Spectral Asymptotics

of non-commutative Laplacian

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- Origin of Matrix Geometry
- Relation to Finsler Geometry
- Non-commutative Exterior Calculus
- Non-comm Laplacians and Dirac Operator
- Spectral Asymptotics and Heat Invariants
- Non-commutative Einstein equations

Origin of Matrix Geometry

Spectral Geometry: *What geometrical information can be derived from the spectra of differential operators on a manifold?*

A linear elliptic second-order partial differential operator determines a geometry on the manifold

Laplace type operator $\Rightarrow$ Riemannian geometry

Non-Laplace type operator $\Rightarrow$ **matrix geometry**

Main idea:
Riemannian metric
$\Rightarrow$ *matrix-valued symmetric* 2*-tensor field*

Vector Bundles

Smooth compact orientable spin manifold M without boundary, $\dim M = n$

Complex Hermitian vector bundle $\mathcal{S}$, $\dim S = N$

Bundle of endomorphisms $\mathrm{End}\,(\mathcal{S})$

Group of unitary endomorphisms $G(\mathcal{S})$

Vector-valued and *endo-valued* p-forms and p-vectors

$$\Lambda_p = \wedge^p T^* M \otimes \mathcal{S}\,, \qquad \Lambda^p = \wedge^p TM \otimes \mathcal{S}$$

$$E_p = \wedge^p T^* M \otimes \mathrm{End}\,(\mathcal{S})\,, \quad E^p = \wedge^p TM \otimes \mathrm{End}\,(\mathcal{S})$$

Vector bundles of *densities* $\mathcal{V}[w]$ of weight w

Non-Commutative Metric

Self-adj endo-valued vector

$$\Gamma \in C^{\infty}(TM \otimes \mathrm{End}(\mathcal{S})[0])$$

Non-commutative Dirac matrices Γ^{μ}

Self-adj endo-valued tensor

$$a \in C^{\infty}(TM \otimes TM \otimes \mathrm{End}(\mathcal{S})[0])$$

Non-commutative metric

$$a^{\mu\nu} = \frac{1}{2}(\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu})$$

Non-Commutative Metric (CONT)

Anti-symmetric tensors

$$A^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p} = \mathrm{Alt}_{\mu_1\cdots\mu_p}\,\mathrm{Alt}_{\nu_1\cdots\nu_p}\,a^{\mu_1\nu_1}\cdots a^{\mu_p\nu_p}$$

Linear mapping $A : \Lambda_p \to \Lambda^p$

$$(A\varphi)^{\mu_1\cdots\mu_p} = A^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\varphi_{\nu_1\cdots\nu_p}\,,$$

Hamiltonian

$$H(x,\xi) = a(x,\xi,\xi) = [\Gamma(x,\xi)]^2$$

where $\xi \in T^*_x M$

Assumptions:

- The mapping A is an *isomorphism* for any p

- The Hamiltonian $H(x,\xi)$ is *positive-definite* for any $x \in M$ and any $\xi \neq 0$.

Eigenvalues: $h_a(x,\xi)$
(distinct, positive, constant multiplicities)

Hamilton-Jacobi Theory

Hamilton-Jacobi equation

$$\det\left[H\left(x, \frac{\partial S}{\partial x}\right) - m^2\mathbb{I}\right] = 0$$

N solutions S_a satisfying its own HJ eq.

$$h_a\left(x, \frac{\partial S_a}{\partial x}\right) = m^2$$

Hamilton equations

$$\frac{dx^\mu}{dt} = \frac{1}{2}\frac{\partial h_a(x,\xi)}{\partial \xi_\mu}$$

$$\frac{d\xi_\mu}{dt} = -\frac{1}{2}\frac{\partial h_a(x,\xi)}{\partial x^\mu}$$

N different **flows** on T^*M

Finsler Geometry

A non-commutative metric naturally defines a *collection of Finsler geometries*

Homogeneity	$h_a(x, \lambda\xi) = \lambda^2 h_a(x, \xi)$
Finsler metrics	$g_a^{\mu\nu}(x, \xi) = \frac{1}{2}\frac{\partial^2 h_a(x, \xi)}{\partial\xi_\mu \partial\xi_\nu}$
Velocity	$u^\mu = g_a^{\mu\nu}(x, \xi)\xi_\nu$
Momentum	$\xi_\mu = g_{a\mu\nu}(x, u)u^\nu$
Covariant metric	$g_{a\mu\nu}(x, u)g_a^{\nu\lambda}(x, \xi) = \delta_\nu^\lambda$
Finsler arc length	$ds_a = \sqrt{g_{a\mu\nu}(x, \dot{x})\dot{x}^\mu \dot{x}^\nu}\, dt$

Connection, Duality and Measure

Anti-self-adj *endo-valued connection* 1*-form*

$$\mathcal{B} \in C^\infty(T^*M \otimes \mathrm{End}(\mathcal{S})[0])$$

Exterior action of $\mathcal{B}$:

$$\mathcal{B} : \Lambda_p\left[\tfrac{1}{2}\right] \to \Lambda_{p+1}\left[\tfrac{1}{2}\right]$$

$$\mathcal{B}\varphi = \mathcal{B} \wedge \varphi$$

Duality

Volume n-form density naturally defines the maps

$$\varepsilon : \Lambda^p[w] \to \Lambda_{n-p}[w+1]$$

$$\tilde{\varepsilon} : \Lambda_p[w] \to \Lambda^{n-p}[w-1]$$

Non-commutative measure

Self-adj non-degenerate *endo-valued density* of weight $\frac{1}{2}$

$$\rho \in C^\infty\left(\mathrm{End}(\mathcal{S})\left[\frac{1}{2}\right]\right)$$

Laplacian and Dirac Operator

(on p-form densities of weight $\frac{1}{2}$)

Gradient (covariant exterior derivative)

$$\mathcal{D} = \rho(d + \mathcal{B})\rho^{-1} : C^\infty\left(\Lambda_p\left[\tfrac{1}{2}\right]\right) \to C^\infty\left(\Lambda_{p+1}\left[\tfrac{1}{2}\right]\right)$$

Non-commutative Laplacian

$$\Delta = -\mathcal{D}^*\mathcal{D} - \mathcal{D}\mathcal{D}^* :\ C^\infty\left(\Lambda_p\left[\tfrac{1}{2}\right]\right) \to C^\infty\left(\Lambda_p\left[\tfrac{1}{2}\right]\right)$$

Non-commutative Dirac operator

$$D = (-1)^{np+1} i\varepsilon\Gamma\tilde{\varepsilon}\mathcal{D} :\ C^\infty\left(\Lambda_p\left[\tfrac{1}{2}\right]\right) \to C^\infty\left(\Lambda_p\left[\tfrac{1}{2}\right]\right)$$

Operators Δ, DD^* and D^*D for $p = 0$

In local coordinates

$$\Delta = \rho^{-1}(\partial_\mu + \mathcal{B}_\mu)\rho a^{\mu\nu}\rho(\partial_\nu + \mathcal{B}_\nu)\rho^{-1}$$

$$DD^* = -\Gamma^\mu\rho(\partial_\mu + \mathcal{B}_\mu)\rho^{-2}(\partial_\nu + \mathcal{B}_\nu)\rho\Gamma^\nu$$

$$D^*D = -\rho^{-1}(\partial_\nu + \mathcal{B}_\nu)\rho\Gamma^\nu\Gamma^\mu\rho(\partial_\mu + \mathcal{B}_\mu)\rho^{-1}$$

Leading symbols

$$\sigma_L(-\Delta; x, \xi) = \sigma_L(D^*D; x, \xi) = \sigma_L(DD^*; x, \xi)$$
$$= H(x, \xi) = [\Gamma(x, \xi)]^2$$

are *positive definite*

Operators $(-\Delta)$, DD^* or D^*D (with some elliptic boundary conditions) are *self-adjoint* *elliptic*

Spectral Functions of an Elliptic Operator

Spectrum of a self-adjoint elliptic operator L with positive-definite leading symbol on a compact manifold is *real*, *discrete*, with *finite multiplicities* and *bounded from below*

For $t > 0$ the **heat semi-group** $\exp(-tL)$ is *trace-class*

Heat trace

$$\mathrm{Tr}\,\exp(-tL) = \sum_{k=1}^{\infty} e^{-t\lambda_k}$$

Spectral asymptotics: as $t \to 0^+$

$$\mathrm{Tr}\,\exp(-tL) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} A_k(L)$$

Asymptotic Expansions

Heat kernel

$$\left(\frac{\partial}{\partial t} + L\right) U(t; x, x') = 0$$

Initial condition

$$U(0; x, x') = \mathbb{I}\, \delta(x, x')$$

Some elliptic boundary conditions

Heat trace

$$\mathrm{Tr}\ \exp(-tL) = \int_M dx\ \ \mathrm{tr}\, U(t; x, x)$$

Asymptotic expansion as $t \to 0$

$$\mathrm{tr}\, U(t; x, x) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(L; x)$$

Spectral Invariants

$$A_k(L) = \int_M dx\, a_k(L; x)$$

Results

For manifolds without boundary all odd coefficients vanish

$$A_{2k+1}(D^*D) = 0$$

Spectral Invariants

$$A_0(D^*D) = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \ \mathrm{tr}\ e^{-H}$$

$$A_2(D^*D) = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}}$$

$$\times \mathrm{tr} \left[\int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} \right.$$

$$\left. - \int_0^1 d\tau_1 e^{-(1-\tau_1)H} D^* D e^{-\tau_1 H} \right]$$

where

$$H = [\Gamma(x,\xi)]^2 , \qquad K = \Gamma(x,\xi) D + D^* \Gamma(x,\xi)$$

Dirichlet Boundary Value Problem

Coordinates $x = (r, \hat{x})$, $\qquad \hat{x} \in \partial M$

Vector ∂_r transversal to ∂M, inward pointing

Cotangent bundle split $T^*M = \mathbb{R} \oplus T^*\partial M$

Covector $\xi = (\omega, \hat{\xi})$, $\qquad \hat{\xi} \in T^*_{\hat{x}}\partial M$

Spectral invariant

$$A_1(L) = -\sqrt{\pi} \int_{\partial M} d\hat{x} \int_{\mathbb{R}^{n-1}} \frac{d\hat{\xi}}{\pi^{\frac{n-1}{2}}}$$

$$\times \int_{w-i\infty}^{w+i\infty} \frac{d\lambda}{2\pi i} e^{-\lambda} \frac{\partial}{\partial \lambda} \log\, \det \Phi(\lambda, \hat{x}, \hat{\xi})$$

where $w << 0$ and

$$\Phi(\lambda, \hat{x}, \hat{\xi}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ H(0, \hat{x}, \omega, \hat{\xi}) - \lambda \right\}^{-1}$$

Non-comm Einstein-Hilbert Functional

$$S_{MG} = -\frac{1}{N}\left\{12A_2(D^*D) + 2\Lambda A_0(D^*D)\right\}$$

Commutative limit $\varkappa \to 0$

$$S_{MG} \to S_{EH} \;=\; \int_M d\,\mathrm{vol}\,(R - 2\Lambda)$$

(**Einstein-Hilbert functional**)

Critical points of S_{EH} are **Einstein spaces**

Critical points of S_{MG} are **non-commutative Einstein spaces**

Open Problems

- Spectral invariants on manifolds with boundary (only A_1 has been computed)

- Development of tools of matrix geometry to the same level of efficiency as in the Riemannian geometry

- Study of non-commutative Einstein spaces

- Consistent study of non-commutative corrections to Einstein equations

- Problems in physics:
  (dark matter, dark energy, space-time singularities, Pioneer anomaly, quantization of gravity)