \text{dist}(x, \partial M) > \varepsilon_1 \}.
\]

We will construct the parametrix on \(M\) by using different approximations in different domains. Strictly speaking, to glue them together in a smooth way one should use ‘smooth characteristic functions’ of different domains (partition of unity) and carry out all necessary estimates. What one has to control is the order of the remainder terms in the limit \(t \to 0\) and their dependence on \(\varepsilon_1\) and \(\varepsilon_2\). Since our task here is not to prove the form of the asymptotic expansion (72), which is known, but rather to compute explicitly the coefficients of the asymptotic expansion, we will not worry about such subtle details. We will compute the asymptotic expansion as \(t \to 0\) in each domain and then take the limit \(\varepsilon_1, \varepsilon_2 \to 0\).

We will use different local coordinates in different domains. In \(M^{\text{int}}\) we do not fix the local coordinates; our treatment will be manifestly covariant.

In \(M_1^{\text{bnd}}\) we choose the local coordinates as follows. Let \(\{ \hat{e}_i \}, (i = 2, \ldots, m)\), be the local frame for the tangent bundle \(T\Sigma_1\) and \(\hat{x} = (\hat{x}^i) = (\hat{x}^2, \ldots, \hat{x}^m), (i = 2, \ldots, m)\), be the local coordinates on \(\Sigma_1\). Let \(r = \text{dist}(x, \Sigma_1)\) be the normal distance to \(\Sigma_1\) (\(r = 0\) being the defining equation of \(\Sigma_1\)), and \(\hat{N} = \partial_{\hat{x}^1}\), be the inward pointing unit normal to \(\Sigma_1\). Then by using the geodesic flow we get the local frame \(\{ N, e_i \}\) for the tangent bundle \(TM\) and the local coordinates \(x = (r, \hat{x})\) on \(M_1^{\text{bnd}}\). The coordinate \(r\) ranges from 0 to \(\varepsilon_1\), \(0 \leq r \leq \varepsilon_1\). The local coordinates in \(M_2^{\text{bnd}}\) are chosen similarly.

Finally, in \(M_0^{\text{bnd}}\) we choose the local coordinates as follows. Let \(\{ \hat{e}_a(\hat{x}) \}, (a = 3, \ldots, m)\), be a local frame for the tangent bundle \(T\Sigma_0\) and let \(\hat{x} = (\hat{x}^a) = (\hat{x}^3, \ldots, \hat{x}^m)\) be the local coordinates on \(\Sigma_0\). To avoid misunderstanding we should stress here that now we use the same notation \(\hat{x}\) to denote coordinates on \(\Sigma_0\) (not on the whole of \(\partial M\)). Let \(\text{dist}_{\hat{x}}(x, \Sigma_0)\) be the distance from a point \(x\) on \(\partial M\) to \(\Sigma_0\) along the boundary \(\partial M\). Then define \(y = \text{dist}_{\hat{x}}(x, \Sigma_0) > 0\) if \(x \in \Sigma_1\) and \(y = -\text{dist}_{\hat{x}}(x, \Sigma_0) < 0\) if \(x \in \Sigma_2\). In other words, \(y = 0\) on \(\Sigma_0\) \((r = y = 0\) being the defining equations of \(\Sigma_0\)), \(y > 0\) on \(\Sigma_1\) and \(y < 0\) on \(\Sigma_2\). Let \(\hat{n}(\hat{x})\) be the unit normal to \(\Sigma_0\) pointing inside \(\Sigma_1\). Then by using the tangential geodesic flow along the boundary (that is normal to \(\Sigma_0\)) we first get the local orthonormal frame \(\{ n(y, \hat{x}), e_a(y, \hat{x}) \}\) for the tangent bundle \(T\partial M\). Further, let the unit normal vector field to the boundary \(N(y, \hat{x})\) be given. Then by using the normal geodesic flow to the boundary we get the local frame \(\{ N(r, y, \hat{x}), n(r, y, \hat{x}), e_a(r, y, \hat{x}) \}\) for the tangent bundle \(TM\) and local coordinates \((r, y, \hat{x})\) on \(M_0^{\text{bnd}}\). The ranges of the coordinates \(r\) and \(y\) are: \(0 \leq r \leq \varepsilon_1\) and \(-\varepsilon_2 \leq y \leq \varepsilon_2\). Finally, we introduce the polar coordinates

\[
r = \rho \cos \theta, \quad y = \rho \sin \theta.
\]

To cover the whole \(M_0^{\text{bnd}}\) the angle \(\theta\) ranges from \(-\pi/2\) to \(\pi/2\) and \(\rho\) ranges from 0 to some \(\varepsilon_3\) (depending on \(\varepsilon_1\) and \(\varepsilon_2\)), \(0 \leq \rho \leq \varepsilon_3\).

The construction of the interior parametrix goes along the same lines as for manifolds without boundary (see lectures 2 and 4). For a finite \(\varepsilon_1 > 0\) the diagonal of the heat kernel has the same asymptotic expansion as for manifolds without boundary. Therefore, by integrating over the interior part \(M^{\text{int}}\) and taking the limit \(\varepsilon_1 \to 0\) we find that the local interior coefficients \(b_k^{(0)}\) are the same as for manifolds without boundary in the smooth case, i.e. \(a_k^{(0)}\), given by (55)-(56).

The Dirichlet parametrix \(U^{\text{bnd}, (1)}(t)\) in \(M_1^{\text{bnd}}\) and Neumann parametrix \(U^{\text{bnd}, (2)}(t)\) in \(M_2^{\text{bnd}}\) are constructed along the same lines as the parametrix of a smooth boundary-value problem described in lecture 5. For finite \(\varepsilon_1, \varepsilon_2 > 0\) the
diagonal of the parametrix has the same kind of asymptotic expansion as \( t \to 0 \) with coefficients being homogeneous functions of \( r/\sqrt{t} \)

\[
U^{\text{bnd},(i)}_{\text{diag}}(t) \sim \sum_{k=0}^{\infty} \frac{1}{t^{(k-m)}} U^{\text{bnd},(i)}_k \left( \frac{r}{\sqrt{t}}; \frac{r}{\sqrt{t}}, \hat{x}, \hat{r} \right).
\]

After integrating the diagonal over \( M_0^{\text{bnd}} \) and taking the limit \( \varepsilon_1, \varepsilon_2 \to 0 \) the contribution of all regular terms will vanish. The singular terms, which behave like distributions near \( \Sigma \), will give the \( \Sigma \) contributions to the boundary heat kernel coefficients \( b_k^{(i)} \). As the result we get the coefficients \( b_k^{(1),1} \) in the form

\[
b_k^{(1),1} = \sum_{n=0}^{k-1} \frac{1}{n!} g_n \lim_{\varepsilon_1 \to 0} \int_0^{\varepsilon_1/\sqrt{t}} d\xi \, \xi^n \times \text{tr}_V U_{k-n}^{\text{bnd},(i)}(1(\xi, \hat{x}, \xi, \hat{x}) \ . \quad (76)
\]

These are the standard boundary heat kernel coefficients for smooth Dirichlet and Neumann boundary conditions. They are listed for example in [22, 23] up to \( k = 4 \). The first two have the form

\[
\begin{align*}
b_0^{(1),1} &= b_0^{(1),2} = 0, \\
b_1^{(1),1} &= -b_1^{(1),2} = -(4\pi)^{-(m-1)/2} \dim V \left( \frac{1}{2} \right), \\
b_2^{(1),1} &= b_2^{(1),2} = (4\pi)^{-m/2} \dim V \left( \frac{1}{2} \right) K ,
\end{align*}
\]

where \( K \) is the trace of the extrinsic curvature (second fundamental form) of the boundary.

The most complicated (and the most interesting) is the case of the mixed parametrix in \( M_0^{\text{bnd}} \) since here the basic problem with frozen coefficients on \( \Sigma_0 \) is two-dimensional. More precisely, in \( M_0^{\text{bnd}} \) the basic problem is on the half-plane. Since the origin is a singular point, we will work in polar coordinates introduced above. One can still use the scaling device described above. Since now we are working in the vicinity of the submanifold \( \Sigma_0 \), the coordinate \( \rho \) should also be scaled, i.e.

\[
\hat{x} \to \hat{x}_0 + \varepsilon(\hat{x} - \hat{x}_0), \quad \hat{r} \to \varepsilon, \quad \rho \to \varepsilon \rho, \\
t \to \varepsilon^2 t,
\]

and similarly for the coordinates \( \hat{x}', \hat{r}' \) and \( \rho' \). Then one needs to expand in \( \varepsilon \) and develop a perturbation theory, which gives a recursion system that determines all coefficients \( B_k \) and \( H_k \). The order at which the log \( t \) terms show up depends on the dimension of the manifold.

We will restrict ourselves to the zeroth order of this perturbation theory only. This is enough for computation of the coefficient \( B_1 \). So, we are going to solve only the basic problem for operators with frozen coefficients at a point \( x_0 \) on \( \Sigma_0 \). We choose normal coordinates on \( \Sigma_0 \) at this point (with \( g_{ab}(0, \theta, \hat{x}_0) = \delta_{ab} \)). Then the zeroth order operator \( F_0 \) has the form

\[
F_0 = -\partial_{\rho}^2 - \frac{1}{\rho} \partial_{\rho} - \frac{1}{\rho^2} \partial_{\theta}^2 - \partial^2
\]

and the zeroth order normal is

\[
N_0 |_{\Sigma_1} = -\frac{1}{\rho} \partial_{\theta} \bigg|_{\theta = \frac{\pi}{2}}, \quad N_0 |_{\Sigma_2} = \frac{1}{\rho} \partial_{\theta} \bigg|_{\theta = -\frac{\pi}{2}}.
\]

Now the boundary operator is discontinuous, and there is a singularity at the origin \( \rho = 0 \).

By separating the ‘free’ semiclassical factor due to \( \Sigma_0 \) we get the ansatz

\[
U^{\text{bnd},(0)}(t|\rho, \theta, \hat{x}; \rho', \theta', \hat{x}') = (4\pi t)^{-(m-2)/2} \times \exp \left( -\frac{|\hat{x} - \hat{x}'|^2}{4t} \right) \Psi(t|\rho, \theta; \rho', \theta'),
\]

(79)

where \( \Psi(t|\rho, \theta; \rho', \theta') \) is a two-dimensional heat kernel determined by the equation

\[
\left( \partial_t - \partial_{\rho}^2 - \frac{1}{\rho} \partial_{\rho} - \frac{1}{\rho^2} \partial_{\theta}^2 \right) \Psi(t|\rho, \theta; \rho', \theta') = 0,
\]

the initial condition

\[
\Psi(0^+|\rho, \theta; \rho', \theta') = \frac{1}{\rho} \delta(\rho - \rho') \delta(\theta - \theta'),
\]

the boundary conditions

\[
\partial_{\theta} \Psi(t|\rho, \theta; \rho', \theta') \bigg|_{\theta = \frac{\pi}{2}} = 0,
\]

\[
\partial_{\theta} \Psi(t|\rho, \theta; \rho', \theta') \bigg|_{\theta = -\frac{\pi}{2}} = 0,
\]

and the asymptotic condition at infinity

\[
\lim_{\rho \to \infty} \Psi(t|\rho, \theta; \rho', \theta') = 0.
\]
Clearly, the heat kernel is also symmetric, \( \Psi(t|\rho, \theta; \rho', \theta') = \Psi(t|\rho', \theta'; \rho, \theta) \). This problem can be solved by separating variables, employing the Hankel transform in the radial coordinate and evaluating a certain spectral series of Bessel functions. As a result, we obtain the mixed leading parametrix in \( M_0^{\text{bnd}} \)

\[
U_0^{\text{bnd},(0)}(t|\rho, \theta; \rho', \theta') = L(t|\rho, \theta; \rho', \theta', \hat{x}') + L(t|\rho, \theta; \hat{x}; \rho', -\theta' - \pi, \hat{x}')
\]

where

\[
L(t|\rho, \theta; \hat{x}; \rho', \theta', \hat{x}') = (4\pi t)^{-m/2} \times \exp \left\{ -\frac{\hat{x}^2}{2t} + \frac{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')}{{4t}} \right\} \times \text{erf} \left( \sqrt{\frac{\rho\rho'}{t} \cos \left( \frac{\theta - \theta'}{2} \right)} \right)
\]

with \( \text{erf}(z) = 2\pi^{-1/2} \int_0^z du e^{-u^2} \) being the error function. The diagonal of the mixed parametrix is easily found to be

\[
U_0^{\text{bnd},(0)}(t) = (4\pi t)^{-m/2} \left\{ 1 - \text{sign}(\theta) \exp \left( -\frac{\rho^2 \cos^2 \theta}{t} \right) - \text{erf} \left( \frac{\rho}{\sqrt{t}} \right) + \text{sign}(\theta) \exp \left( -\frac{\rho'^2 \cos^2 \theta}{t} \right) \text{erfc} \left( \frac{\rho}{\sqrt{t}} \sin \theta \right) \right\},
\]

where \( \text{sign}(x) \) is the sign function, i.e. \( \text{sign}(x) = 1 \) for \( x > 0 \) and \( \text{sign}(x) = -1 \) for \( x < 0 \), and \( \text{erfc}(z) = 1 - \text{erf}(z) = 2\pi^{-1/2} \int_z^\infty du e^{-u^2} \) is the complementary error function.

Finally, we compute the integral of the diagonal of the mixed parametrix over \( M_0^{\text{bnd}} \) and take the limit \( \varepsilon_1, \varepsilon_2 \to 0 \). We have

\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_{M_0^{\text{bnd}}} \text{tr} \, U_0^{\text{bnd},(0)}(t) = \lim_{\varepsilon_3 \to 0} \int_{\Sigma_0} d\rho \rho \int_{-\pi/2}^{\pi/2} d\theta \text{tr} \, U_0^{\text{bnd},(0)}(t),
\]

for some finite \( \varepsilon_3 > \sqrt{\varepsilon_1^2 + \varepsilon_2^2} > 0 \). By computing the integrals and taking the limits we obtain

\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_{M_0^{\text{bnd}}} \text{tr} \, U_0^{\text{bnd},(0)}(t) = -t^{(2-m)/2} (4\pi)^{-m/2} \frac{\pi}{4} \dim V \text{ vol}(\Sigma_0).
\]

This gives exactly the coefficient \( b_2^{(2)} \) in the heat trace asymptotic expansion (73), i.e.

\[
b_2^{(2)} = -(4\pi)^{-(m-2)/2} \dim V \frac{1}{16}.
\]

Acknowledgment and Note Added in Proof

I would like to thank the organizers for their kind invitation to present these lectures. The Zaremba problem considered in section 6 was studied recently in Ref. [35]; it has been shown that the logarithmic terms in the expansion (72) are absent, i.e. \( H_k = 0 \) for any \( k \), which confirms the conjecture of Ref. [33]. Seeley has also shown that the correct setting of the Zaremba problem involves an additional boundary condition along the singular set \( \Sigma_0 \). The general solution as well as the heat kernel asymptotics do depend on this additional condition. Our solution corresponds to the choice of most regular eigenfunctions close to \( \Sigma_0 \). Other solutions contain integrable singularities near \( \Sigma_0 \), which lead to additional contributions to the heat kernel coefficients.

REFERENCES