Noncommutative Deformation of General Relativity

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- Main Idea (Matrix Deformation)
- Origin of Riemannian Geometry
- Hyperbolic Systems and Causal Structure
- Finsler Geometry and Matrix Geometry
- Invariant Action Functionals

<u>Main Idea: Matrix Deformation</u>

Einstein General Relativity	⇒	*Matrix General Relativity (MGR)* *Gravitational Chromodynamics GCD*
Riemannian Geometry	⇒	*Matrix Geometry*
Riemannian metric	⇒	*matrix-valued tensor field* (collection of *Finsler metrics*)

New features:

- dynamical *causal structure* of spacetime (collection of light cones)
- new *gravicolor degrees of freedom*
- new gauge symmetry and corresponding charges

Origing of Riemannian Geometry

light propagation

$\Downarrow$

wave equation

$\Downarrow$

Hamilton-Jacobi equation

$\Downarrow$

characteristics of the wave equation

$\Downarrow$

causal structure of spacetime

$\Downarrow$

(pseudo)-Riemannian metric of spacetime

$\Downarrow$

basic notions of general relativity

Origin of Matrix Geometry

Propagation of *collection of fields*
(with *internal structure*)

$$\Downarrow$$

Hyperbolic system of linear second-order
partial differential equations

$$\Downarrow$$

Matrix Geometry

Linear Hyperbolic System

Linear self-adjoint hyperbolic PDO with *matrix-valued* coefficients

$$L\varphi = [a^{\mu\nu}(x)\partial_\mu\partial_\nu + b^\mu(x)\partial_\mu + c(x)]\,\varphi = 0$$

Covariance under diffeomorphisms and gauge transformations

$$\varphi(x) \longrightarrow U(x)\varphi(x)$$

Matrix-valued contravariant symmetric two-tensor

$$a^{\mu\nu} = a^{\nu\mu}\,, \qquad (a^{\mu\nu})^\dagger = a^{\mu\nu}$$

Principal symbol (*matrix Hamiltonian*)

$$H(x,\xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu$$

Strict Hyperbolicity: for any x and $\xi = (\lambda, \zeta_k)$ with $\zeta \neq 0$ the roots $\lambda_j(x,\zeta)$ of the characteristic equation

$$\det H(x,\xi) = 0$$

are *real*

Causal Structure

Eigenvalues $h_i(x, \xi)$ of the matrix Hamiltonian $H(x, \xi)$ are *real* and *distinct* (or with *constant multiplicities*)

Hamilton-Jacobi equations $h_i\left(x, \frac{\partial S}{\partial x}\right) = 0$

Characteristics and Hamiltonian systems

$$\frac{dx^\mu}{ds} = \frac{\partial}{\partial \xi_\mu} h_i(x, \xi)\,, \qquad \frac{d\xi_\mu}{ds} = -\frac{\partial}{\partial x^\mu} h_i(x, \xi)$$

Null geodesics	$h_i(x, \nu) = 0$
Causal Cones	$\mathcal{C}_i(x)$
Past and Future	$\mathcal{I}_i^-(x)$ and $\mathcal{I}_i^+(x)$
Interior of the cone	$\mathcal{I}_i(x) = \mathcal{I}_i^-(x) \cup \mathcal{I}_i^+(x)$
Exterior of the cone	$\mathcal{E}_i(x)$

Causal Set

$$\mathcal{I}(x) = \bigcup_{i=1}^{s} \mathcal{I}_i(x)$$

Absolute Past and Future

$$\mathcal{I}^{\pm}(x) = \bigcup_{i=1}^{s} \mathcal{I}_i^{\pm}(x)$$

Acausal (Causally Disconnected) Set

$$\mathcal{E}(x) = \bigcap_{i=1}^{s} \mathcal{E}_i(x)$$

Causal Structure of spacetime

$$M = \mathcal{I}^{-}(x) \cup \mathcal{I}^{+}(x) \cup \partial\mathcal{I}(x) \cup \mathcal{E}(x)$$

Remark: Causal structure is *dynamic* (varies from point to point)

Finsler Geometry

Homogeneity	$h_i(x, \lambda\xi) = \lambda^2 h_i(x, \xi)$
Finsler form	$h_i(x, \xi) = g_i^{\mu\nu}(x, \xi)\xi_\mu\xi_\nu$
Finsler metrics	$g_i^{\mu\nu}(x, \xi) = \frac{1}{2}\frac{\partial^2 h_i(x, \xi)}{\partial\xi_\mu\partial\xi_\nu}$
Homogeneity	$g_i^{\mu\nu}(x, \lambda\xi) = g_i^{\mu\nu}(x, \xi)$
Hyperbolicity	$\text{sign} g_i^{\mu\nu}(x, \xi) = (- + \cdots +)$
Remark:	same hyperbolic direction
Tangent vector	$\dot{x}^\mu = g_i^{\mu\nu}(x, \xi)\xi_\nu$
	$\xi_\mu = g_{i\mu\nu}(x, \dot{x})\dot{x}^\nu$
Covariant metric	$g_{i\mu\nu}(x, \dot{x}) g_i^{\nu\alpha}(x, \xi) = \delta_\nu^\alpha$
Interval	$ds_i^2 = g_{i\mu\nu}(x, \dot{x}) dx^\mu dx^\nu$

Matrix Geometry

Matrix connection $\mathcal{A}^{\mu}{}_{\alpha\beta} = \left(\mathcal{A}^{\mu}{}_{\alpha\beta}{}^{A}{}_{B}\right)$

Yang-Mills field $\mathcal{B}_{\mu}$

Compatibility condition

$$\partial_{\mu} a^{\alpha\beta} + [\mathcal{B}_{\mu}, a^{\alpha\beta}] + \mathcal{A}^{\alpha}{}_{\lambda\mu} a^{\lambda\beta} + \mathcal{A}^{\beta}{}_{\lambda\mu} a^{\alpha\lambda} = 0$$

Symmetry condition in the commutative limit

$$\begin{aligned} \mathcal{A}^{\alpha}{}_{\lambda\mu} = \frac{1}{2} b_{\lambda\sigma} \Big\{ \Big[& a^{\alpha\gamma}\partial_{\gamma} a^{\rho\sigma} + a^{\alpha\gamma}[\mathcal{B}_{\gamma}, a^{\rho\sigma}] \\ & - a^{\rho\gamma}\partial_{\gamma} a^{\sigma\alpha} - a^{\rho\gamma}[\mathcal{B}_{\gamma}, a^{\sigma\alpha}] \\ & - a^{\sigma\gamma}\partial_{\gamma} a^{\alpha\rho} - a^{\sigma\gamma}[\mathcal{B}_{\gamma}, a^{\alpha\rho}] \Big] \Big\} b_{\rho\mu} \end{aligned}$$

where $b_{\mu\nu} = (b_{\mu\nu}{}^{A}{}_{B})$ is defined by

$$a^{\mu\nu} b_{\nu\lambda} = b_{\mu\nu} a^{\nu\lambda} = \delta^{\mu}_{\lambda} \mathbb{I}$$

Matrix Curvature

$$\begin{aligned}
\mathcal{R}^{\lambda}{}_{\alpha\mu\nu} &= \partial_\mu \mathcal{A}^{\lambda}{}_{\alpha\nu} + [\mathcal{B}_\mu, \mathcal{A}^{\lambda}{}_{\alpha\nu}] \\
&\quad -\partial_\nu \mathcal{A}^{\lambda}{}_{\alpha\mu} - [\mathcal{B}_\nu, \mathcal{A}^{\lambda}{}_{\alpha\mu}] \\
&\quad +\mathcal{A}^{\lambda}{}_{\beta\mu}\mathcal{A}^{\beta}{}_{\alpha\nu} - \mathcal{A}^{\lambda}{}_{\beta\nu}\mathcal{A}^{\beta}{}_{\alpha\mu}
\end{aligned}$$

Yang-Mills Curvature

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + [\mathcal{B}_\mu, \mathcal{B}_\nu]$$

Matrix Torsion

$$\mathcal{T}^{\lambda}{}_{\mu\nu} = \mathcal{A}^{\lambda}{}_{\mu\nu} - \mathcal{A}^{\lambda}{}_{\nu\mu}$$

Invariant Action Functionals

Matrix measure (not unique)

$$\rho = (\psi^\dagger \psi)^{-1/4}$$

where

$$\psi = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \varepsilon_{\nu_1 \ldots \nu_n} a^{\mu_1 \nu_1} \ldots a^{\mu_n \nu_n}$$

Action (not unique)

$$\begin{aligned} S \;=\; & \frac{1}{16\pi G N} \int dx \; \mathrm{tr}\, \rho \, (a^{\nu\mu} \mathcal{R}^{\alpha}{}_{\mu\alpha\nu} - 2\Lambda) \\ & + (\mathrm{torsion})^2 \\ & + \mathrm{Yang-Mills} \\ & + \mathrm{matter} \end{aligned}$$

Invariant Functionals from Spectral Asymptotics

Invariant elliptic self-adjoint second-order PDO F with positive definite *non-scalar principal symbol* $H(x,\xi)$

Heat trace asymptotic expansion as $t \to 0$

$$\mathrm{Tr}_{L^2} \exp(-tF) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} A_k$$

Spectral Invariants A_k

For operators with *scalar leading symbol*

$$A_0 = \int_M dx g^{1/2} N, \qquad A_1 = \int_M dx g^{1/2} \,\mathrm{tr}\, \left(\frac{1}{6} R - Q\right)$$

Invariant Action Functional of Matrix General Relativity

$$S = \frac{1}{16\pi G N} (6A_1 - 2\Lambda A_0)$$

Conclusions

Spontaneous breakdown of gauge symmetry

Broken phase: one tensor field (metric of the space-time)

Unbroken phase: no preferred metric in the usual sense

Confinement of gravicolor degrees of freedom

Only the invariants (*graviwhite states*) are visible at large distances