

Spectral Asymptotics **in Matrix Geometry**

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- Origin of Matrix Geometry
- Relation to Finsler Geometry
- Non-commutative Exterior Calculus
- Non-comm Laplacians and Dirac Operator
- Spectral Asymptotics and Heat Invariants
- Non-commutative Einstein equations

Origin of Matrix Geometry

Laplace operator \Rightarrow Riemannian geometry

2nd-order self-adj
elliptic PDO acting on
sections of a *vector
bundle* \Rightarrow Matrix geometry

Main Idea:

Riemannian metric \Rightarrow *matrix-valued tensor field*
 \Downarrow
(collection of *Finsler metrics*)

Vector Bundles

Complex vector space

$$S = \mathbb{C}^N$$

Linear endomorphisms

$$\text{End}(S) = \text{Mat}(N, \mathbb{C})$$

Gauge Group of
unitary endo's

$$G(S) = U(N)$$

Smooth compact orient n -dim M
manifold, no boundary

Vector-valued and endo-valued tensors

$$\Lambda_p = \wedge^p T_x M^* \otimes S, \quad \Lambda^p = \wedge^p T_x M \otimes S$$

$$E_p = \wedge^p T_x M^* \otimes \text{End}(S), \quad E^p = \wedge^p T_x M \otimes \text{End}(S)$$

Vector bundle of densities of $V[w]$
weight w with fiber V

Non-commutative Metric

Self-adj endo- $a \in C^\infty(TM \otimes TM \otimes \text{End}(S)[0])$
valued symm
 tensor field

Assumptions:

- Linear mapping $A : \Lambda_p \rightarrow \Lambda^p$

$$(A\varphi)^{\mu_1 \dots \mu_p} = A^{\mu_1 \dots \mu_p \nu_1 \dots \nu_p} \varphi_{\nu_1 \dots \nu_p},$$

where

$$A^{\mu_1 \dots \mu_p \nu_1 \dots \nu_p} = \text{Alt}_{\mu_1 \dots \mu_p} \text{Alt}_{\nu_1 \dots \nu_p} a^{\mu_1 \nu_1} \dots a^{\mu_p \nu_p}$$

is an isomorphism for any p and defines an inner product on Λ_p

- **Eigenvalues** $h_i(x, \xi)$ of the non-comm **Hamiltonian**

$$H(x, \xi) = a^{\mu\nu}(x) \xi_\mu \xi_\nu$$

are positive with constant multiplicities
 for any $x \in M$ and $\xi \in T_x^*M$, $\xi \neq 0$

Characteristics and Hamiltonian Systems

Hamilton-Jacobi equations

$$h_i\left(x, \frac{\partial S}{\partial x}\right) = 0$$

Characteristics and Hamiltonian systems

$$\frac{dx^\mu}{ds} = \frac{\partial}{\partial \xi_\mu} h_i(x, \xi)$$

$$\frac{d\xi_\mu}{ds} = -\frac{\partial}{\partial x^\mu} h_i(x, \xi)$$

Finsler Geometry

Homogeneity $h_i(x, \lambda\xi) = \lambda^2 h_i(x, \xi)$

Finsler form $h_i(x, \xi) = g_i^{\mu\nu}(x, \xi)\xi_\mu\xi_\nu$

Finsler metrics $g_i^{\mu\nu}(x, \xi) = \frac{1}{2} \frac{\partial^2 h_i(x, \xi)}{\partial \xi_\mu \partial \xi_\nu}$

Homogeneity $g_i^{\mu\nu}(x, \lambda\xi) = g_i^{\mu\nu}(x, \xi)$

Ellipticity $g_i^{\mu\nu}(x, \xi)$ is positive definite

Tangent vector $\dot{x}^\mu = g_i^{\mu\nu}(x, \dot{x})\dot{\xi}_\nu$

$$\dot{\xi}_\mu = g_{i\mu\nu}(x, \dot{x})\dot{x}^\nu$$

Covariant metric $g_{i\mu\nu}(x, \dot{x})g_i^{\nu\alpha}(x, \dot{x}) = \delta_\nu^\alpha$

Interval $ds_i^2 = g_{i\mu\nu}(x, \dot{x})dx^\mu dx^\nu$

Non-commutative Dirac Matrices

Self-adj endo-valued vector field $\Gamma \in C^\infty(E^1[0])$

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2a^{\mu\nu}$$

Self-adj non-deg endo-valued density

$\rho \in C^\infty(\text{End}(S) \left[\frac{1}{2} \right])$ of weight $\frac{1}{2}$

Comutative limit:

$$\Gamma^\mu = \gamma^\mu, \quad a^{\mu\nu} = g^{\mu\nu} \mathbb{I}, \quad \rho = g^{1/4} \mathbb{I},$$

where γ^μ are Dirac matrices, $g^{\mu\nu}$ is a Riemannian metric and $g = [\det g^{\mu\nu}]^{-1}$

Non-commutative Deformation

$$\Gamma^\mu = \gamma^\mu + \kappa \alpha^\mu, \quad a^{\mu\nu} = g^{\mu\nu} \mathbb{I} + \kappa h^{\mu\nu}, \quad \rho = g^{1/4} e^{\kappa \phi}$$

κ is a **deformation papameter**

Gauge Connection

Anti-self-adj endo-valued 1-form $\mathcal{B} \in C^\infty(E_1[0])$
that transforms under gauge transformations
 $U \in G(S)$ as

$$\mathcal{B}'_\mu = U\mathcal{B}_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

Natural actions of \mathcal{B} :

$$\mathcal{B} : \Lambda_p \left[\frac{1}{2} \right] \rightarrow \Lambda_{p+1} \left[\frac{1}{2} \right]$$

$$(\mathcal{B}\varphi)_{\mu_1 \dots \mu_{p+1}} = (p+1)\mathcal{B}_{[\mu_1} \varphi_{\mu_2 \dots \mu_{p+1}]}$$

$$\tilde{\mathcal{B}} : \Lambda^p \left[\frac{1}{2} \right] \rightarrow \Lambda^{p-1} \left[\frac{1}{2} \right]$$

$$(\tilde{\mathcal{B}}\varphi)^{\mu_1 \dots \mu_{p-1}} = \mathcal{B}_\mu \varphi^{\mu \mu_1 \dots \mu_{p-1}}$$

Gauge Curvature

$$\mathcal{R} = d\mathcal{B} + [\mathcal{B}, \mathcal{B}]$$

Non-commutative Exterior Calculus

Exterior derivative (**gradient**) on *tensors*

$$d : C^\infty(\Lambda_p[0]) \rightarrow C^\infty(\Lambda_{p+1}[0])$$

$$(d\varphi)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\partial_{[\mu_1}\varphi_{\mu_2 \cdots \mu_p]}$$

Coderivative (**divergence**) on *densities* of weight 1

$$\tilde{d} : C^\infty(\Lambda^p[1]) \rightarrow C^\infty(\Lambda^{p-1}[1])$$

$$(\tilde{d}\varphi)^{\mu_1 \cdots \mu_{p-1}} = \partial_\mu \varphi^{\mu \mu_1 \cdots \mu_{p-1}}$$

Covariant Gradient

$$\mathcal{D} = \rho(d + \mathcal{B})\rho^{-1} : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_{p+1}\left[\frac{1}{2}\right]\right)$$

Covariant Divergence

$$\tilde{\mathcal{D}} = \rho^{-1}(\tilde{d} + \tilde{\mathcal{B}})\rho : C^\infty\left(\Lambda^p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda^{p-1}\left[\frac{1}{2}\right]\right)$$

Non-comm Laplacian and Dirac Operator

Adjoint Gradient

$$\bar{\mathcal{D}} = -A^{-1}\tilde{\mathcal{D}}A : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_{p-1}\left[\frac{1}{2}\right]\right)$$

Non-commutative Laplacian

$$\Delta = -\bar{\mathcal{D}}\mathcal{D} - \mathcal{D}\bar{\mathcal{D}} : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right)$$

Natural actions of Γ

$$\Gamma : C^\infty\left(\Lambda^p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda^{p+1}\left[\frac{1}{2}\right]\right)$$

$$(\Gamma\varphi)^{\mu_1\dots\mu_{p+1}} = (p+1)\Gamma^{[\mu_1}\varphi^{\mu_2\dots\mu_{p+1}]}$$

$$\tilde{\Gamma} : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_{p-1}\left[\frac{1}{2}\right]\right)$$

$$(\tilde{\Gamma}\varphi)_{\mu_1\dots\mu_{p-1}} = \Gamma^\mu\varphi_{\mu\mu_1\dots\mu_{p-1}}$$

Non-commutative Dirac operator

$$D = i\tilde{\Gamma}\mathcal{D} : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right)$$

$$\bar{D} = iA^{-1}\tilde{\mathcal{D}}\Gamma A : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \rightarrow C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right)$$

Operators on Endo-Valued Densities ($p = 0$)

$$\Delta = \rho^{-1}(\tilde{d} + \tilde{\mathcal{B}})\rho A \rho(d + \mathcal{B})\rho^{-1}$$

$$D = i\tilde{\Gamma}\mathcal{D}, \quad \bar{D} = i\tilde{\mathcal{D}}\Gamma$$

In local coordinates

$$\Delta = \rho^{-1}(\partial_\mu + \mathcal{B}_\mu)\rho a^{\mu\nu}\rho(\partial_\nu + \mathcal{B}_\nu)\rho^{-1}$$

$$D = i\Gamma^\mu\rho(\partial_\mu + \mathcal{B}_\mu)\rho^{-1}, \quad \bar{D} = i\rho^{-1}(\partial_\mu + \mathcal{B}_\nu)\rho\Gamma^\mu$$

$$D\bar{D} = -\Gamma^\mu\rho(\partial_\mu + \mathcal{B}_\mu)\rho^{-2}(\partial_\nu + \mathcal{B}_\nu)\rho\Gamma^\nu$$

$$\bar{D}D = -\rho^{-1}(\partial_\nu + \mathcal{B}_\nu)\rho\Gamma^\nu\Gamma^\mu\rho(\partial_\mu + \mathcal{B}_\mu)\rho^{-1}$$

Leading symbols

$$\sigma_L(D; x, \xi) = \sigma_L(\bar{D}; x, \xi) = -\Gamma(x, \xi) = -\Gamma^\mu(x)\xi_\mu$$

$$\begin{aligned} \sigma_L(-\Delta; x, \xi) &= \sigma_L(\bar{D}D; x, \xi) = \sigma_L(D\bar{D}; x, \xi) \\ &= H(x, \xi) = [\Gamma(x, \xi)]^2 \end{aligned}$$

Spectral Asymptotics

Heat Semigroup $\exp(-t\bar{D}D)$ is trace-class

$$\mathrm{Tr}_{L^2} \exp(-t\bar{D}D) = \sum_{k=1}^{\infty} e^{-\lambda_k(\bar{D}D)} = \int_M dx \, \mathrm{tr}_S U(t; x, x)$$

Asymptotic expansions as $t \rightarrow 0^+$

$$U(t; x, x) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} a_k(\bar{D}D; x)$$

$$\mathrm{Tr}_{L^2} \exp(-t\bar{D}D) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(2k-n)/2} A_k(\bar{D}D)$$

Spectral Invariants

$$A_k(\bar{D}D) = \int_M dx \, \mathrm{tr}_S a_k(\bar{D}D; x)$$

Index of the Dirac Operator

Operators $\bar{D}D$ and $D\bar{D}$ are non-negative

$\bar{D}D$ and $D\bar{D}$ have the same non-zero spectrum

Zero eigenspaces $\text{Ker}(\bar{D}D)$ and $\text{Ker}(D\bar{D})$ could be different

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$$\begin{aligned}\text{Ind}(D) &= \dim \text{Ker}(\bar{D}) - \dim \text{Ker}(D) \\ &= \text{Tr}_{L^2} \exp(-tD\bar{D}) - \text{Tr}_{L^2} \exp(-t\bar{D}D) \\ &= (4\pi)^{-n/2} \left[A_{n/2}(D\bar{D}) - A_{n/2}(\bar{D}D) \right]\end{aligned}$$

Spectral invariants of $\bar{D}D$ and $D\bar{D}$ except $A_{n/2}$ are equal

$$A_k(D\bar{D}) = A_k(\bar{D}D), \quad k \neq \frac{n}{2}$$

Results

Spectral Invariants

$$A_0(\bar{D}D) = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \operatorname{tr}_S e^{-H}$$

$$A_1(\bar{D}D) = \int_M dx \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \operatorname{tr}_S \left[\int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \int_0^1 d\tau_1 e^{-(1-\tau_1)H} \bar{D}D e^{-\tau_1 H} \right]$$

where

$$K = -\Gamma(\xi)D - \bar{D}\Gamma(\xi)$$

Non-commutative Einstein-Hilbert Functional

$$S_{MG} = -\frac{1}{N} \left\{ 12A_1(\bar{D}D) + 2\Lambda A_0(\bar{D}D) \right\}$$

Commutative limit $\kappa \rightarrow 0$

$$S_{EH} = \int_M d\text{vol} (R - 2\Lambda)$$

gives the **Einstein-Hilbert functional** with cosmological constant Λ

Einstein Spaces are extremals of S_{EH}

Variation of S_{MG} gives **non-commutative Einstein equations**

Extremals of S_{MG} are **non-commutative Einstein Spaces**