

Heat Kernel on Homogeneous Bundles **over Symmetric Spaces**

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- Symmetric spaces
- Homogeneous vector bundles
- Heat semi-group
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- Examples (S^2 and H^2)

Motivation

- Quantum Field Theory
 - effective action, propagators
- Statistical Physics
 - partition function, correlation functions
- Integrable Systems
 - isospectrality, KdV hierarchy
- Spectral Asymptotics
- Spectral Geometry
- Topology
 - index theorems

Laplacians on Vector Bundles

Riemannian manifold (M, g) without boundary

Vector bundle \mathcal{V} over M

Connection ∇ on \mathcal{V}

Laplacian

$$\Delta = -\nabla^* \nabla = g^{\mu\nu} \nabla_\mu \nabla_\nu$$

- The operator $(-\Delta)$ is *elliptic* and *self-adjoint*.
- It has a *discrete real* spectrum *bounded from below* with *finite multiplicities*.

Heat Kernel

Heat semigroup

$$U(t) = \exp(t\Delta) : L^2(\mathcal{V}) \rightarrow L^2(\mathcal{V})$$

Heat equation

$$(\partial_t - \Delta)U(t; x, x') = 0$$

Initial condition

$$U(0^+; x, x') = \delta(x, x')$$

Heat kernel

$$\begin{aligned} U(t; x, x') &= \exp(t\Delta)\delta(x, x') \\ &= \sum_{n=1}^{\infty} e^{-t\lambda_n} \varphi_n(x) \otimes \varphi_n^*(x') \end{aligned}$$

Heat kernel diagonal

$$U^{\text{diag}}(t; x) = U(t; x, x) = \sum_{n=1}^{\infty} e^{-t\lambda_n} |\varphi_n(x)|^2$$

Spectral Invariants

Heat trace

$$\mathrm{Tr}_{L^2} \exp(t\Delta) = \sum_{n=1}^{\infty} e^{-t\lambda_n} = \int_M d\mathrm{vol} \, \mathrm{tr}_V U^{\mathrm{diag}}(t)$$

Zeta function

$$\begin{aligned} \zeta(s, \lambda) &= \mathrm{Tr}_{L^2} (-\Delta - \lambda)^{-s} \\ &= \sum_{n=1}^{\infty} (\lambda_n - \lambda)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} e^{t\lambda} \mathrm{Tr}_{L^2} \exp(t\Delta) \end{aligned}$$

Determinant

$$\log \mathrm{Det}_{L^2} (-\Delta - \lambda) = -\frac{\partial}{\partial s} \zeta(s, \lambda) \Big|_{s=0}$$

Heat Kernel Asymptotics

As $t \rightarrow 0^+$

Diagonal

$$U^{\text{diag}}(t; x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k a_k(x)$$

Trace

$$\text{Tr}_{L^2} \exp(t\Delta) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k A_k$$

Spectral Invariants

$$A_k = \int_M d\text{vol} \text{tr}_V a_k$$

Heat Invariants

Low-order

$$a_0 = I$$

$$a_1 = \frac{1}{6}R$$

$$a_2 = \frac{1}{72}R^2 - \frac{1}{180}|\text{Ric}|^2 + \frac{1}{180}|\text{Riem}|^2 + \frac{1}{12}|\Omega|^2 \\ + \frac{1}{30}\Delta R$$

Remark. Local invariants a_k are *homogeneous polynomials of the curvature* of degree k .

Explicit calculation. a_3 : Gilkey (1975), a_4 : IGA (1987), a_5 : Yajima (2004), Van de Ven (1998)

Parallel Curvature

Parallel curvature (locally symmetric spaces)

$$\nabla_f R^{ab}{}_{cd} = 0$$

Constraints

$$R^{fg}{}_{ea} R^e{}_{bcd} - R^{fg}{}_{eb} R^e{}_{acd} + R^{fg}{}_{ec} R^e{}_{dab} - R^{fg}{}_{ed} R^e{}_{cab} = 0$$

Parallel bundle curvature

$$\nabla_f \Omega_{ab} = 0$$

Constraints

$$[\Omega_{cd}, \Omega_{ab}] - R^f{}_{acd} \Omega_{fb} - R^f{}_{bcd} \Omega_{af} = 0$$

Goal: Compute all heat kernel coefficients for bundles with parallel curvature

Curvature Algebra of a Symmetric Space

Curvature tensor

$$R_{abcd} = \beta_{ik} E^i_{ab} E^k_{cd}$$

where E^i_{ab} , $i = 1, \dots, p$, anti-symmetric matrices, and β_{ik} is a symmetric nondegenerate $p \times p$ matrix

Holonomy algebra, \mathcal{H} ,

$$[D_i, D_k] = F^j_{ik} D_j$$

where F^j_{ik} are the structure constants, and the $n \times n$ matrices $D_i = (D^a_{ib})$, are defined by

$$D^a_{ib} = -\beta_{ik} E^k_{cb} \delta^{ca}$$

Curvature algebra, \mathcal{G} ,

$$[C_A, C_B] = C^C_{AB} C_C$$

where $A, B, C = 1, \dots, N$ with $N = p + n$, and the $N \times N$ matrices $(C_A)^B_C = C^B_{AC}$ are defined by

$$C^i_{ab} = E^i_{ab}, \quad C^a_{ib} = -C^a_{bi} = D^a_{ib}, \quad C^i_{kl} = F^i_{kl}$$

Killing Vectors Fields

Killing vector fields $\xi_A = (P_a, L_i)$ (generators of isometries)

$$P_a = \left(\sqrt{K} \cot \sqrt{K} \right)^b{}_a \frac{\partial}{\partial y^b},$$

$$L_i = -D^b{}_{ia} y^a \frac{\partial}{\partial y^b},$$

where y^a are normal coordinates and

$$K^a{}_b = R^a{}_{cbd} y^c y^d$$

Killing vector fields form a representation of the curvature algebra \mathcal{G}

$$[\xi_A, \xi_B] = C^C{}_{AB} \xi_C$$

and satisfy the equation

$$\gamma^{AB} \xi_A^\mu \xi_B^\nu = g^{\mu\nu}$$

where $(\gamma^{AB}) = (\gamma_{AB})^{-1}$ and

$$(\gamma_{AB}) = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}$$

Twisted Spin-tensor Bundles

Twisted spin-tensor bundle $\mathcal{V} = \mathcal{T} \otimes \mathcal{W}$

Spin-tensor bundle \mathcal{T} realizing a representation of the spin group $\text{Spin}(n)$ with generators Σ_{ab}

Associated vector bundle \mathcal{W} with the structure group G_{YM}

Bundle curvature 2-form

$$\Omega = \frac{1}{2} R_{ab} \Sigma^{ab} + \mathcal{F},$$

Curvature of the connection on the bundle \mathcal{W}

$$\mathcal{F} = dA + A \wedge A$$

Homogeneous Vector Bundles

General symmetric space has the structure

$$M = M_0 \times M_s, \quad M_s = M_+ \times M_-$$

where M_0 is a Euclidean space and M_s is a *semi-simple space*

The structure group G_{YM} of the twisted spin-tensor bundle \mathcal{V} has a subgroup $Z \times H$, where Z is an Abelian group and H is the holonomy group. That is, the holonomy algebra \mathcal{H} is an ideal of the Lie algebra \mathcal{G}_{YM} of the structure group

Bundle curvature

$$\Omega_{ab} = \frac{1}{2} R^{cd}{}_{ab} G_{cd} + \mathcal{B}_{ab}$$

where

$$G_{ab} = \Sigma_{ab} \otimes \mathbb{I}_X + \mathbb{I}_\Sigma \otimes X_{ab},$$

X_{ab} are the generators of $\mathcal{SO}(n)$ in some representation, and \mathcal{B}_{ab} takes values in an Abelian ideal of the gauge algebra \mathcal{G}_{YM}

Twisted Lie Derivatives

Twisted Lie derivative along ξ_A

$$\mathcal{L}_A = \mathcal{L}_{\xi_A} = \nabla_{\xi_A} + \frac{1}{2}\xi_A^a{}_{;b}G^b{}_a$$

Theorem. The operators \mathcal{L}_A form an algebra that is a direct sum of a nilpotent ideal and a semisimple algebra

$$[\mathcal{L}_A, \mathcal{L}_B] = C^C{}_{AB}\mathcal{L}_C + \mathcal{B}_{AB},$$

where

$$\mathcal{B}_{AB} = \begin{pmatrix} \mathcal{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem. The Laplacian Δ on the bundle \mathcal{V} has the form

$$\Delta = \mathcal{L}^2 - \mathcal{R}^2$$

where

$$\mathcal{L}^2 = \gamma^{AB}\mathcal{L}_A\mathcal{L}_B$$

and

$$\mathcal{R}^2 = \frac{1}{4}R^{abcd}G_{ab}G_{cd}$$

Heat Semigroup

Theorem. The heat semigroup $\exp(t\Delta)$ can be represented in the form

$$\begin{aligned} \exp(t\Delta) &= (4\pi t)^{-N/2} \det_{TM} \left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}} \right)^{-1/2} \\ &\times \exp \left(-t\mathcal{R}^2 + \frac{1}{6}R_G t \right) \\ &\times \int_{\mathbb{R}_{\text{reg}}^N} dk |\gamma|^{1/2} \det_g \left(\frac{\sinh[C(k)/2]}{C(k)/2} \right)^{1/2} \\ &\times \exp \left\{ -\frac{1}{4t} \langle k, \gamma t\mathcal{B} \coth(t\mathcal{B})k \rangle \right\} \exp[\mathcal{L}(k)] \end{aligned}$$

where $|\gamma| = \det \gamma$, $\mathcal{B} = (\mathcal{B}_{AB})$,

$$R_G = -\frac{1}{4}\gamma^{AB}C^C{}_{AD}C^D{}_{BC}$$

$$C(k) = C_A k^A, \quad \mathcal{L}(k) = \mathcal{L}_A k^A$$

Regularization and Analytical Continuation

Natural splitting of canonical coordinates

$$(k^A) = (p^a, \omega^i)$$

Coordinates $p = (p^a) \in \mathbb{R}^n$ are real

Coordinates $\omega = (\omega^j) \in \mathbb{R}_{\text{reg}}^p$ are complex

$$\omega \mapsto e^{i\pi/4} \omega$$

or V -shaped contour

$$\omega \mapsto \exp \left\{ i \frac{\pi}{4} [2 - \text{sign}(\omega)] \right\} \omega$$

For $t < 0$ all integrals are well defined and define analytic functions of t in the complex plane with a cut along the negative imaginary axis.

Heat Kernel Diagonal and Heat Trace

Heat trace

$$\text{Tr}_{L^2} \exp(t\Delta) = \text{vol}(M) \text{tr}_V U^{\text{diag}}(t)$$

where tr_V is the fiber trace

Heat kernel diagonal

$$U^{\text{diag}}(t) = \exp(t\Delta) \delta(x, x') \Big|_{x=x'}$$

Lemma.

$$\begin{aligned} & \exp[\mathcal{L}(k)] \delta(x, x') \Big|_{x=x'} \\ &= \det_{TM} \left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2} \right)^{-1} \exp[\mathcal{R}(\omega)] \delta(p) \end{aligned}$$

where

$$D(\omega) = D_i \omega^i, \quad \mathcal{R}(\omega) = -\frac{1}{2} D^a{}_{ib} G^b{}_a \omega^i$$

Theorem. The heat kernel diagonal has the form

$$\begin{aligned}
 U^{\text{diag}}(t) &= (4\pi t)^{-n/2} \det_{TM} \left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}} \right)^{-1/2} \\
 &\times \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H - \mathcal{R}^2 \right) t \right\} \\
 &\times \int_{\mathbb{R}_{\text{reg}}^p} \frac{d\omega}{(4\pi t)^{p/2}} |\beta|^{1/2} \exp \left\{ -\frac{1}{4t} \langle \omega, \beta\omega \rangle \right\} \Psi(\omega)
 \end{aligned}$$

where

$$\begin{aligned}
 \Psi(\omega) &= \cosh [\mathcal{R}(\omega)] \det_{\mathcal{H}} \left(\frac{\sinh [F(\omega)/2]}{F(\omega)/2} \right)^{1/2} \\
 &\times \det_{TM} \left(\frac{\sinh [D(\omega)/2]}{D(\omega)/2} \right)^{-1/2}
 \end{aligned}$$

Here $|\beta| = \det \beta_{ij}$, $F(\omega) = (F(\omega)^k_j)$ is a matrix defined by

$$F(\omega)^k_j = F^k_{ij} \omega^i$$

and

$$R_H = -\frac{1}{4} \beta^{ij} F^k_{il} F^l_{jk}$$

Heat Kernel Asymptotics

Gaussian average over the holonomy algebra

$$\langle f(\omega) \rangle = \int_{\mathbb{R}_{\text{reg}}^p} \frac{d\omega}{(4\pi)^{p/2}} |\beta|^{1/2} \exp\left(-\frac{1}{4} \langle \omega, \beta \omega \rangle\right) f(\omega)$$

Gaussian average of polynomials

$$\langle \omega_1^i \dots \omega^{i_{2k+1}} \rangle = 0$$

$$\langle \omega^{i_1} \dots \omega^{i_{2k}} \rangle = \frac{(2k)!}{k!} \beta^{(i_1 i_2 \dots \beta^{i_{2k-1} i_{2k}})}$$

Heat kernel diagonal

$$U^{\text{diag}}(t) = (4\pi t)^{-n/2} \det_{TM} \left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}} \right)^{-1/2} \\ \times \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H - \mathcal{R}^2 \right) t \right\} \langle \Psi(\sqrt{t} \omega) \rangle$$

This gives a generating function for all heat kernel coefficients a_k for any locally symmetric space

Algebraic Representation

Creation and annihilation operators

$$[b^j, b_k^*] = \delta_k^j, \quad [b^j, b^k] = [b_j^*, b_k^*] = 0$$

Vacuum vector

$$\langle 0|0\rangle = 1, \quad b^j|0\rangle = \langle 0|b_k^* = 0$$

Gaussian average is the vacuum expectation value

$$\langle f(\omega) \rangle = \langle 0|f(b) \exp\langle b^*, \beta b^* \rangle|0\rangle,$$

where $\langle b^*, \beta b^* \rangle = \beta^{jk} b_j^* b_k^*$.

This should be computed by the so-called normal ordering, that is, by simply commuting the operators b_j through the operators b_k^* until they hit the vacuum vector giving zero.

Heat Kernel on $S^2 = SO(3)/SO(2)$

Heat kernel diagonal

$$\begin{aligned} U^{\text{diag}}(t) &= \frac{1}{4\pi t} \exp \left[\left(\frac{1}{4} + \alpha^2 \right) \frac{t}{r^2} \right] \\ &\times \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp \left(-\frac{\omega^2}{4} \right) \frac{\omega \sqrt{-t}/(2r)}{\sinh \left[\omega \sqrt{-t}/(2r) \right]} \\ &\times \cosh \left(\alpha \omega \sqrt{-t}/r \right) \end{aligned}$$

where r is the radius and $\alpha \in \mathbb{Z}$ or $\alpha \in \frac{1}{2} + \mathbb{Z}$.

Sum over closed geodesics

$$\begin{aligned} U^{\text{diag}}(t) &= \frac{1}{4\pi t} \exp \left[\left(\frac{1}{4} + \alpha^2 \right) \frac{t}{r^2} \right] \sum_{k=-\infty}^{\infty} (-1)^k \\ &\times \int_0^{2\pi r/\sqrt{t}} \frac{d\omega}{\sqrt{4\pi}} \exp \left[-\frac{1}{4} \left(\omega + \frac{2\pi r}{\sqrt{t}} k \right)^2 \right] \\ &\times \frac{\sqrt{t}}{2r} \frac{\left(\omega + \frac{2\pi r}{\sqrt{t}} k \right)}{\sin \left[\omega \sqrt{t}/(2r) \right]} \cos \left(\alpha \omega \sqrt{t}/r \right). \end{aligned}$$

Heat Kernel on $H^2 = SO(1, 2)/SO(2)$

$$\begin{aligned} U^{\text{diag}}(t) &= \frac{1}{4\pi t} \exp \left[- \left(\frac{1}{4} + \alpha^2 \right) \frac{t}{a^2} \right] \\ &\times \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp \left(-\frac{\omega^2}{4} \right) \frac{\omega\sqrt{t}/(2a)}{\sinh \left[\omega\sqrt{t}/(2a) \right]} \\ &\times \cosh \left(\alpha\omega\sqrt{t}/a \right) \end{aligned}$$

where a is the pseudo-radius

Duality

$$a \mapsto ir \quad (\text{ or } t \mapsto -t)$$

$$U_{S^2}^{\text{diag}}(t) = U_{H^2}^{\text{diag}}(t) \Big|_{a=ir}$$

Spectral Representation

$$U_{H^2}^{\text{diag}}(t) = \int_{-\infty}^{\infty} \frac{d\nu}{4\pi a^2} \mu(\nu) \exp \left\{ - \left(\frac{1}{4} + \alpha^2 + \nu^2 \right) \frac{t}{a^2} \right\}$$

where (Plancherel (or Harish-Chandra) measure)

$$\mu(\nu) = \nu \tanh \nu \quad \text{for integer } \alpha = m,$$

and

$$\mu(\nu) = \nu \coth \nu \quad \text{for half-integer } \alpha = m + \frac{1}{2}.$$

$$U_{S^2}^{\text{diag}}(t) = \frac{1}{4\pi r^2} \sum_{k=0}^{\infty} d_k \exp(-\lambda_k t),$$

where for integer $\alpha = m$:

$$d_k = k + \frac{1}{2}, \quad \lambda_k = \frac{1}{r^2} \left[\left(k + \frac{1}{2} \right)^2 - \frac{1}{4} - m^2 \right]$$

and for half-integer $\alpha = m + \frac{1}{2}$:

$$d_k = k, \quad \lambda_k = \frac{1}{r^2} \left[k^2 - \frac{1}{4} - \left(m + \frac{1}{2} \right)^2 \right]$$