4.2 Simplicial Homology Groups

4.2.1 Simplicial Complexes

- Let $p_0, p_1, \ldots p_k$ be $k+1$ points in $\mathbb{R}^n$, with $k \le n$. We identify points in $\mathbb{R}^n$ with the vectors that point to them.
- Assume that they are independent, that is, do not lie in a $(k-1)$-dimensional hyperplane, or that the vectors $v_{ij} = p_j - p_i$ are linearly independent.
- The *k*-**simplex** $\sigma_k = \langle p_0, p_1, \ldots, p_k \rangle$ is the compact subset of $\mathbb{R}^n$ defined by

$$\sigma_k = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=0}^{k} c_i p_i, \quad \text{with} \quad c_i \ge 0, \sum_{i=0}^{k} c_i = 1 \right\}$$

- For any j, $0 \le j \le k$, a subset of $j+1$ points defines a j-simplex called the *j*-**face**.
- A 0-simplex is a point, called a **vertex**.
- A 1-simplex is a line segment, called an **edge**.
- A 2-simplex is the interior of a triangle.
- The 3-simplex is a **tetrahedron**.
- A **simplicial complex** is a set K of finitely many simplexes such that:
  - every face of every simplex of K belongs to K,
  - the intersection of any two simplexes in K is either empty or is a common face.
- A subset $|K|$ of $\mathbb{R}^n$ which is the union of all simplexes in a complex K is called a **polyhedron**.
- A simplicial complex K and a homeomorphism

  $$F : |K| \to X$$

  to a topological space X is called a **triangulation** of X.
- A topological space X is called **triangulable** if there is a triangulation of X.

- An unoriented k-simplex $\langle p_0, p_1, \ldots, p_k\rangle$ can be oriented as follows. An **oriented k-simplex** $(p_0, p_1, \ldots, p_k)$ changes sign under a permutation of any two points. Let φ be a permutation of points $\{p_0, p_1, \ldots, p_k\}$. Then

$$(p_{\varphi(0)}, p_{\varphi(1)}, \ldots, p_{\varphi(k)}) = (\operatorname{sign}\varphi)(p_0, p_1, \ldots, p_k),$$

where $\operatorname{sign}\varphi$ is the parity of the permutation φ.

4.2.2 Simplicial Homology Groups

- Let K be an n-dimensional simplicial complex.
- Let N_p is the number of p-simplexes in K.
- A p**-chains** is a formal sum

$$c = \sum_{i=1}^{N_p} c_i \sigma_{p,i}$$

where $\sigma_{p,i}$ are p-simplexes in K and $c_i \in \mathbb{Z}$.

- **Remark.** We can define chains over any Abelian group, for example, $\mathbb{R}$ or $\mathbb{Z}_2$.
- This allows to define the Abelian group structure: addition, zero, opposite.
- The p**-chain group** $C_p(K)$ of K is a free Abelian group generated by the oriented k-simplexes of K,

$$C_p(K) \simeq \bigoplus_{i=1}^{N_p} \mathbb{Z}$$

- By definition $C_p(K) = 0$ for $p > n$.
- The **boundary operator** is a homomorphism

$$\partial_p : C_p(K) \to C_{p-1}(K)$$

defined as follows.

- The **boundary** of an oriented p-simplex $\sigma_p = (p_0, p_1, \ldots, p_p)$ is a $(p-1)$-chain defined by

$$\partial_p \sigma_p = \sum_{i=0}^{p} (-1)^i (p_0, p_1, \ldots, \hat{p}_i, \ldots, p_k)$$

  where $\hat{p}_i$ is omitted.

- The boundary of a p-chain is defined by linearity.

- The **chain complex** is a sequence of free Abelian groups and homomorphisms

$$0 \overset{i}{\to} C_n(K) \overset{\partial_n}{\to} C_{n-1}(K) \overset{\partial_{n-1}}{\to} \cdots \overset{\partial_1}{\to} C_0(K) \overset{\partial_0}{\to} 0$$

  where $i :\hookrightarrow C_n(K)$ is the inclusion map.

- A p-chain z such that

$$\partial_p z = 0$$

  is called a p**-cycle**.

- The p-cycles form a free Abelian subgroup of $C_p(K)$ called the p**-cycle group**

$$Z_p(K) = \operatorname{Ker} \partial_p$$

- A p-chain b such that

$$b = \partial_{p+1} c$$

  for some $(p+1)$-chain c, is called a p**-boundary**.

- The p-boundaries form a free Abelian subgroup of $C_p(K)$ called the p**-boundary group**

$$B_p(K) = \operatorname{Im} \partial_{p+1}$$

- **Proposition.** The boundary of a boundary vanishes, that is,

$$\partial_p \partial_{p+1} = 0$$

- **Corollary.** Every boundary is a cycle, that is,

$$B_p(K) \subset Z_p(K)$$

- The p-homology group $H_p(K)$ is defined by

$$H_p(K) = Z_p(K)/B_p(K)$$

  It is not necessarily free Abelian.

- We say that two p-cycles are **homologous** if they differ by a boundary.
- Homology is an equivalence relation.
- The equivalence classes of the homology are called **homology classes.**
- The homology groups are the sets of homology classes.
- **Theorem.** Homology groups are topological invariants. In particular,
  - The homology groups of different triangulations of the same topological space are isomorphic.
  - The homology groups of any triangulations of homeomorphic topological spaces are isomorphic.
- Therefore, the homology groups of a triangulable topological space (which is not necessary a polyhedron) are defined to be the homology groups of some triangulation.
- **Spheres**.

$$H_0(S^1) = H_1(S^1) = \mathbb{Z}.$$
$$H_0(S^2) = H_2(S^2) = \mathbb{Z}, \qquad H_1(S^2) = 0.$$

- **Theorem.** For any connected simplicial complex K

$$H_0(K) = \mathbb{Z}\,.$$

- **Möbius Strip**.

$$H_0(K) = \mathbb{Z}, \qquad H_1(K) = \mathbb{Z}, \qquad H_2(K) = 0\,.$$

- **Real Projective Space** $\mathbb{R}P^2$.

$$H_0(\mathbb{R}P^2) = \mathbb{Z}, \qquad H_1(\mathbb{R}P^2) = \mathbb{Z}_2, \qquad H_2(\mathbb{R}P^2) = 0\,.$$

- The homology group over $\mathbb{Z}$ is not necessarily free Abelian group but may include the torsion.

- **Torus T^2.**
$$H_0(T^2) = H_2(T^2) = \mathbb{Z}, \qquad H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}.$$

- **Surface Σ_g of genus g.**
$$H_0(\Sigma_g) = H_2(\Sigma_g) = \mathbb{Z}, \qquad H_1(\Sigma_g) = \bigoplus_{i=1}^{2g} \mathbb{Z}.$$

- **Klein Bottle K^2.**
$$H_0(K^2) = \mathbb{Z}, \qquad H_2(K^2) = 0, \qquad H_1(K^2) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

- **Theorem.** The homology groups of a disconnected simplicial complex are equal to the direct sum of the homology groups of its connected components.

- **Corollary.** If a complex K has m connected components, then
$$H_0(K) = \bigoplus_{i=1}^{m} \mathbb{Z}.$$

- **Corollary.** For a complex K
$$H_0(K) = \mathbb{Z}$$
if and only if K is connected.

- A general homology group over $\mathbb{Z}$ has the form
$$H_p(K) = \bigoplus_{i=1}^{m} \oplus \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}$$

- The number of generators of H_p counts the number of $(p+1)$ dimensional holes in the polyhedron $|K|$.

- The **torsion subgroup** $\mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}$ measures the twisting in the polyhedron $|K|$.

- The homology groups over $\mathbb{R}$ or $\mathbb{Z}_2$ do not have torsion.
- The homology groups $H_p(K, \mathbb{R})$ are finite-dimensional vector spaces.
- The dimension of the vector spaces $H_p(K, \mathbb{R})$ are called **Betti numbers**

$$b_p(K) = \dim H_p(K, \mathbb{R})$$

- The Betti numbers are equal to the ranks of the free Abelian parts of the homology groups over $\mathbb{Z}$.
- The **Euler characteristic** of a simplicial complex K with N_p p-simplexes is an integer defined by

$$\chi(K) = \sum_{p=0}^{n} (-1)^p \dim C_p(K, \mathbb{R}) = \sum_{p=0}^{n} (-1)^p N_p.$$

- **Theorem.** The Euler characteristic of a simplicial complex K is equal to

$$\chi(K) = \sum_{p=0}^{n} (-1)^p \dim H_p(K, \mathbb{R}) = \sum_{p=0}^{n} (-1)^p b_p(K).$$

- The Euler characteristic is a topological invariant.
- The Euler characteristic of a topological space does not depend on the triangulation, so, it can be defined for any triangulation.