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**Riemannian Geometry,
Lie Groups
and Geometric Analysis**

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We present the theory of Riemannian geometry, Lie groups and geometric analysis in the form suitable for local calculations.

1 Tensor Algebra

1.1 Basis

Let M be an n -dimensional orientable Riemannian manifold with the metric g . Let x^μ , $\mu = 1, \dots, n$, be the local coordinates on M . Then ∂_μ and dx^μ form coordinate bases in the tangent and the cotangent spaces. Let $e_a = e^\mu{}_a \partial_\mu$ be a basis in the tangent space and $\sigma^a = \sigma^a{}_\mu dx^\mu$ be the dual basis in the cotangent space, so that

$$\sigma^a(e_b) = \sigma^a{}_\mu e^\mu{}_b = \delta^a{}_b, \quad (1.1)$$

then also

$$e^\mu{}_a \sigma^a{}_\nu = \delta^\mu{}_\nu. \quad (1.2)$$

We will use Greek indices to label the components of tensors with respect to the coordinate basis and Latin indices to label the components of tensors with respect to a generic basis (that we will call a local frame). We also use Einstein summation convention and sum over repeated indices.

1.2 Tensor Fields

A tensor field of type (p, q) has the form

$$\begin{aligned} T &= T_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p}(x) dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q} \otimes \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_p} \\ &= T_{b_1 \dots b_q}^{a_1 \dots a_p}(x) \sigma^{b_1} \otimes \dots \otimes \sigma^{b_q} \otimes e_{a_1} \otimes \dots \otimes e_{a_p}, \end{aligned} \quad (1.3)$$

where

$$T_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} = T(dx^{\mu_1}, \dots, dx^{\mu_p}, \partial_{\nu_1}, \dots, \partial_{\nu_q}), \quad (1.4)$$

$$T_{b_1 \dots b_q}^{a_1 \dots a_p} = T(\sigma^{a_1}, \dots, \sigma^{a_p}, e_{b_1}, \dots, e_{b_q}), \quad (1.5)$$

The components of the tensor T are related by

$$T_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} = T_{b_1 \dots b_q}^{a_1 \dots a_p} \sigma^{b_1}{}_{\nu_1} \dots \sigma^{b_q}{}_{\nu_q} e^{\mu_1}{}_{a_1} \dots e^{\mu_p}{}_{a_p}. \quad (1.6)$$

The vector space of tensors of type (p, q) is denoted by T_q^p . Its dimension is obviously

$$\dim T_q^p = n^{p+q}. \quad (1.7)$$

The (r, s) -contraction of tensors of type (p, q) is the map

$$\text{tr}_s^r : T_q^p \rightarrow T_{q-1}^{p-1} \quad (1.8)$$

defined by

$$\text{tr}_s^r T = T_{\nu_1 \dots \nu_{s-1} \alpha \nu_{s+1} \dots \nu_q}^{\mu_1 \dots \mu_{r-1} \alpha \mu_{r+1} \dots \mu_{p-1}} dx^{\nu_1} \otimes \dots \widehat{dx^{\nu_s}} \dots \otimes dx^{\nu_q} \otimes \partial_{\mu_1} \otimes \dots \widehat{\partial_{\mu_r}} \dots \otimes \partial_{\mu_p}, \quad (1.9)$$

where the hat indicates that the corresponding factor is missing.

1.3 Transformation Properties

Let $x' = x'(x)$ be a diffeomorphism. Then

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}, \quad \partial'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu}, \quad (1.10)$$

where $\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}}$. Let us denote the Jacobian of the diffeomorphism by

$$J(x) = \det \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} \right). \quad (1.11)$$

The frame components of the tensor do not change and the coordinate components of the tensor transform as

$$T'^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}(x) \quad (1.12)$$

A *pseudo-tensor* is a collection of two tensors, one for each orientation of the manifold, such that when the orientation is reversed the tensor changes sign, that is, it transforms under a diffeomorphism according to

$$T'(x') = \text{sign}(J)T(x). \quad (1.13)$$

A *tensor density of weight w* of type (p, q) is a geometric object T which transforms under a diffeomorphism according to

$$T'(x') = |J(x)|^{-w}T(x). \quad (1.14)$$

A *pseudo-tensor density of weight w* of type (p, q) is a geometric object T which transforms under a diffeomorphism according to

$$T'(x') = \text{sign}(J)|J(x)|^{-w}T(x). \quad (1.15)$$

1.4 Riemannian Metric

The Riemannian metric is a tensor field of type $(0, 2)$

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \gamma_{ab} \sigma^a \otimes \sigma^b, \quad (1.16)$$

with symmetric and non-degenerate matrix of the components $g_{\mu\nu}$ or γ_{ab} , which are related by

$$g_{\mu\nu} = \gamma_{ab} \sigma^a{}_\mu \sigma^b{}_\nu. \quad (1.17)$$

The inverse (or contravariant, or dual) metric tensor is a tensor of type $(2, 0)$ defined by

$$g^{-1} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu = \gamma^{ab} e_a \otimes e_b, \quad (1.18)$$

where the matrix $(g^{\mu\nu})$ is inverse to $(g_{\mu\nu})$ and the matrix (γ^{ab}) is the inverse to (γ_{ab}) . The local frame is orthonormal if

$$\gamma_{ab} = \delta_{ab}. \quad (1.19)$$

The metric is used to lower and raise the tensor indices, for example,

$$A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu = g^{\mu\nu} A_\nu, \quad (1.20)$$

and so on.

In the following we will also use the following notation for the determinant of the metric

$$|g| = \det(g_{\mu\nu}) = \det(\gamma_{ab}) [\det(\sigma^a{}_\mu)]^2. \quad (1.21)$$

It is easy to see that $\sqrt{|g|}$ is a scalar density of weight 1. Therefore, if T is a density of weight w , then $|g|^{-w} T$ is a true tensor.

1.5 Permutations

Let S_p be the symmetric group of order p . The order of the symmetric group S_p is

$$|S_p| = p!. \quad (1.22)$$

The elements of the group S_p are called permutations. The *sign of a permutation* $\varphi \in S_p$, denoted by $\text{sign}(\varphi)$ (or simply $(-1)^\varphi$), is defined by

$$\text{sign}(\varphi) = (-1)^\varphi = \begin{cases} +1, & \text{if } \varphi \text{ is even,} \\ -1, & \text{if } \varphi \text{ is odd.} \end{cases} \quad (1.23)$$

Every permutation $\varphi \in S_p$ defines a map

$$\varphi : T_p \rightarrow T_p, \quad (1.24)$$

which assigns to every tensor T of type $(0, p)$ a new tensor $\varphi(T)$, of type $(0, p)$ by: for any collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$

$$\varphi(T)(\mathbf{v}_1, \dots, \mathbf{v}_p) = T(\mathbf{v}_{\varphi(1)}, \dots, \mathbf{v}_{\varphi(p)}). \quad (1.25)$$

The components of the tensor $\varphi(T)$ are obtained by the action of the permutation φ on the indices of the tensor T

$$\varphi(T)_{a_1 \dots a_p} = T_{a_{\varphi(1)} \dots a_{\varphi(p)}}. \quad (1.26)$$

The *symmetrization* of the tensor T of the type $(0, p)$ is defined by

$$\text{Sym}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \varphi(T). \quad (1.27)$$

The symmetrization of the components is denoted by parenthesis, that is, the components of the symmetrized tensor $\text{Sym}(T)$ are given by

$$[\text{Sym}(T)]_{a_1 \dots a_p} = T_{(a_1 \dots a_p)} = \frac{1}{p!} \sum_{\varphi \in S_p} T_{a_{\varphi(1)} \dots a_{\varphi(p)}}. \quad (1.28)$$

The *anti-symmetrization* of the tensor T of the type $(0, p)$ is defined by

$$\text{Alt}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \text{sign}(\varphi) \varphi(T). \quad (1.29)$$

The anti-symmetrization is denoted by square brackets, that is, $\text{Alt}(T)$ are given by

$$\text{Alt}(T)_{a_1 \dots a_p} = T_{[a_1 \dots a_p]} = \frac{1}{p!} \sum_{\varphi \in S_p} \text{sign}(\varphi) T_{a_{\varphi(1)} \dots a_{\varphi(p)}}. \quad (1.30)$$

A tensor T of type $(0, p)$ is called *symmetric* if for any permutation $\varphi \in S_p$

$$\varphi(T) = T, \quad (1.31)$$

and *anti-symmetric* if for any permutation $\varphi \in S_p$

$$\varphi(T) = \text{sign}(\varphi)T. \quad (1.32)$$

A symmetric tensor is invariant under the permutation of any two indices, and an anti-symmetric changes sign under the permutation of any two indices.

An anti-symmetric tensor of type $(0, p)$ is called a *p-form*. The vector space of all *p*-forms is denoted by Λ_p . Its dimension is equal to

$$\dim \Lambda_p = \binom{n}{p}. \quad (1.33)$$

Similarly one can define permutation, symmetrization, and anti-symmetrization of tensors of type $(p, 0)$. An anti-symmetric tensor of type $(p, 0)$ is called a poly-vector (or *p-vector*). Even more generally, for a tensor of type (p, q) one can define the permutations of the upper and lower indices separately, which means that the group $S_p \times S_q$ acts on the vector space T_q^p . One can also define partial permutation of a tensor, leaving some indices fixed. For example,

$$A_{[ef|gh|r]}^{(a|b|c)d} \quad (1.34)$$

means that there is symmetrization over upper indices a and c and anti-symmetrization over all lower indices excluding indices g and h . The vertical lines denote the indices excluded from the (anti)-symmetrization.

1.6 Alternating Tensors

Let (a_1, \dots, a_p) and (b_1, \dots, b_p) be two *p*-tuples of integers such that

$$1 \leq a_1, \dots, a_p, b_1, \dots, b_p \leq n. \quad (1.35)$$

The *generalized Kronecker symbol* is defined by

$$\delta_{b_1 \dots b_p}^{a_1 \dots a_p} = \begin{cases} 1 & \text{if } (a_1, \dots, a_p) \text{ is an even permutation of } (b_1, \dots, b_p), \\ -1 & \text{if } (a_1, \dots, a_p) \text{ is an odd permutation of } (b_1, \dots, b_p), \\ 0 & \text{otherwise .} \end{cases} \quad (1.36)$$

One can easily check that

$$\delta_{b_1 \dots b_p}^{a_1 \dots a_p} = p! \delta_{[a_1}^{a_1} \dots \delta_{b_p]}^{b_p} = \det \begin{pmatrix} \delta_{b_1}^{a_1} & \dots & \delta_{b_p}^{a_1} \\ \vdots & \ddots & \vdots \\ \delta_{b_1}^{a_p} & \dots & \delta_{b_p}^{a_p} \end{pmatrix} \quad (1.37)$$

Thus, the Kronecker symbols are the components of a tensor of type (p, p) , which are anti-symmetric separately in upper indices and the lower indices. Obviously, the Kronecker symbols vanish for $p > n$.

The contraction of Kronecker symbols gives Kronecker symbols of lower rank. More precisely, one can show that for any $1 \leq p, q \leq n$, there holds

$$\delta_{b_1 \dots b_p c_1 \dots c_q}^{a_1 \dots a_p c_1 \dots c_q} = \frac{(n-p)!}{(n-p-q)!} \delta_{b_1 \dots b_p}^{a_1 \dots a_p}. \quad (1.38)$$

In particular, for any $1 \leq q \leq n$ we have

$$\delta_{a_1 \dots a_q}^{a_1 \dots a_q} = \frac{n!}{(n-q)!}. \quad (1.39)$$

and

$$\delta_{a_1 \dots a_n}^{a_1 \dots a_n} = n!. \quad (1.40)$$

The product of Kronecker tensors is also equal to a Kronecker tensor, more precisely,

$$\frac{1}{r!} \delta_{c_1 \dots c_p m_1 \dots m_r}^{a_1 \dots a_p b_1 \dots b_r} \delta_{b_1 \dots b_r}^{k_1 \dots k_r} = \delta_{c_1 \dots c_p m_1 \dots m_r}^{a_1 \dots a_p k_1 \dots k_r} \quad (1.41)$$

The completely anti-symmetric (alternating) *Levi-Civita symbols* are defined by

$$\varepsilon_{a_1 \dots a_n} = \delta_{a_1 \dots a_n}^{1 \dots n}, \quad \varepsilon^{a_1 \dots a_n} = \delta_{1 \dots n}^{a_1 \dots a_n}, \quad (1.42)$$

so that

$$\varepsilon^{a_1 \dots a_n} = \varepsilon_{a_1 \dots a_n} = \begin{cases} 1 & \text{if } (a_1, \dots, a_n) \text{ is an even permutation of } (1, \dots, n), \\ -1 & \text{if } (a_1, \dots, a_n) \text{ is an odd permutation of } (1, \dots, n), \\ 0 & \text{otherwise.} \end{cases} \quad (1.43)$$

They satisfy the identity

$$\varepsilon^{a_1 \dots a_n} \varepsilon_{b_1 \dots b_n} = \delta_{b_1 \dots b_n}^{a_1 \dots a_n}. \quad (1.44)$$

The contraction of this identity over k indices gives a useful identity

$$\frac{1}{k!} \varepsilon^{a_1 \dots a_{n-k} m_1 \dots m_k} \varepsilon_{b_1 \dots b_{n-k} m_1 \dots m_k} = \delta_{b_1 \dots b_{n-k}}^{a_1 \dots a_{n-k}}, \quad (1.45)$$

in particular,

$$\varepsilon^{a_1 \dots a_n} \varepsilon_{a_1 \dots a_n} = n!. \quad (1.46)$$

It is easy to see that there holds also

$$\frac{1}{(n-p)!} \varepsilon^{b_1 \dots b_p c_1 \dots c_{n-p}} \delta_{c_1 \dots c_{n-p}}^{a_1 \dots a_{n-p}} = \varepsilon^{b_1 \dots b_p a_1 \dots a_{n-p}} \quad (1.47)$$

By using these tensors the anti-symmetrization can be written as

$$T_{[a_1 \dots a_p]} = \frac{1}{p!} \delta_{a_1 \dots a_p}^{b_1 \dots b_p} T_{b_1 \dots b_p}. \quad (1.48)$$

Therefore, if A and B are anti-symmetric tensors of type $(0, p)$ and $(p, 0)$, then

$$A_{a_1 \dots a_p} = \frac{1}{p!} \delta_{a_1 \dots a_p}^{b_1 \dots b_p} A_{b_1 \dots b_p}, \quad (1.49)$$

$$B^{a_1 \dots a_p} = \frac{1}{p!} \delta_{b_1 \dots b_p}^{a_1 \dots a_p} B^{b_1 \dots b_p}. \quad (1.50)$$

1.7 Determinants

Let $\text{Mat}(n, \mathbb{R})$ be the set of $n \times n$ real matrices. The *determinant* is a map $\det : \text{Mat}(n, \mathbb{R}) \rightarrow \mathbb{R}$ that assigns to each matrix $A = (A^i_j)$ a real number $\det A$ defined by

$$\det A = \sum_{\varphi \in S_n} \text{sign}(\varphi) A^1_{\varphi(1)} \cdots A^n_{\varphi(n)}, \quad (1.51)$$

It is easy to see now that the determinant of a matrix $A = (A^i_j)$ can be written as

$$\begin{aligned} \det A &= \varepsilon^{a_1 \dots a_n} A^1_{a_1} \cdots A^n_{a_n} \\ &= \varepsilon_{b_1 \dots b_n} A^{b_1}_1 \cdots A^{b_n}_n \\ &= \frac{1}{n!} \delta_{b_1 \dots b_n}^{a_1 \dots a_n} A^{b_1}_{a_1} \cdots A^{b_n}_{a_n}. \end{aligned} \quad (1.52)$$

Moreover, one can express the *characteristic determinants* (and the *elementary symmetric functions*) in terms of Kronecker symbols. Let $A = (A^\mu_\nu)$ be a matrix. Then one can show that the *sum of all $k \times k$ principal minors of $\det A$* is

$$M_k = \frac{1}{k!} \delta_{\nu_1 \dots \nu_k}^{\mu_1 \dots \mu_k} A^{\nu_1}_{\mu_1} \cdots A^{\nu_k}_{\mu_k}. \quad (1.53)$$

Furthermore, let $\mathbb{I} = (\delta^\mu_\nu)$ be the unit matrix and λ_k , $k = 1, \dots, n$, be the *roots of the characteristic equation*

$$\det(A - \lambda \mathbb{I}) = 0. \quad (1.54)$$

Then there holds

$$\begin{aligned} \det(A - \lambda \mathbb{I}) &= \prod_{k=1}^n (\lambda_k - \lambda) = \sum_{k=0}^n (-\lambda)^{n-k} F_k \\ &= \sum_{k=0}^n (-\lambda)^{n-k} M_k, \end{aligned} \quad (1.55)$$

where F_k are the *elementary symmetric functions of the roots*

$$F_0 = 1, \quad (1.56)$$

$$F_1 = \lambda_1 + \cdots + \lambda_n, \quad (1.57)$$

$$F_2 = \lambda_1 \lambda_2 + \cdots + \lambda_{n-1} \lambda_n, \quad (1.58)$$

$$\dots \quad (1.59)$$

$$F_n = \lambda_1 \cdots \lambda_n. \quad (1.60)$$

Another useful equation is the formula for the cofactor C^μ_ν of an element A^μ_ν in the determinant such that

$$\det A = C^\nu_\mu A^\mu_\nu, \quad (1.61)$$

and

$$C^\mu_\nu = \frac{\partial \det A}{\partial A^\nu_\mu}. \quad (1.62)$$

One can show that this cofactor has the form

$$C^\mu_\nu = \frac{1}{(n-1)!} \delta_{\nu \beta_1 \dots \beta_{n-1}}^{\mu \alpha_1 \dots \alpha_{n-1}} A^{\beta_1}_{\alpha_1} \cdots A^{\beta_{n-1}}_{\alpha_{n-1}}. \quad (1.63)$$

1.8 Exterior Algebra

The *exterior algebra* Λ (or *Grassmann algebra*) is the set of all forms of all degrees, that is,

$$\Lambda = \Lambda_0 \oplus \cdots \oplus \Lambda_n. \quad (1.64)$$

The dimension of the exterior algebra is obviously.

$$\dim \Lambda = \sum_{p=0}^n \binom{n}{p} = 2^n. \quad (1.65)$$

The exterior product is a map

$$\wedge : \Lambda \times \Lambda \rightarrow \Lambda, \quad (1.66)$$

such that for any p -form α and any q -form β the exterior product $\alpha \wedge \beta$ is a $(p+q)$ -form

$$\alpha \wedge \beta = \frac{(p+q)!}{p!q!} \text{Alt}(\alpha \otimes \beta). \quad (1.67)$$

In components

$$(\alpha \wedge \beta)_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[a_1 \dots a_p} \beta_{a_{p+1} \dots a_{p+q}]}, \quad (1.68)$$

which means

$$\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{[a_1 \dots a_p} \beta_{a_{p+1} \dots a_{p+q}]} \sigma^{a_1} \wedge \cdots \wedge \sigma^{a_{p+q}}. \quad (1.69)$$

We note a useful formula for the exterior product of a 1-form with a q -form β ; it has $q+1$ terms

$$\begin{aligned} (\alpha \wedge \beta)_{ca_1 \dots a_q} &= (q+1) \alpha_{[c} \beta_{a_1 \dots a_q]} = \alpha_c \beta_{a_1 \dots a_q} + q \alpha_{[a_1} \beta_{|c| a_2 \dots a_q]}, \\ &= \alpha_c \beta_{a_1 \dots a_q} + \alpha_{a_1} \beta_{a_2 \dots a_q c} + \cdots + \alpha_{a_q} \beta_{c \dots a_{q-1}} \end{aligned} \quad (1.70)$$

where all terms come with the plus sign if the indices are permuted in the cyclic order. Another useful formula is for the exterior product of a 2-form α and a q -form β . It has $\frac{(q+2)(q+1)}{2}$ terms

$$\begin{aligned} (\alpha \wedge \beta)_{cda_1 \dots a_q} &= \frac{(q+2)(q+1)}{2} \alpha_{[cd} \beta_{a_1 \dots a_q]} \\ &= \alpha_{cd} \beta_{a_1 \dots a_q} - q \alpha_{c[a_1} \beta_{|d| a_2 \dots a_q]} + q \alpha_{d[a_1} \beta_{|c| a_2 \dots a_q]} \\ &\quad + \frac{q(q-1)}{2} \alpha_{[a_1 a_2} \beta_{|cd| a_3 \dots a_q]}, \end{aligned} \quad (1.71)$$

The exterior product has the following commutation property

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \quad (1.72)$$

where p and q are the degrees of the forms α and β . In particular, this means that the exterior square of any p -form α of odd degree p (in particular, for any 1-form) vanishes

$$\alpha \wedge \alpha = 0. \quad (1.73)$$

1.9 Linear Transformations

We list below a couple of useful formulas. Let α^a be a collection of 1-forms

$$\alpha^a = A^a_b \sigma^b, \quad (1.74)$$

where σ^a is the basis 1-forms and $A = (A^a_b)$ is a matrix. Then

$$\alpha^1 \wedge \cdots \wedge \alpha^n = (\det A) \sigma^1 \wedge \cdots \wedge \sigma^n. \quad (1.75)$$

This has an immediate corollary. Let $x'^\mu = x'^\mu(x)$, $i = 1, \dots, n$, be a diffeomorphism. Then

$$dx'^1 \wedge \cdots \wedge dx'^n = \det \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) dx^1 \wedge \cdots \wedge dx^n. \quad (1.76)$$

More generally, let α^j , $1 \leq j \leq p$, be a collections of 1-forms and \mathbf{v}_i , $1 \leq i \leq p$, be a collection of vectors. Then

$$(\alpha^1 \wedge \cdots \wedge \alpha^p)(\mathbf{v}_1, \dots, \mathbf{v}_p) = \det A. \quad (1.77)$$

where A is a $p \times p$ matrix defined by

$$A^j_i = \alpha^j(\mathbf{v}_i). \quad (1.78)$$

This can be used to show that a collection of 1-forms α^j , $1 \leq j \leq p$, is linearly dependent if and only if

$$\alpha^1 \wedge \cdots \wedge \alpha^p = 0. \quad (1.79)$$

1.10 Volume Form

It is important to realize that the Levi-civita symbols are not tensors! Rather, $\varepsilon_{\mu_1 \dots \mu_n}$ represents the components of a pseudo- n -form (that is, a pseudo-tensor density of type $(0, n)$) of weight (-1) , and $\varepsilon^{\mu_1 \dots \mu_n}$ represents the components of a pseudo- n -vector (that is, a pseudo-tensor density of type $(n, 0)$) of weight 1. To make (pseudo)-tensors out of them we define

$$E_{\mu_1 \dots \mu_n} = \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_n}, \quad E^{\mu_1 \dots \mu_n} = \frac{1}{\sqrt{|g|}} \varepsilon^{\mu_1 \dots \mu_n}, \quad (1.80)$$

where $|g| = \det g_{\mu\nu}$. Then $E^{\mu_1 \dots \mu_n}$ represents the components of a pseudo- n -vector and $E_{\mu_1 \dots \mu_n}$ represents the components of a pseudo- n -form

$$d\text{vol} = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n \quad (1.81)$$

called the Riemannian volume element (or the *volume form*).

Notice that now

$$E^{\mu_1 \dots \mu_n} = g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} E_{\nu_1 \dots \nu_n}, \quad (1.82)$$

as it should be. Also notice that in the generic basis the volume form is defined by

$$d\text{vol} = \sqrt{|\gamma|} \sigma^1 \wedge \dots \wedge \sigma^n, \quad (1.83)$$

where $|\gamma| = \det(\gamma_{ab})$, so that its components in this basis are

$$E_{a_1 \dots a_n} = e^{\mu_1}_{a_1} \dots e^{\mu_n}_{a_n} E_{\mu_1 \dots \mu_n} = \sqrt{|\gamma|} \varepsilon_{a_1 \dots a_n}. \quad (1.84)$$

Recall that since $g_{\mu\nu} = \gamma_{ab} \sigma^a_{\mu} \sigma^b_{\nu}$, then

$$\sqrt{|g|} = \sqrt{|\gamma|} \det(\sigma^a_{\mu}). \quad (1.85)$$

1.11 Duality and Hodge Star Operator

The volume form allows one to define the *duality* of p -forms and $(n - p)$ -vectors. For each p -form α one assigns the *dual* $(n - p)$ -vector $\tilde{\alpha}$ by

$$\tilde{\alpha}^{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} E^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} \alpha_{\nu_1 \dots \nu_p}. \quad (1.86)$$

Similarly, for each p -vector A one assigns the *dual* $(n-p)$ -form \tilde{A} by

$$\tilde{A}_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} E_{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} A^{\nu_1 \dots \nu_p} . \quad (1.87)$$

Now, by lowering and raising the indices of the dual forms we can define the duality of forms and poly-vectors separately. We define the natural inner product $\langle \cdot, \cdot \rangle$ on the space of p forms by

$$\langle \alpha, \beta \rangle = \frac{1}{p!} g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} \alpha_{\mu_1 \dots \mu_n} \beta_{\nu_1 \dots \nu_n} . \quad (1.88)$$

Then the *Hodge star operator*

$$* : \Lambda_p \rightarrow \Lambda_{n-p}$$

maps any p -form α to a $(n-p)$ -form $*\alpha$ *dual* to α defined so that for any p -form β

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle d\text{vol} , \quad (1.89)$$

or

$$*(\alpha \wedge *\beta) = \langle \alpha, \beta \rangle . \quad (1.90)$$

In particular,

$$*1 = d\text{vol} , \quad *d\text{vol} = 1 . \quad (1.91)$$

This means that for any p -form α and a $(n-p)$ -form β we have

$$\langle *\beta, \alpha \rangle = \langle \beta, *^{-1}\alpha \rangle , \quad (1.92)$$

therefore, for any two p -forms α and β

$$\langle *\beta, *\alpha \rangle = \langle \beta, \alpha \rangle . \quad (1.93)$$

In components, this means that

$$\begin{aligned} (*\alpha)_{\lambda_1 \dots \lambda_{n-p}} &= \frac{1}{p!} E^{\nu_1 \dots \nu_p}{}_{\lambda_1 \dots \lambda_{n-p}} \alpha_{\nu_1 \dots \nu_p} \\ &= \frac{1}{p!} \varepsilon_{\mu_1 \dots \mu_p \lambda_1 \dots \lambda_{n-p}} \sqrt{|g|} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \alpha_{\nu_1 \dots \nu_p} \\ &= \frac{1}{p!} \frac{1}{\sqrt{|g|}} g_{\lambda_1 \mu_1} \dots g_{\lambda_{n-p} \mu_{n-p}} \varepsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} \alpha_{\nu_1 \dots \nu_p} . \end{aligned} \quad (1.94)$$

The inverse operator is then

$$\begin{aligned}
(*^{-1}\alpha)_{\lambda_1 \dots \lambda_{n-p}} &= \frac{1}{p!} E_{\lambda_1 \dots \lambda_{n-p}}^{\nu_1 \dots \nu_p} \alpha_{\nu_1 \dots \nu_p} \\
&= \frac{1}{p!} \varepsilon_{\lambda_1 \dots \lambda_{n-p} \mu_1 \dots \mu_p} \sqrt{|g|} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \alpha_{\nu_1 \dots \nu_p} \quad (1.95) \\
&= \frac{1}{p!} \frac{1}{\sqrt{|g|}} g_{\lambda_1 \mu_1} \dots g_{\lambda_{n-p} \mu_{n-p}} \varepsilon^{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} \alpha_{\nu_1 \dots \nu_p} .
\end{aligned}$$

Notice that the difference between $*$ and its inverse $*^{-1}$ is in the position of the free indices $(\lambda_1, \dots, \lambda_{n-p})$: in the direct transformation they are the last indices of the alternating tensor, whereas for the inverse transformation they are the first indices of the alternating tensor. Therefore, they can only differ by a sign. By moving $(n-p)$ indices through p remaining indices of the alternating tensor we obtain

$$*^2 \alpha = (-1)^{p(n-p)} \alpha = (-1)^{p(n+1)} \alpha . \quad (1.96)$$

In particular, for even p and any n or for odd n and any p

$$*^2 = \text{Id} . \quad (1.97)$$

1.12 Interior Product

The *interior product* with a vector \mathbf{v} is a map

$$i_{\mathbf{v}} : \Lambda_{p-1} \rightarrow \Lambda_{p-1} \quad (1.98)$$

such that for any p -form α (with $p \geq 1$) the interior product $i_{\mathbf{v}}\alpha$ is a $(p-1)$ -form defined for any vectors $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ by,

$$i_{\mathbf{v}}\alpha(\mathbf{v}_1, \dots, \mathbf{v}_{p-1}) = \alpha(\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}) , \quad (1.99)$$

and if $p = 0$, then by definition

$$i_{\mathbf{v}}\alpha = 0 . \quad (1.100)$$

In particular, if $p = 1$, then $i_{\mathbf{v}}\alpha$ is a scalar

$$i_{\mathbf{v}}\alpha = \alpha(\mathbf{v}) . \quad (1.101)$$

In components,

$$(i_{\mathbf{v}}\alpha)_{a_1\dots a_{p-1}} = v^b \alpha_{ba_1\dots a_{p-1}}. \quad (1.102)$$

A map $L : \Lambda \rightarrow \Lambda$ is called a *derivation* if for any two forms α and β

$$L(\alpha \wedge \beta) = (L\alpha) \wedge \beta + \alpha \wedge (L\beta), \quad (1.103)$$

and an *anti-derivation* if

$$L(\alpha \wedge \beta) = (L\alpha) \wedge \beta + (-1)^p \alpha \wedge (L\beta), \quad (1.104)$$

where p is the degree of the form α . Now, it is easy to see that the interior product is an anti-derivation of the exterior algebra.

By using the Hodge star one can define the interior product in a different way. Let \mathbf{v} be a vector and α the corresponding one-form defined by $\alpha(\mathbf{w}) = g(\mathbf{v}, \mathbf{w})$, that is, in components, $v^\mu = g^{\mu\nu} \alpha_\nu$. Then for any p -form β

$$*^{-1}(\alpha \wedge *\beta) = (-1)^{p-1} i_{\mathbf{v}}\beta, \quad (1.105)$$

or

$$\alpha \wedge *\beta = (-1)^{p-1} * i_{\mathbf{v}}\beta. \quad (1.106)$$

Indeed, we have

$$\begin{aligned} (*^{-1}(\alpha \wedge *\beta))_{\lambda_1\dots\lambda_{p-1}} &= \frac{1}{(n-p+1)!} (n-p+1) \frac{1}{p!} \\ &\quad E_{\lambda_1\dots\lambda_{p-1}\delta\nu_1\dots\nu_{n-p}} v^\delta E^{\gamma\mu_1\dots\mu_{p-1}\nu_1\dots\nu_{n-p}} \beta_{\gamma\mu_1\dots\mu_{p-1}} \\ &= \frac{1}{p!} \delta_{\lambda_1\dots\lambda_{p-1}\delta}^{\gamma\mu_1\dots\mu_{p-1}} v^\delta \beta_{\gamma\mu_1\dots\mu_{p-1}} \\ &= v^\delta \beta_{\lambda_1\dots\lambda_{p-1}\delta} = (-1)^{p-1} v^\delta \beta_{\delta\lambda_1\dots\lambda_{p-1}}. \end{aligned} \quad (1.107)$$

2 Tensor Analysis

2.1 Exterior Derivative

The *exterior derivative* (or gradient) is a map

$$d : \Lambda_p \rightarrow \Lambda_{p+1}, \quad (2.1)$$

which assigns to any p -form α a $(p + 1)$ -form $d\alpha$ defined for any vectors $\mathbf{v}_1, \dots, \mathbf{v}_{p+1}$ by

$$\begin{aligned} (d\alpha)(\mathbf{v}_1, \dots, \mathbf{v}_{p+1}) &= \sum_{k=1}^{p+1} (-1)^{k-1} \mathbf{v}_k(\alpha(\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_k, \dots, \mathbf{v}_{p+1})) \\ &+ \sum_{1 \leq i < k \leq p+1} (-1)^{i+k} \alpha([\mathbf{v}_i, \mathbf{v}_k], \mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \widehat{\mathbf{v}}_k, \dots, \mathbf{v}_{p+1}) \end{aligned} \quad (2.2)$$

Here, as usual a hat means that the corresponding term is omitted. One can show that the exterior derivative is an anti-derivation on the exterior algebra, that is,

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta), \quad (2.3)$$

where p is the degree of the form α .

This takes especially simple form in coordinate basis. Let α be a p -form

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \quad (2.4)$$

Then

$$d\alpha = \frac{1}{p!} \partial_{[\mu_1} \alpha_{\mu_2 \dots \mu_{p+1}]} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}. \quad (2.5)$$

Notice that the anti-symmetrization sign can be omitted here since the basis is already anti-symmetric. In components

$$\begin{aligned} (d\alpha)_{\mu_1 \mu_2 \dots \mu_{p+1}} &= (p+1) \partial_{[\mu_1} \alpha_{\mu_2 \dots \mu_{p+1}]} \\ &= \sum_{k=1}^{p+1} (-1)^{k-1} \partial_{\mu_k} \alpha_{\mu_1 \dots \widehat{\mu}_k \dots \mu_{p+1}}, \\ &= \partial_{\mu_1} \alpha_{\mu_2 \dots \mu_{p+1}} + \text{cyclic permutation of indices.} \end{aligned} \quad (2.6)$$

By using this definition it is easy to show that

$$d^2 = 0. \quad (2.7)$$

For 0-form f and a 1-form α the exterior derivative has the form

$$(df)(\mathbf{v}) = \mathbf{v}(f), \quad (2.8)$$

$$(d\alpha)(\mathbf{v}, \mathbf{w}) = \mathbf{v}(\alpha(\mathbf{w})) - \mathbf{w}(\alpha(\mathbf{v})) - \alpha([\mathbf{v}, \mathbf{w}]), \quad (2.9)$$

which in local coordinates reads

$$df = \partial_\mu f dx^\mu, \quad (2.10)$$

$$d\alpha = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu. \quad (2.11)$$

2.2 Coderivative

The *coderivative* (or divergence) is a linear map

$$\delta : \Lambda_p \rightarrow \Lambda_{p-1} \quad (2.12)$$

defined by

$$\delta = (-1)^{p+1} *^{-1} d* = (-1)^{pn+1} * d* . \quad (2.13)$$

Since $*^2 = \pm \text{Id}$ and $d^2 = 0$ we immediately have

$$\delta^2 = 0. \quad (2.14)$$

Also, we see that, for any 0-form f , $*f$ is an n -form and, therefore, $d*f = 0$, i.e. a coderivative of any 0-form is zero

$$\delta f = 0. \quad (2.15)$$

In local coordinates this takes the form

$$\begin{aligned} (\delta\alpha)_{\lambda_1 \dots \lambda_{p-1}} &= (-1)^{p+1} \frac{1}{(n-p+1)!} E_{\lambda_1 \dots \lambda_{p-1}}^{\rho\mu_1 \dots \mu_{n-p}} \\ &\quad \times (n-p+1) \partial_\rho \left(\frac{1}{p!} E^{\beta_1 \dots \beta_{p-1}\sigma}_{\mu_1 \dots \mu_{n-p}} \alpha_{\beta_1 \dots \beta_{p-1}\sigma} \right) \\ &= (-1)^{p+1} \frac{1}{p!(n-p)!} \frac{1}{\sqrt{|g|}} g_{\lambda_1 \kappa_1} \dots g_{\lambda_{p-1} \kappa_{p-1}} \varepsilon^{\kappa_1 \dots \kappa_{p-1} \gamma \nu_1 \dots \nu_{n-p}} \\ &\quad \times \partial_\gamma \left(\varepsilon_{\mu_1 \dots \mu_{p-1} \rho \nu_1 \dots \nu_{n-p}} \sqrt{|g|} g^{\mu_1 \beta_1} \dots g^{\mu_{p-1} \beta_{p-1}} g^{\rho\sigma} \alpha_{\beta_1 \dots \beta_{p-1}\sigma} \right) \\ &= (-1)^{p+1} \frac{1}{p!(n-p)!} \varepsilon^{\kappa_1 \dots \kappa_{p-1} \gamma \nu_1 \dots \nu_{n-p}} \varepsilon_{\mu_1 \dots \mu_{p-1} \rho \nu_1 \dots \nu_{n-p}} \\ &\quad \times \frac{1}{\sqrt{|g|}} g_{\lambda_1 \kappa_1} \dots g_{\lambda_{p-1} \kappa_{p-1}} \partial_\gamma \left(\sqrt{|g|} g^{\mu_1 \beta_1} \dots g^{\mu_{p-1} \beta_{p-1}} g^{\rho\sigma} \alpha_{\beta_1 \dots \beta_{p-1}\sigma} \right). \end{aligned} \quad (2.16)$$

Finally, by using the formula for the product of alternating tensors we obtain

$$\begin{aligned}
(\delta\alpha)_{\lambda_1\dots\lambda_{p-1}} &= (-1)^{p+1} g_{\lambda_1\mu_1} \cdots g_{\lambda_{p-1}\mu_{p-1}} \\
&\quad \times \frac{1}{\sqrt{|g|}} \partial_\rho \left(\sqrt{|g|} g^{\mu_1\beta_1} \cdots g^{\mu_{p-1}\beta_{p-1}} g^{\rho\sigma} \alpha_{\beta_1\dots\beta_{p-1}\sigma} \right) \\
&= g_{\lambda_1\mu_1} \cdots g_{\lambda_{p-1}\mu_{p-1}} \frac{1}{\sqrt{|g|}} \partial_{\mu_p} \left(\sqrt{|g|} g^{\mu_p\beta_p} g^{\mu_1\beta_1} \cdots g^{\mu_{p-1}\beta_{p-1}} \alpha_{\beta_p\beta_1\beta_2\dots\beta_{p-1}} \right)
\end{aligned} \tag{2.17}$$

For a 1-form this reads

$$\delta\alpha = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \alpha_\nu \right). \tag{2.18}$$

By using these formulas one can show that δ is the negative adjoint of the exterior derivative d with respect to the Riemannian volume element in the sense that for any p -form α and a $(p+1)$ -form β

$$(\beta, d\alpha)_{L^2} = -(\delta\beta, \alpha)_{L^2}. \tag{2.19}$$

2.3 Hodge Laplacian

Let $C^\infty(\Lambda_p)$ be the space of smooth p -forms. The Hodge-Laplacian is an operator

$$\Delta_{\text{Hodge}} : C^\infty(\Lambda_p) \rightarrow C^\infty(\Lambda_p), \tag{2.20}$$

defined by

$$\Delta_{\text{Hodge}} = (d + \delta)^2 = d\delta + \delta d. \tag{2.21}$$

This operator is obviously self-adjoint. Moreover, since $\delta = (-1)^{p+1} *^{-1} d*$ it commutes with the star operator,

$$*\Delta_{\text{Hodge}} = \Delta_{\text{Hodge}} * . \tag{2.22}$$

It is a second order partial differential operator. We can now obtain the expression in local coordinates for it. Let ω be a p -form. Then

$$\begin{aligned}
&(d\delta\omega)_{\alpha_1\dots\alpha_p} \\
&= p \partial_{[\alpha_1} \left[g_{\alpha_2|\mu_2|} \cdots g_{\alpha_p|\mu_p|} \frac{1}{\sqrt{|g|}} \partial_{\mu_1} \left(\sqrt{|g|} g^{\mu_1\beta_1} g^{\mu_2\beta_2} \cdots g^{\mu_p\beta_p} \omega_{\beta_1\beta_2\dots\beta_p} \right) \right],
\end{aligned} \tag{2.23}$$

where the last anti-symmetrization goes over all α_i and the indices μ_i are excluded from it. Next,

$$\begin{aligned} (\delta d\omega)_{\alpha_1 \dots \alpha_p} &= (p+1)g_{\alpha_1 \mu_1} \cdots g_{\alpha_p \mu_p} \\ &\times \frac{1}{\sqrt{|g|}} \partial_{\mu_{p+1}} \left(\sqrt{|g|} g^{\mu_{p+1} \beta_{p+1}} g^{\mu_1 \beta_1} \cdots g^{\mu_p \beta_p} \partial_{[\beta_{p+1}} \omega_{\beta_1 \dots \beta_p]} \right). \end{aligned} \quad (2.24)$$

Thus the Hodge Laplacian on p -forms has the form

$$\begin{aligned} &(\Delta_{\text{Hodge}} \omega)_{\alpha_1 \dots \alpha_p} \\ &= \left\{ p \partial_{[\alpha_1} g_{\alpha_2 |\mu_2|} \cdots g_{\alpha_p |\mu_p|]} \frac{1}{\sqrt{|g|}} \partial_{\mu_1} \sqrt{|g|} g^{\mu_1 \beta_1} g^{\mu_2 \beta_2} \cdots g^{\mu_p \beta_p} \right. \\ &\quad \left. + (p+1) g_{\alpha_1 \mu_1} \cdots g_{\alpha_p \mu_p} \frac{1}{\sqrt{|g|}} \partial_{\mu_{p+1}} \sqrt{|g|} g^{[\mu_{p+1} | \beta_{p+1}} g^{|\mu_1| \beta_1} \cdots g^{|\mu_p| \beta_p]} \partial_{\beta_{p+1}} \right\} \omega_{\beta_1 \dots \beta_p}, \end{aligned} \quad (2.25)$$

where there is an anti-symmetrization over all β_i in the second term and the indices μ_j are excluded from anti-symmetrization.

On scalars, for $p = 0$ this takes the familiar form

$$\Delta_{\text{Hodge}} = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu. \quad (2.26)$$

On one-forms, for $p = 1$ this reads

$$(\Delta_{\text{Hodge}} \omega)_\alpha = \left\{ \partial_\alpha \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\beta} + 2g_{\alpha\mu} \frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} g^{\nu[\gamma} g^{\beta]\mu} \partial_\gamma \right\} \omega_\beta. \quad (2.27)$$

We will describe below how the Hodge Laplacian is related to the Bochner Laplacian defined with the help of the Levi-Civita connection.

2.4 Lie Derivative

Let \mathbf{X} be a vector field. Let $C^\infty(T_q^p)$ be the set of smooth sections of a tensor bundle of type (p, q) . The Lie derivative with respect to X is a map

$$\mathcal{L}_X : C^\infty(T_q^p) \rightarrow C^\infty(T_q^p) \quad (2.28)$$

defines as follows.

The Lie derivative of a function f is equal to the directional derivative of f along the vector field, that is,

$$\mathcal{L}f = X(f). \quad (2.29)$$

The Lie derivative of a vector field Y is equal to the commutator of vector fields

$$\mathcal{L}_X Y = [X, Y]. \quad (2.30)$$

The Lie derivative of α with respect to X is a 1-form $L_X \alpha$ defined for any vector Y by

$$(\mathcal{L}_X \alpha)(Y) = X(\alpha(Y)) - \alpha([X, Y]), \quad (2.31)$$

so that the Leibnitz rule holds

$$\mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X \alpha)(Y) + \alpha(\mathcal{L}_X Y) \quad (2.32)$$

Similarly, for any p -form α the Lie derivative is defined by

$$\begin{aligned} (\mathcal{L}_X \alpha)(Y_1, \dots, Y_p) &= \mathcal{L}_X (\alpha(Y_1, \dots, Y_p)) \\ &\quad - \sum_{i=1}^p \alpha(Y_1, \dots, [X, Y_i], \dots, Y_p). \end{aligned} \quad (2.33)$$

Then one can show that Lie derivative $L_X \Lambda_p \rightarrow \Lambda_p$ is a derivation of the exterior algebra and Leibnitz rule holds

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta). \quad (2.34)$$

The Lie derivative of any tensors is defined similarly, so that Leibnitz rule is satisfied, in general,

$$\mathcal{L}_X(T \otimes R) = (\mathcal{L}_X T) \otimes R + T \otimes (\mathcal{L}_X R). \quad (2.35)$$

We list now the expressions for Lie derivative in local coordinates. For a tensor of type (p, q) we obtain

$$\begin{aligned} (\mathcal{L}_X T)_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} &= X^\alpha \partial_\alpha T_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} + \sum_{k=1}^q T_{\nu_1 \dots \nu_{k-1} \alpha \nu_{k+1} \dots \nu_q}^{\mu_1 \dots \mu_p} \partial_{\nu_k} X^\alpha \\ &\quad - \sum_{k=1}^p T_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_{k-1} \alpha \mu_{k+1} \dots \mu_p} \partial_\alpha X^{\mu_k}. \end{aligned} \quad (2.36)$$

For a p -form β this reads

$$(\mathcal{L}_{\mathbf{X}}\beta)_{\nu_1\dots\nu_p} = X^\alpha \partial_\alpha \beta_{\nu_1\dots\nu_p} + \beta_{\alpha\nu_2\dots\nu_p} \partial_{i_1} X^j + \dots + \beta_{\nu_1\dots\nu_{p-1}\alpha} \partial_{\nu_p} X^\alpha. \quad (2.37)$$

In particular, for a 1-form β we have

$$(\mathcal{L}_{\mathbf{X}}\beta)_\nu = X^\alpha \partial_\alpha \beta_\nu + \beta_\alpha \partial_\nu X^\alpha. \quad (2.38)$$

Another important formula is the expression for the Lie derivative of a tensor g of type $(0, 2)$ (not necessarily antisymmetric)

$$(\mathcal{L}_{\mathbf{X}}g)_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu X^\alpha + g_{\alpha\nu} \partial_\mu X^\alpha. \quad (2.39)$$

By using the definition of the Lie derivative one can prove the following properties

$$\mathcal{L}_{\mathbf{X}+\mathbf{Y}} = \mathcal{L}_{\mathbf{X}} + \mathcal{L}_{\mathbf{Y}}, \quad (2.40)$$

$$\mathcal{L}_{c\mathbf{X}} = c\mathcal{L}_{\mathbf{X}}, \quad (2.41)$$

$$\mathcal{L}_{\mathbf{X}}\mathbf{Y} = -\mathcal{L}_{\mathbf{Y}}\mathbf{X}, \quad (2.42)$$

$$[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] = \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]}. \quad (2.43)$$

The Lie derivative on forms has, in addition, the following properties

$$\mathcal{L}_{\mathbf{X}}d = d\mathcal{L}_{\mathbf{X}}, \quad (2.44)$$

$$\mathcal{L}_{\mathbf{X}} = i_{\mathbf{X}}d + di_{\mathbf{X}}, \quad (2.45)$$

$$[\mathcal{L}_{\mathbf{X}}, i_{\mathbf{Y}}] = i_{[\mathbf{X}, \mathbf{Y}]}. \quad (2.46)$$

Let $d\text{vol}$ be the Riemannian volume form, let α be a one-form and \mathbf{v} be the corresponding vector, such that $v^\mu = g^{\mu\nu}\alpha_\nu$. Then the scalar

$$\text{div } \mathbf{v} = \delta\alpha = *^{-1}d*\alpha = *\mathcal{L}_{\mathbf{v}}d\text{vol} = *d(i_{\mathbf{v}}d\text{vol}) \quad (2.47)$$

is called the *divergence* of the vector field \mathbf{v} .

2.5 Affine Connection, Torsion and Curvature

Let e_a be a basis of vector fields. Then the *commutation coefficients* $\theta^a{}_{bc}$ are defined by

$$[e_a, e_b] = \theta^c{}_{ab}e_c. \quad (2.48)$$

The commutation coefficients are

$$\theta^a{}_{bc} = \sigma^a{}_\nu (e_b^\mu \partial_\mu e_c^\nu - e_c^\mu \partial_\mu e_b^\nu) . \quad (2.49)$$

An affine connection ∇ is defined by

$$\nabla_{e_a} e_b = \omega^c{}_{ba} e_c . \quad (2.50)$$

where $\omega^c{}_{ba}$ are the *coefficients of the affine connection*. Let σ^a be the dual basis of one-forms. Then the coefficients of the affine connection are given by

$$\omega^c{}_{ba} = \sigma^c(\nabla_{e_a} e_b) . \quad (2.51)$$

The *torsion* of the affine connection is a tensor of type (1, 2) defined for any one-form α and two vector fields X and Y by

$$T(\alpha, X, Y) = \alpha(\nabla_X Y - \nabla_Y X - [X, Y]) . \quad (2.52)$$

The components of the torsion and the curvature tensors are

$$T^a{}_{bc} = \omega^a{}_{cb} - \omega^a{}_{bc} - \theta^a{}_{bc} . \quad (2.53)$$

The connection is *torsion-free* (or symmetric) if for any two vector fields X and Y

$$\nabla_X Y - \nabla_Y X = [X, Y] , \quad (2.54)$$

that is,

$$\omega^a{}_{cb} - \omega^a{}_{bc} = -\theta^a{}_{cb} . \quad (2.55)$$

The *curvature* of the affine connection is a tensor of type (1, 3) defined for any one-form α and three vector fields X , Y and Z by

$$\text{Riem}(\alpha, Z, X, Y) = \alpha \left(([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) Z \right) . \quad (2.56)$$

The components of the curvature tensor are

$$R^a{}_{bcd} = e_c(\omega^a{}_{bd}) - e_d(\omega^a{}_{bc}) + \omega^a{}_{fc} \omega^f{}_{bd} - \omega^a{}_{fd} \omega^f{}_{bc} - \omega^a{}_{bf} \theta^f{}_{dc} . \quad (2.57)$$

The connection is *flat* if for any three vector fields X , Y and Z

$$[\nabla_X, \nabla_Y] Z = \nabla_{[X, Y]} Z . \quad (2.58)$$

The curvature is *parallel* if for any vector fields X, Y, Z, V and a one-form α ,

$$\begin{aligned} V[\text{Riem}(\alpha, Z, X, Y)] &= \text{Riem}(\alpha, \nabla_V Z, X, Y) + \text{Riem}(\alpha, Z, \nabla_V X, Y) \\ &\quad + \text{Riem}(\alpha, Z, X, \nabla_V Y) + \text{Riem}(\nabla_V \alpha, Z, X, Y). \end{aligned} \quad (2.59)$$

Let g be a Riemannian metric. The affine connection ∇ is said to be *compatible with the metric g* if for any three vector fields X, Y and Z ,

$$Z[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \quad (2.60)$$

2.6 Levi-Civita Connection

On any Riemannian manifold there is a unique connection, called the *Levi-Civita connection*, which is torsion-free and compatible with the metric. One can show that the coefficients of the Levi-Civita connection are

$$\begin{aligned} \omega^a{}_{bc} &= \frac{1}{2} \gamma^{ad} [e_b(\gamma_{cd}) + e_c(\gamma_{bd}) - e_d(\gamma_{bc})] \\ &\quad + \frac{1}{2} (\gamma^{af} \gamma_{ce} \theta^e{}_{fb} + \gamma^{af} \gamma_{be} \theta^e{}_{fc} - \theta^a{}_{bc}), \end{aligned} \quad (2.61)$$

where

$$\gamma_{ab} = g(e_a, e_b) \quad (2.62)$$

are the components of the metric tensor and $(\gamma^{ab}) = (\gamma_{ab})^{-1}$ is the inverse matrix.

In coordinate basis the Levi-Civita connection is described by the Christoffel coefficients

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}). \quad (2.63)$$

The relation of Christoffel symbols with the coefficients of the affine connection is

$$\Gamma^\alpha{}_{\beta\gamma} = \omega^a{}_{bc} e^\alpha{}_a \sigma^b{}_\beta \sigma^c{}_\gamma - \sigma^b{}_\beta \partial_\gamma e^\alpha{}_b. \quad (2.64)$$

In an orthonormal basis, when $\gamma_{ab} = \delta_{ab}$, the Levi-Civita connection is described by

$$\omega_{abc} = \frac{1}{2} (\theta_{cab} + \theta_{bac} - \theta_{abc}). \quad (2.65)$$

This also be written in the form

$$\omega^{ab}{}_c = e^\mu{}_c e^{a\nu} \partial_{[\mu} \sigma^b{}_{\nu]} - e^\mu{}_c e^{b\nu} \partial_{[\mu} \sigma^a{}_{\nu]} + e^{a\nu} e^{b\lambda} \partial_{[\lambda} \sigma^c{}_{\nu]} \quad (2.66)$$

Notice that the Christoffel coefficients are symmetric in lower indices

$$\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}, \quad (2.67)$$

and the connection in orthonormal basis is anti-symmetric in the first two indices

$$\omega_{abc} = -\omega_{bac}. \quad (2.68)$$

2.7 Covariant Derivatives

Let \mathbf{X} be a vector field and T_q^p be the bundle of tensors of type (p, q) . The *covariant derivative* of a tensor field T of type (p, q) is a linear operator

$$\nabla : C^\infty(T_q^p M) \rightarrow C^\infty(T_{q+1}^p M) \quad (2.69)$$

that assigns to a tensor field T of type (p, q) a new tensor field ∇T of type $(p, q + 1)$.

First of all, the covariant derivative of a 1-form α on a manifold M is a tensor $\nabla\alpha$ of type $(0, 2)$ such that for any two vector fields \mathbf{X} and \mathbf{Y}

$$(\nabla\alpha)(\mathbf{X}, \mathbf{Y}) = (\nabla_{\mathbf{X}}\alpha)(\mathbf{Y}) = \mathbf{X}[\alpha(\mathbf{Y})] - \alpha(\nabla_{\mathbf{X}}\mathbf{Y}). \quad (2.70)$$

Then the covariant derivative of a tensor T of type (p, q) is a tensor ∇T of type $(p, q + 1)$ such that for any vector fields $\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q$ and 1-forms ω_1, ω_p

$$\begin{aligned} (\nabla T)(\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p) &= (\nabla_{\mathbf{X}}T)(\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p) \\ &= \mathbf{X}[T(\mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p)] \\ &\quad - \sum_{i=1}^q T(\mathbf{Y}_1, \dots, \nabla_{\mathbf{X}}\mathbf{Y}_i, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p) \\ &\quad - \sum_{j=1}^p T(\mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \nabla_{\mathbf{X}}\omega_j, \dots, \omega_p). \end{aligned}$$

Let \mathbf{e}_i be a basis of vector fields and σ^i be the dual basis of 1-forms. The covariant derivative of a 1-form has the form

$$(\nabla_i\alpha)_k = \mathbf{e}_i(\alpha_k) - \omega^l{}_{ki}\alpha_l \quad (2.71)$$

In the coordinate frame this simplifies to

$$(\nabla_i \alpha)_k = \partial_i \alpha_k - \omega^l_{ki} \alpha_l. \quad (2.72)$$

The covariant derivative of a tensor of type (p, q) in a coordinate basis has the form

$$\nabla_i T_{k_1 \dots k_q}^{j_1 \dots j_p} = \partial_i T_{k_1 \dots k_q}^{j_1 \dots j_p} + \sum_{m=1}^p \omega^{j_m}_{li} T_{k_1 \dots k_q}^{j_1 \dots j_{m-1} l j_{m+1} \dots j_p} - \sum_{n=1}^q \omega^l_{k_n i} T_{k_1 \dots k_{n-1} l k_{n+1} \dots k_q}^{j_1 \dots j_p} \quad (2.73)$$

Let T be a tensor field on M . We say that T is *parallel* along \mathbf{X}

$$\nabla_{\mathbf{X}} T = 0. \quad (2.74)$$

Let $C^\infty(T_q^p)$ be the space of smooth (p, q) tensors. The Bochner Laplacian on such tensors is a second-order partial differential operator

$$\Delta_{\text{Bochner}} : C^\infty(T_q^p) \rightarrow C^\infty(T_q^p), \quad (2.75)$$

defined by

$$\Delta_{\text{Bochner}} = g^{\mu\nu} \nabla_\mu \nabla_\nu. \quad (2.76)$$

Obviously, the metric, and, therefore, the volume form is parallel, which means that the Hodge star operator $*$ commutes with the covariant derivative and with the Bochner Laplacian, in particular; therefore,

$$\Delta_{\text{Bochner}} * = * \Delta_{\text{Bochner}}. \quad (2.77)$$

Also, one can show that

2.8 Ricci Identities

The commutators of covariant derivatives of tensors are expressed in terms of the curvature and the torsion. In a coordinate basis for a torsion-free connection we have the following identities (called the Ricci identities):

$$[\nabla_i, \nabla_j] Y^k = R^k_{lij} Y^l \quad (2.78)$$

$$[\nabla_i, \nabla_j] \alpha_k = -R^l_{kij} \alpha_l \quad (2.79)$$

$$[\nabla_i, \nabla_j] T_{k_1 \dots k_q}^{j_1 \dots j_p} = \sum_{m=1}^p R^{j_m}_{lij} T_{k_1 \dots k_q}^{j_1 \dots j_{m-1} l j_{m+1} \dots j_p} - \sum_{n=1}^q R^l_{k_n ij} T_{k_1 \dots k_{n-1} l k_{n+1} \dots k_q}^{j_1 \dots j_p}. \quad (2.80)$$

2.9 Normal Coordinates

Let (M, g) be a Riemannian manifold and Γ_{jk}^i be the Christoffel coefficients defining the Levi-Civita connection in a coordinate basis. Let x_0 be a point in M and x^i be a local coordinate system in a coordinate patch about x_0 . We expand the metric in a Taylor series at the point x_0

$$g_{ij}(x) = g_{ij}(x_0) + [\partial_k g_{ij}](x_0)(x^k - x_0^k) + \frac{1}{2}[\partial_k \partial_l g_{ij}](x_0)(x^k - x_0^k)(x^l - x_0^l) + O((x - x_0)^3) \quad (2.81)$$

The matrix $g_{ij}(x_0)$ is a constant real symmetric matrix with real eigenvalues. There exists a coordinate system such that

$$g_{ij}(x_0) = \delta_{ij}, \quad [\partial_k g_{ij}](x_0) = 0, \quad (2.82)$$

so that the Taylor series has the form

$$g_{ij}(x) = \delta_{ij} + \frac{1}{2}[\partial_k \partial_l g_{ij}](x_0)(x^k - x_0^k)(x^l - x_0^l) + O((x - x_0)^3). \quad (2.83)$$

Such coordinates are called *Riemann normal coordinates*.

In Riemann normal coordinates the Christoffel symbols vanish at x_0

$$\Gamma_{jk}^i(x_0) = 0 \quad (2.84)$$

and the curvature of the Levi-Civita connection at x_0 is expressed in terms of second derivatives of the metric at x_0 ,

$$R_{ijkl}(x_0) = \frac{1}{2} \left\{ \partial_k \partial_j g_{il} - \partial_l \partial_j g_{ik} - \partial_i \partial_l g_{kj} + \partial_k \partial_i g_{jl} \right\} \Big|_{x_0}, \quad (2.85)$$

so that the Taylor series of the metric at x_0 has the form

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(x_0)(x^k - x_0^k)(x^l - x_0^l) + O((x - x_0)^3). \quad (2.86)$$

2.10 Properties of the Curvature Tensor

Let (M, g) be an n -dimensional Riemannian manifold. We will restrict ourselves to the Levi-Civita connection below. We define some new curvature tensors. The *Ricci tensor*

$$R_{ij} = R^k_{ikj}. \quad (2.87)$$

The *scalar curvature*

$$R = g^{ij} R_{ij} = g^{ij} R^k{}_{ikj}. \quad (2.88)$$

The *Einstein tensor*

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R. \quad (2.89)$$

The *trace-free Ricci tensor*

$$E_{ij} = R_{ij} - \frac{1}{n} g_{ij} R. \quad (2.90)$$

The *Weyl tensor* (for $n > 2$)

$$\begin{aligned} C^{ij}{}_{kl} &= R^{ij}{}_{kl} - \frac{4}{n-2} R^{[i}{}_{[k} \delta^{j]}{}_{l]} + \frac{2}{(n-1)(n-2)} R \delta^{[i}{}_{[k} \delta^{j]}{}_{l]} \\ &= R^{ij}{}_{kl} - \frac{4}{n-2} E^{[i}{}_{[k} \delta^{j]}{}_{l]} - \frac{2}{n(n-1)} R \delta^{[i}{}_{[k} \delta^{j]}{}_{l]} \end{aligned} \quad (2.91)$$

The Riemann curvature tensor of the Levi-Civita connection has the following symmetry properties

$$R_{ijkl} = -R_{ijlk} \quad (2.92)$$

$$R_{ijkl} = -R_{jikl} \quad (2.93)$$

$$R_{ijkl} = R_{klij} \quad (2.94)$$

$$R^i{}_{[jkl]} = R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0 \quad (2.95)$$

$$R_{ij} = R_{ji} \quad (2.96)$$

In particular this gives a useful identity

$$R^i{}_{[k}{}^j{}_{l]} = R^{[i}{}_{[k}{}^j{}_{l]} = \frac{1}{2} R^{ij}{}_{kl}. \quad (2.97)$$

The Weyl tensor has the same symmetry properties as the Riemann tensor and all its contractions vanish, that is,

$$C^i{}_{jik} = 0. \quad (2.98)$$

The number of algebraically independent components of the Riemann tensor of the Levi-Civita connection is equal to

$$\frac{n^2(n^2-1)}{12}. \quad (2.99)$$

In dimension $n = 2$ the Riemann tensor has only one independent component determined by the scalar curvature

$$R^{12}{}_{12} = \frac{1}{2}R. \quad (2.100)$$

The trace-free Ricci tensor E_{ij} vanishes, that is

$$\begin{aligned} R^{ij}{}_{kl} &= R\delta^{[i}{}_{[k}\delta^{j]}{}_{l]} \\ R_{ij} &= \frac{1}{2}Rg_{ij}. \end{aligned} \quad (2.101)$$

In dimension $n = 3$ the Riemann tensor has six independent components determined by the Ricci tensor R_{ij} . The Weyl tensor C_{ijkl} vanishes, that is,

$$R^{ij}{}_{kl} = 4R^{[i}{}_{[k}\delta^{j]}{}_{l]} + R\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}. \quad (2.102)$$

2.11 Bianchi Identities

Let (M, g) be an n -dimensional Riemannian manifold. We will restrict ourselves to the Levi-Civita connection below. The Riemann tensor satisfies the following identities

$$\nabla_{[m}R^{ij}{}_{kl]} = \nabla_m R^{ij}{}_{kl} + \nabla_k R^{ij}{}_{lm} + \nabla_l R^{ij}{}_{mk} = 0. \quad (2.103)$$

These identities are called the *Bianchi identities*. They have many corollaries: the divergences of the Riemann tensor and the Ricci tensor have the form

$$\nabla_i R^{ij}{}_{kl} = \nabla_k R_l^j - \nabla_l R_k^j, \quad (2.104)$$

$$\nabla_i R_j^i = \frac{1}{2}\nabla_j R. \quad (2.105)$$

The divergence of the Einstein tensor vanishes

$$\nabla_i G_j^i = 0. \quad (2.106)$$

By using the Bianchi identities one can simplify the Laplacian of the Riemann tensor, $\Delta R^{ij}{}_{kl} = \nabla^m \nabla_m R^{ij}{}_{kl}$ in terms of the derivatives of the Ricci tensor and the scalar curvature.

2.12 Cartan's Structural Equations

Let us fix an orthonormal frame σ^i . One can show that

$$d\sigma^i = -\frac{1}{2}\theta^i_{jk}\sigma^j \wedge \sigma^k. \quad (2.107)$$

Recall that the Levi-Civita connection in an orthonormal frame is

$$\omega_{ijk} = \frac{1}{2}(\theta_{kij} + \theta_{jik} - \theta_{ijk}). \quad (2.108)$$

Now we define the *connection 1-forms*

$$\mathcal{A}^i_j = \omega^i_{jk}\sigma^k \quad (2.109)$$

and the *curvature 2-forms*

$$\mathcal{F}^i_j = \frac{1}{2}R^i_{jkl}\sigma^k \wedge \sigma^l. \quad (2.110)$$

Then the equation

$$d\sigma^i = \omega^i_{jk}\sigma^j \wedge \sigma^k \quad (2.111)$$

can be written as

$$d\sigma^i + \mathcal{A}^i_j \wedge \sigma^j = 0. \quad (2.112)$$

This is called *Cartan's first structural equation*.

The curvature 2-forms are obtained from the connection 1-forms by *Cartan's second structural equation*

$$\mathcal{F}^i_j = d\mathcal{A}^i_j + \mathcal{A}^i_k \wedge \mathcal{A}^k_j. \quad (2.113)$$

This equation is equivalent to the expression for the curvature components and can be obtained from that.

Finally, *Cartan's third structural equation*

$$d\mathcal{F}^i_j + \mathcal{A}^i_k \wedge \mathcal{F}^k_j - \mathcal{F}^i_k \wedge \mathcal{A}^k_j = 0. \quad (2.114)$$

is equivalent to Bianchi identities.

Cartan structural equations can be written in a very compact way by introducing the *covariant exterior derivative* acting on vector valued and matrix valued forms. Let α be a p -form valued in a vector space V (in our case $V = \mathbb{R}^n$). Such a form is called a *twisted form*. Let α^i be the components

of this form in a fixed basis in V . That is, α^i is a p -form for each $i = 1, \dots, n$. Then the covariant exterior derivative

$$\mathcal{D} : \Lambda_p \otimes V \rightarrow \Lambda_{p+1} \otimes V$$

is defined by

$$(\mathcal{D}\alpha)^i = d\alpha^i + \mathcal{A}^i_j \wedge \alpha^j$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha + \mathcal{A} \wedge \alpha$$

with obvious notation.

Let V^* be the dual vector space to V (in our case it is again \mathbb{R}^n). We consider p -forms valued in V^* (covectors) and naturally extend the operator \mathcal{D} to such forms

$$\mathcal{D} : \Lambda_p \otimes V^* \rightarrow \Lambda_{p+1} \otimes V^*$$

by

$$(\mathcal{D}\alpha)_i = d\alpha_i - (-1)^p \alpha_j \wedge \mathcal{A}^j_i$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha - (-1)^p \alpha \wedge \mathcal{A}.$$

Finally, we consider matrix-valued p -forms valued in $V \otimes V^*$ and extend the operator \mathcal{D} to such forms

$$\mathcal{D} : \Lambda_p \otimes V \otimes V^* \rightarrow \Lambda_{p+1} \otimes V \otimes V^*$$

by

$$(\mathcal{D}\alpha)^i_j = d\alpha^i_j + \mathcal{A}^i_k \wedge \alpha^k_j - (-1)^p \alpha^i_k \wedge \mathcal{A}^k_j$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha + \mathcal{A} \wedge \alpha - (-1)^p \alpha \wedge \mathcal{A}.$$

Now, let $\sigma = (\sigma^i)$, $\mathcal{F} = (\mathcal{F}^i_j)$ and $\alpha = (\alpha^i)$ be an arbitrary vector-valued 1-form. Then

$$\begin{aligned} \mathcal{D}\sigma &= 0 \\ \mathcal{D}^2\alpha &= \mathcal{F} \wedge \alpha \\ \mathcal{D}\mathcal{F} &= 0. \end{aligned}$$

2.13 Connection on Vector Bundles

Now, let \mathcal{V} be a vector bundle over a manifold M with a structure group G and $C^\infty(\mathcal{V})$ be the space of its smooth sections. Let the sections of the vector bundle \mathcal{V} be given locally by a column-vector $\varphi = (\varphi^A)$, $A = 1, \dots, N$. A connection on the vector bundle \mathcal{V} is an first-order partial differential operator

$$\nabla : C^\infty(\mathcal{V}) \rightarrow C^\infty(TM \otimes \mathcal{V}), \quad (2.115)$$

given by

$$\nabla\varphi = d\varphi + \mathcal{A} \otimes \varphi, \quad (2.116)$$

where $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ is the connection one-form valued in the Lie algebra of the structure group. In local coordinates this takes the form

$$(\nabla\varphi)_\mu = (\mathbb{I}\partial_\mu + \mathcal{A}_\mu)\varphi, \quad (2.117)$$

where \mathbb{I} is the unit matrix and the matrix \mathcal{A}_μ acts on the column-vector φ on the left; we omit \mathbb{I} below for simplicity.

This enables one to define the generalized Laplacian on the vector bundle. It has the form

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu, \quad (2.118)$$

which in local coordinates takes the form

$$\Delta\varphi = \frac{1}{\sqrt{|g|}}(\partial_\mu + \mathcal{A}_\mu)\sqrt{|g|}g^{\mu\nu}(\partial_\nu + \mathcal{A}_\nu)\varphi. \quad (2.119)$$

2.14 Weitzenböck Endomorphism

Since the Levi-Civita connection is compatible with the metric and it is torsion free it allows to rewrite all formulas for the exterior derivative, co-derivative, and Lie derivative in terms of covariant derivatives, which simplifies them greatly. In particular, the exterior derivative and the co-derivative of p -forms are

$$(d\alpha)_{\mu_1\mu_2\dots\mu_{p+1}} = (p+1)\nabla_{[\mu_1}\alpha_{\mu_2\dots\mu_{p+1}]}, \quad (2.120)$$

$$(\delta\alpha)_{\lambda_1\dots\lambda_{p-1}} = \nabla^\mu\alpha_{\mu\lambda_1\dots\lambda_{p-1}}. \quad (2.121)$$

Therefore, the Hodge Laplacian on p -forms has the form

$$(\Delta_{\text{Hodge}}\omega)_{\alpha_1\dots\alpha_p} = p\nabla_{[\alpha_1}\nabla^{\beta}\omega_{\beta|\alpha_2\dots\alpha_p]} + (p+1)\nabla^\beta\nabla_{[\beta}\omega_{\alpha_1\dots\alpha_p]}. \quad (2.122)$$

This can be easily simplified to

$$(\Delta_{\text{Hodge}}\omega)_{\alpha_1\dots\alpha_p} = \Delta_{\text{Bochner}}\omega_{\alpha_1\dots\alpha_p} + p[\nabla_{[\alpha_1}, \nabla^{\beta]}]\omega_{|\beta|\alpha_2\dots\alpha_p]} . \quad (2.123)$$

Now by commuting covariant derivatives and using the Ricci identities one can express Hodge Laplacians on p -forms are related to the Bochner Laplacians defined above by a formula

$$\Delta_{\text{Hodge}} = \Delta_{\text{Bochner}} - W , \quad (2.124)$$

where W is an endomorphism of the bundle Λ_p linear in the curvature called *Weitzenböck endomorphisms*. We obtain

$$(W\omega)_{\alpha_1\dots\alpha_p} = pR^{\beta}_{[\alpha_1}\omega_{|\beta|\alpha_2\dots\alpha_p]} + p\sum_{k=2}^p R^{\mu}_{[\alpha_k\alpha_1}\omega_{|\beta|\alpha_2\dots\alpha_{k-1}|\mu|\alpha_{k+1}\dots\alpha_p]} , \quad (2.125)$$

where the indices μ and β are excluded from anti-symmetrization. Now, permuting the indices α_i and using the property of the Riemann tensor

$$R^{\mu}_{[\rho}\omega^{\nu]}_{\sigma]} = \frac{1}{2}R^{\mu\nu}_{\rho\sigma} , \quad (2.126)$$

we get

$$(W\omega)_{\alpha_1\dots\alpha_p} = pR^{\nu}_{[\alpha_1}\omega_{|\nu|\alpha_2\dots\alpha_p]} - \frac{p(p-1)}{2}R^{\mu\nu}_{[\alpha_1\alpha_2}\omega_{|\mu\nu|\alpha_3\dots\alpha_p]} . \quad (2.127)$$

Of course, for $p = 0$ the Weitzenböck endomorphism is equal to zero. Also, the second term linear in Riemann curvature is present only for $p \geq 2$. Let us list particular formulas for one-forms

$$(W\omega)_{\alpha} = R^{\beta}_{\alpha}\omega_{\beta} . \quad (2.128)$$

and for two-forms,

$$(W\omega)_{\alpha\gamma} = R^{\beta}_{\alpha}\omega_{\beta\gamma} - R^{\beta}_{\gamma}\omega_{\beta\alpha} - R^{\mu\beta}_{\alpha\gamma}\omega_{\mu\beta} . \quad (2.129)$$

This defines the bilinear form on the space of p -forms, Λ_p . For $p = 1$

$$\langle\beta, W\omega\rangle = \text{Ric}(\beta, \omega) , \quad (2.130)$$

For $p \geq 2$

$$\begin{aligned}\langle \beta, W\omega \rangle &= \frac{1}{p!} \beta^{\alpha_1 \dots \alpha_p} (W\omega)_{\alpha_1 \dots \alpha_p} \\ &= \frac{1}{p!} \beta^{\rho\sigma\alpha_3 \dots \alpha_p} W_{\rho\sigma}{}^{\mu\nu} \omega_{\mu\nu\alpha_3 \dots \alpha_p},\end{aligned}\quad (2.131)$$

where

$$W^{\mu\nu}{}_{\rho\sigma} = pR^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \frac{p(p-1)}{2} R^{\mu\nu}{}_{\rho\sigma}.\quad (2.132)$$

We will need the traces

$$W^{\mu\nu}{}_{\rho\nu} = \frac{p}{4} [R\delta^\mu{}_\rho + (n-2p)R^\mu{}_\rho]\quad (2.133)$$

and

$$W^{\mu\nu}{}_{\mu\nu} = \frac{p(n-p)}{2} R.\quad (2.134)$$

Therefore, for an n -form $\omega = *f$, where f is a function, we have

$$\langle \omega, W\omega \rangle = \frac{2}{n(n-1)} f^2 W^{\mu\nu}{}_{\mu\nu} = \frac{p(n-p)}{n(n-1)} f^2 R = 0,\quad (2.135)$$

which is equal to zero for $p = n$. Thus, $W = 0$ for $p = n$. Similarly, for a $(n-1)$ -form $\omega = *\alpha$, where α is a 1-form, we have

$$\begin{aligned}\langle \omega, W\omega \rangle &= \frac{2}{(n-1)(n-2)} \alpha_\kappa [W^{\mu\nu}{}_{\mu\nu} \delta^\kappa{}_\lambda - 2W^{\kappa\nu}{}_{\lambda\nu}] \alpha^\lambda \\ &= \frac{p}{(n-1)(n-2)} \alpha_\kappa [(n-1-p)R\delta^\kappa{}_\lambda - (n-2p)R^\kappa{}_\lambda] \alpha^\lambda,\end{aligned}\quad (2.136)$$

which for $p = n-1$ is equal to just the Ricci tensor as it should be by duality,

$$\langle \omega, W\omega \rangle = R^\lambda{}_\kappa \alpha^\kappa \alpha_\lambda.\quad (2.137)$$

This is a manifestation of the general property. Since both Laplacians commute with the star operator $*$, then, in general, for any $(n-p)$ -form ω

$$\langle *\omega, W_{(p)} *\omega \rangle = \langle \omega, *^{-1} W_{(p)} *\omega \rangle = \langle \omega, W_{(n-p)} \omega \rangle.\quad (2.138)$$

Therefore, we obtain a non-trivial duality property of the Weitzenböck endomorphism

$$*^{-1}W_{(p)}* = W_{(n-p)}. \quad (2.139)$$

This means

$$(n-p+2)(n-p+1)W_{(p)}{}^{\mu\nu}{}_{[\mu\nu\omega\gamma_1\dots\gamma_{n-p}]} = p(p-1)W_{(n-p)}{}^{\mu\nu}{}_{[\gamma_1\gamma_2\omega]_{\mu\nu}|\gamma_3\dots\gamma_{n-p]}}. \quad (2.140)$$

We have

$$\begin{aligned} & (n-p+2)W_{(p)}{}^{\mu\nu}{}_{[\mu\nu\omega\gamma_1\dots\gamma_{n-p}]} \\ &= W_{(p)}{}^{\mu\nu}{}_{\mu\nu\omega\gamma_1\dots\gamma_{n-p}} - 2(n-p)W_{(p)}{}^{\mu\nu}{}_{\mu[\gamma_1\omega]_{\nu}|\gamma_2\dots\gamma_{n-p}} \\ &+ \frac{(n-p)(n-p-1)}{2}W_{(p)}{}^{\mu\nu}{}_{[\gamma_1\gamma_2\omega]_{\mu\nu}|\gamma_3\dots\gamma_{n-p}}. \end{aligned} \quad (2.141)$$

This means that the Weitzenböck tensor satisfies a non-trivial identity

$$\begin{aligned} & (n-p+1)\left\{W_{(p)}{}^{\alpha\beta}{}_{\alpha\beta}\delta^{[\mu}{}_{[\gamma_1}\delta^{\nu]}{}_{\gamma_2]} - 2(n-p)W_{(p)}{}^{\alpha[\mu}{}_{\alpha[\gamma_1}\delta^{\nu]}{}_{\gamma_2]} \right. \\ & \left. + \frac{(n-p)(n-p-1)}{2}W_{(p)}{}^{\mu\nu}{}_{\gamma_1\gamma_2}\right\} = p(p-1)W_{(n-p)}{}^{\mu\nu}{}_{\gamma_1\gamma_2}. \end{aligned} \quad (2.142)$$

By using the decomposition of the curvature tensor for $n \geq 3$

$$R^{ij}{}_{kl} = C^{ij}{}_{kl} + \frac{4}{n-2}R^{[i}{}_{[k}\delta^{j]}{}_{l]} - \frac{2}{(n-1)(n-2)}R\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}, \quad (2.143)$$

where $C^{ij}{}_{kl}$ is the Weyl tensor (recall that for $n = 3$ Weyl tensor vanishes) we can rewrite it in the form

$$W^{\mu\nu}{}_{\rho\sigma} = -\frac{p(p-1)}{2}C^{\mu\nu}{}_{\rho\sigma} + \frac{p(n-2p)}{(n-2)}R^{\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} + \frac{p(p-1)}{(n-1)(n-2)}R\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}. \quad (2.144)$$

and, further, in terms of irreducible components of the Riemann tensor (Weyl tensor, traceless Ricci tensor and scalar curvature)

$$W^{\mu\nu}{}_{\rho\sigma} = -\frac{p(p-1)}{2}C^{\mu\nu}{}_{\rho\sigma} + \frac{p(n-2p)}{(n-2)}E^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} + \frac{p(n-p)}{n(n-1)}R\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}. \quad (2.145)$$

Notice that the scalar curvature term vanishes for $p = 0$ and $p = n$ and is positive for $p \neq 0, n$ and *positive* scalar curvature. Also, a remarkable property is that the traceless Ricci term vanishes precisely in the middle dimension $p = n/2$ (for even n). The Weyl tensor term vanishes for $p = 0$ and $p = 1$ and is positive for negative Weyl curvature.

The importance of Weitzenböck formulas lies in the integral identity

$$\begin{aligned} \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 &= -(\omega, \Delta_{\text{Hodge}}\omega)_{L^2} \\ &= -(\omega, \Delta_{\text{Bochner}}\omega)_{L^2} + (\omega, W\omega)_{L^2} \\ &= \|\nabla\omega\|_{L^2}^2 + (\omega, W\omega)_{L^2}. \end{aligned} \quad (2.146)$$

Therefore, if W is strictly positive uniformly throughout the manifold so that

$$(\omega, W\omega)_{L^2} = \int_M d\text{vol} \langle \omega, W\omega \rangle > 0, \quad (2.147)$$

then the above expression is strictly positive, which means that there are no harmonic forms.

For example, for one-forms this means that there cannot exist harmonic one-forms on a manifold with strictly positive Ricci curvature. In general, positive Ricci curvature and negative sectional curvature work against harmonic forms. If Ricci curvature is strictly positive and sectional curvature is strictly negative then there are no harmonic forms for $1 \leq p \leq n - 1$, that is, their Betti numbers in corresponding dimension are equal to zero

$$B_p(M) = 0 \quad \text{for} \quad p = 1, 2, \dots, (n - 1). \quad (2.148)$$

Since there is exactly one harmonic 0-form (a constant) on compact manifolds and the dual n -form then

$$B_0(M) = B_n(M) = 1, \quad (2.149)$$

and, therefore, the Euler characteristic of such manifolds is

$$\chi(M) = \sum_{k=0}^n (-1)^k B_k(M) = 1 + (-1)^n, \quad (2.150)$$

which is equal to 2 for even n and to zero for odd n . Thus such manifolds cannot have a rich topology. The sphere S^n is such a manifold with positive constant curvature. That is why, $\chi(S^{2n}) = 2$ and $\chi(S^{2n+1}) = 0$.

Let M be a compact Lie group, $C^a{}_{bc}$ be the structure constants and

$$\gamma_{ab} = -C^c{}_{ad}C^d{}_{bc}. \quad (2.151)$$

be the corresponding (positive-definite) Cartan-Killing form (it is the negative of the usual one). Then Riemann tensor of the bi-invariant metric is

$$R_{abcd} = \frac{1}{4}\gamma_{fg}C^f{}_{ab}C^g{}_{cd}, \quad (2.152)$$

and the Ricci tensor is

$$R_{ab} = \frac{1}{4}\gamma_{ab}. \quad (2.153)$$

The Weitzenböck tensor is now

$$W^{\mu\nu}{}_{\rho\sigma} = \frac{p}{4}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \frac{p(p-1)}{8}\gamma^{\nu\lambda}C^\mu{}_{\lambda\alpha}C^\alpha{}_{\rho\sigma}. \quad (2.154)$$

2.15 Conformal Transformations

Let $\omega \in C^\infty(M)$ be a smooth function on M . A conformal transformation of the metric is defined by $g_{\mu\nu} \mapsto \bar{g}_{\mu\nu}$ with

$$\bar{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}, \quad (2.155)$$

so that

$$\bar{g}^{\mu\nu} = e^{-2\omega}g^{\mu\nu}, \quad \bar{g} = \det \bar{g}_{\mu\nu} = e^{2n\omega} \det g_{\mu\nu} = e^{2n\omega}g. \quad (2.156)$$

We introduce the following notation for the derivatives of the function ω

$$\omega_\mu = \nabla_\mu\omega, \quad \omega_{\mu\nu} = \nabla_\mu\nabla_\nu\omega. \quad (2.157)$$

Christoffel symbols of the transformed metric are

$$\bar{\Gamma}^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} + K^\alpha{}_{\mu\nu}, \quad (2.158)$$

where

$$K^\alpha{}_{\mu\nu} = \delta_\mu^\alpha\omega_\nu + \delta_\nu^\alpha\omega_\mu - g_{\mu\nu}\omega^\alpha. \quad (2.159)$$

Therefore the Riemann tensor transforms under the conformal transformations by

$$\begin{aligned} \bar{R}^\alpha{}_{\beta\mu\nu} &= R^\alpha{}_{\beta\mu\nu} + \nabla_\mu K^\alpha{}_{\beta\nu} - \nabla_\nu K^\alpha{}_{\beta\mu} + K^\alpha{}_{\gamma\mu}K^\gamma{}_{\beta\nu} - K^\alpha{}_{\gamma\nu}K^\gamma{}_{\beta\mu} \\ &= R^\alpha{}_{\beta\mu\nu} + \delta^\alpha{}_\nu\omega_{\mu\beta} - \delta^\alpha{}_\mu\omega_{\nu\beta} + g_{\beta\mu}\omega^\alpha{}_\nu - g_{\beta\nu}\omega^\alpha{}_\mu \\ &\quad + \delta^\alpha{}_\mu\omega_\nu\omega_\beta - \delta^\alpha{}_\nu\omega_\mu\omega_\beta + \delta^\alpha{}_\nu g_{\mu\beta}\omega_\gamma\omega^\gamma - \delta^\alpha{}_\mu g_{\nu\beta}\omega_\gamma\omega^\gamma \\ &\quad + g_{\beta\nu}\omega_\mu\omega^\alpha - g_{\beta\mu}\omega_\nu\omega^\alpha. \end{aligned} \quad (2.160)$$

The Ricci tensor transforms as

$$\begin{aligned}\bar{R}_{\beta\nu} &= R_{\beta\nu} - g_{\beta\nu}\omega^\mu{}_\mu - (n-2)\omega_{\nu\beta} \\ &\quad + (n-2)\omega_\nu\omega_\beta - (n-2)g_{\nu\beta}\omega_\gamma\omega^\gamma, \end{aligned} \quad (2.161)$$

and the scalar curvature transforms as

$$\bar{R} = e^{-2\omega} \{R - 2(n-1)\omega^\mu{}_\mu - (n-1)(n-2)\omega_\gamma\omega^\gamma\}, \quad (2.162)$$

Notice that

$$\omega^\mu{}_\mu = \Delta\omega, \quad \omega_\gamma\omega^\gamma = |\nabla\omega|^2. \quad (2.163)$$

The scalar Laplacian $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ can be shown to transform by

$$\bar{\Delta} = e^{-2\omega} \{\Delta + (n-2)\omega^\mu\nabla_\mu\}, \quad (2.164)$$

which can be written in the form

$$\bar{\Delta} = e^{-[(n+2)/2]\omega} \Delta e^{[(n-2)/2]\omega} - e^{-2\omega} \left\{ \frac{(n-2)^2}{4} \omega^\mu\omega_\mu + \frac{n-2}{2} \omega^\mu{}_\mu \right\}, \quad (2.165)$$

Let $Y : C^\infty(M) \rightarrow C^\infty(M)$ be the Yamabe operator acting on smooth functions and defined by

$$Y = -\Delta + \frac{(n-2)}{4(n-1)}R. \quad (2.166)$$

Then, it is easy to see that it transforms covariantly, that is,

$$\bar{Y} = e^{-[(n+2)/2]\omega} Y e^{[(n-2)/2]\omega}. \quad (2.167)$$

This leads to the fact that the spectrum (in fact, all spectral invariants) of the Yamabe operator is conformally invariant.

We also compute the transformation of the Hodge Laplacian on p -forms

$$\begin{aligned}(\bar{\Delta}_{\text{Hodge}}\sigma)_{\alpha_1\dots\alpha_p} &= e^{-2\omega} \left\{ \Delta_{\text{Hodge}}\sigma_{\alpha_1\dots\alpha_p} - p(n-2p+2)\omega_{[\alpha_1}\nabla^{\beta}\sigma_{|\beta|\alpha_2\dots\alpha_p]} \right. \\ &\quad + p(n-2p)\omega^\beta\nabla_{[\alpha_1}\sigma_{|\beta|\alpha_2\dots\alpha_p]} + (p+1)(n-2p-2)\omega^\beta\nabla_{[\beta}\sigma_{\alpha_1\dots\alpha_p]} \\ &\quad \left. + p(n-2p)\omega^\beta_{[\alpha_1}\sigma_{|\beta|\alpha_2\dots\alpha_p]} - p(n-2p)(n-2p+2)\omega^\beta\omega_{[\alpha_1}\sigma_{|\beta|\alpha_2\dots\alpha_p]} \right\} \end{aligned} \quad (2.168)$$

This can be computed to

$$\begin{aligned}
(\bar{\Delta}_{\text{Hodge}}\sigma)_{\alpha_1\dots\alpha_p} &= e^{-2\omega} \left\{ \Delta_{\text{Hodge}}\sigma_{\alpha_1\dots\alpha_p} + (n - 2p - 2)\omega^\beta \nabla_\beta \sigma_{\alpha_1\dots\alpha_p} \right. \\
&\quad - p(n - 2p + 2)\omega_{[\alpha_1} \nabla^{\beta} \sigma_{|\beta|\alpha_2\dots\alpha_p]} + 2p\omega^\beta \nabla_{[\alpha_1} \sigma_{|\beta|\alpha_2\dots\alpha_p]} \\
&\quad + p(n - 2p)\omega^\beta_{[\alpha_1} \sigma_{|\beta|\alpha_2\dots\alpha_p]} \\
&\quad \left. - p(n - 2p)(n - 2p + 2)\omega^\beta \omega_{[\alpha_1} \sigma_{|\beta|\alpha_2\dots\alpha_p]} \right\}. \tag{2.169}
\end{aligned}$$

2.16 Homology of Some Manifolds

Let $P(M)$ be the Poincaré polynomial defined by

$$P(M, t) = \sum_{p=0}^n t^p B_p(M), \tag{2.170}$$

where B_p are Betti numbers. Recall that Euler characteristic is

$$\chi(M) = P(M, -1) = \sum_{p=0}^n (-1)^p B_p(M). \tag{2.171}$$

Then for the product of two manifolds $M_1 \times M_2$ there holds

$$P(M_1 \times M_2, t) = P(M_1, t)P(M_2, t). \tag{2.172}$$

Also, if M_1 and M_2 are compact Lie groups with the same Lie algebra then

$$P(M_1, t) = P(M_2, t). \tag{2.173}$$

In particular,

$$P(S^n, t) = t^n + 1, \tag{2.174}$$

$$P(SO(4), t) = P(S^3 \times S^3, t) = (t^3 + 1)(t^3 + 1) = t^6 + 2t^3 + 1 \tag{2.175}$$

$$P(CP^n, t) = 1 + t^2 + \dots + t^{2n}. \tag{2.176}$$

Recall that

$$CP^n = S^{2n+1}/U(1) = U(n+1)/[U(n) \times U(1)]. \tag{2.177}$$

3 Lie Groups

3.1 Introduction

Let G be a Lie group of dimension n . Let x^μ , ($\mu = 1, \dots, n$) be local coordinates on G such that 0 corresponds to the identity element e of the group, that is, $e^\mu = 0$. We will quantities that transform as tensors at an arbitrary point as well as quantities that transform as tensors at the identity element. That is why, we will use two different sets of indices. The indices that belong to the tangent space at identity will be denoted by small Latin letters, while the indices that belong to the tangent space at an arbitrary point will be denoted by Greek letters.

Let $F : G \times G \rightarrow G$ be the group multiplication map and $\varphi : G \rightarrow G$ is the inverse map given locally by

$$(xy)^\mu = F^\mu(x, y), \quad (x^{-1})^\mu = \varphi^\mu(x). \quad (3.1)$$

They satisfy the identities

$$F^\mu(x, 0) = F^\mu(0, x) = x^\mu, \quad (3.2)$$

$$F^\mu(x, \varphi(x)) = F^\mu(\varphi(x), x) = 0, \quad (3.3)$$

$$F^\mu(x, F(y, z)) = F^\mu(F(x, y), z). \quad (3.4)$$

The function $F(x, y)$ is analytic function of both x and y ; therefore, it can be expanded in the Taylor series

$$F^a(x, y) = \sum_{k, m=0}^{\infty} \frac{1}{k!m!} F^a_{b_1 \dots b_m c_1 \dots c_k} x^{b_1} \dots x^{b_m} y^{c_1} \dots y^{c_k}, \quad (3.5)$$

where

$$F^a_{b_1 \dots b_m c_1 \dots c_k} = \frac{\partial^m}{\partial x^{b_1} \dots \partial x^{b_m}} \frac{\partial^k}{\partial y^{c_1} \dots \partial y^{c_k}} F^a(x, y) \Big|_{x=y=0}. \quad (3.6)$$

3.2 Differential Identities

First of all, we introduce a special notation for the antisymmetric second partial derivatives

$$C^a_{bc} = F^a_{bc} - F^a_{cb} = \left\{ \frac{\partial^2}{\partial x^b \partial y^c} - \frac{\partial^2}{\partial x^c \partial y^b} \right\} F^a(x, y) \Big|_{x=y=0}. \quad (3.7)$$

These constants, called the *structure constants*, constitute one of the most important characterizations of the group. They cannot be arbitrary but satisfy a very important identity, called the *Jacobi identity*, which can be obtained by differentiating one of the basic group identities (the associativity one) with respect to x, y and z , letting $x = y = z = 0$ and anti-symmetrizing over α, β and γ . It has the form

$$C^a{}_{[b|f]}C^f{}_{cd]} = 0, \quad (3.8)$$

where the index f is excluded from the anti-symmetrization. Let C_a denote $n \times n$ matrices with the entries determined by the structure constants, $(C_a)^b{}_c = C^b{}_{ac}$. Then the Jacobi identity can be rewritten in the form

$$[C_a, C_b] = C^c{}_{ab}C_c. \quad (3.9)$$

This identity will play a role later in the discussion of the Lie algebra and the adjoint representation of the Lie algebra and the group.

Next, we introduce useful notation for the first partial derivatives of the function F

$$L^\alpha{}_b(x) = \left. \frac{\partial}{\partial y^b} F^\alpha(y, x) \right|_{y=0}, \quad (3.10)$$

$$R^\alpha{}_b(x) = \left. \frac{\partial}{\partial y^b} F^\alpha(x, y) \right|_{y=0}, \quad (3.11)$$

$$X^a{}_\beta(x) = \left. \frac{\partial}{\partial x^\beta} F^a(x, y) \right|_{y=\varphi(x)}, \quad (3.12)$$

$$Y^a{}_\beta(x) = \left. \frac{\partial}{\partial x^\beta} F^a(y, x) \right|_{y=\varphi(x)}. \quad (3.13)$$

Recall that $\varphi(x)$ just denotes the coordinates of the inverse element x^{-1} , therefore, when $y = \varphi(x)$ then $z = F(x, y) = 0$. These quantities form $n \times n$ matrices $L = (L^\alpha{}_b)$, $R = (R^\alpha{}_b)$, $X = (X^a{}_\beta)$ and $Y = (Y^a{}_\beta)$. By differentiating the basic identities one can show

$$L(0) = R(0) = X(0) = Y(0) = \mathbb{I}, \quad (3.14)$$

where \mathbb{I} is the unit matrix. Therefore, these matrices are invertible, at least in the neighborhood of the identity.

Now, by differentiating the basic identities we obtain

$$\frac{\partial F^\alpha(x, y)}{\partial x^\beta} L^\beta_\gamma(x) = L^\alpha_\gamma(F(x, y)), \quad (3.15)$$

$$\frac{\partial F^\alpha(y, x)}{\partial x^\beta} R^\beta_\gamma(x) = R^\alpha_\gamma(F(y, x)). \quad (3.16)$$

These identities express a very important property of the matrices L and R , namely, that the vector fields defined by these matrices are invariant under the left or right action of the group; we will say more about this below when discussing actions of the group.

Let us denote the inverse matrices of L and R by \tilde{X} and \tilde{Y} , that is,

$$L\tilde{X} = \tilde{X}L = \mathbb{I}, \quad R\tilde{Y} = \tilde{Y}R = \mathbb{I}. \quad (3.17)$$

Let $z^\alpha = F^\alpha(x, y)$. Then the above equations can be written in the form

$$\frac{\partial z^\alpha}{\partial x^\beta} = L^\alpha_a(z)\tilde{X}^a_\beta(x), \quad (3.18)$$

$$\frac{\partial z^\alpha}{\partial y^\beta} = R^\alpha_a(z)\tilde{Y}^a_\beta(y). \quad (3.19)$$

These equations show that the matrices L and R , if known, enable one to find the function $F(x, y)$ by solving these equations with the appropriate initial conditions.

Now, letting $y = \varphi(x)$ so that $z = F(x, y) = 0$ in these equations we get

$$\left. \frac{\partial}{\partial x^\beta} F^a(x, y) \right|_{y=\varphi(x)} = \tilde{X}^a_\beta(x), \quad (3.20)$$

$$\left. \frac{\partial}{\partial x^\beta} F^a(y, x) \right|_{y=\varphi(x)} = \tilde{Y}^a_\beta(x). \quad (3.21)$$

Thus

$$\tilde{X} = X, \quad \tilde{Y} = Y, \quad (3.22)$$

that is, the matrices X and Y introduced above are exactly the inverses of the matrices L and R .

Differentiating the basic group identities further one can obtain further differential identities. The most important of these identities are

$$\partial_\mu X^a_\nu - \partial_\nu X^a_\mu = C^a_{bc} X^b_\mu X^c_\nu, \quad (3.23)$$

$$\partial_\mu Y^a_\nu - \partial_\nu Y^a_\mu = -C^a_{bc} Y^b_\mu Y^c_\nu. \quad (3.24)$$

These identities can be rewritten in the coordinate free form by introducing the one-forms

$$X^a = X^a{}_{\mu} dx^{\mu}, \quad Y^a = Y^a{}_{\mu} dx^{\mu}.$$

Then they take the form

$$dX^a = \frac{1}{2} C^a{}_{bc} X^b \wedge X^c, \quad (3.25)$$

$$dY^a = -\frac{1}{2} C^a{}_{bc} Y^b \wedge Y^c. \quad (3.26)$$

This can be written in an even more compact form. Let us define matrix-valued one-forms

$$C(X) = C_a X^a, \quad C(Y) = C_a Y^a. \quad (3.27)$$

Then the above equations take a very simple form

$$dC(X) = \frac{1}{2} C(X) \wedge C(X) \quad (3.28)$$

$$dC(Y) = -\frac{1}{2} C(Y) \wedge C(Y). \quad (3.29)$$

Here the wedge product of two matrix-valued one-forms A and B is defined by

$$A \wedge B = A \otimes B - B \otimes A, \quad (3.30)$$

where \otimes involves the matrix multiplication as well; to avoid any confusion we write it explicitly

$$(A \wedge B)^i{}_j = A^i{}_k \otimes B^k{}_j - B^i{}_k \otimes A^k{}_j. \quad (3.31)$$

By using these identities one can also obtain similar identities for the matrices L and R

$$L^{\mu}{}_a \partial_{\mu} L^{\nu}{}_b - L^{\mu}{}_b \partial_{\mu} L^{\nu}{}_a = -C^c{}_{ab} L^{\nu}{}_c, \quad (3.32)$$

$$R^{\mu}{}_a \partial_{\mu} R^{\nu}{}_b - R^{\mu}{}_b \partial_{\mu} R^{\nu}{}_a = C^c{}_{ab} R^{\nu}{}_c. \quad (3.33)$$

These identities can be rewritten in the coordinate free form by introducing the vector fields

$$L_a = L^{\mu}{}_a \partial_{\mu}, \quad R_a = R^{\mu}{}_a \partial_{\mu}.$$

Then they take the form

$$[L_a, L_b] = -C^c{}_{ab}L_c, \quad (3.34)$$

$$[R_a, R_b] = C^c{}_{ab}R_c. \quad (3.35)$$

Recall that the commutator of the vector field is nothing but the Lie derivative. Therefore, these identities can be written in the form

$$\mathcal{L}_{L_a}L_b = -C^c{}_{ab}L_c, \quad (3.36)$$

$$\mathcal{L}_{R_a}R_b = C^c{}_{ab}R_c, \quad (3.37)$$

where \mathcal{L}_ξ is the Lie derivative along the vector field ξ . This leads then to the invariance properties of the one-forms X^a and Y^b with respect to the vector fields L_a and R_b . We obtain

$$\mathcal{L}_{L_a}X^b = C^b{}_{ac}X^c, \quad (3.38)$$

$$\mathcal{L}_{R_a}Y^b = -C^b{}_{ac}Y^c. \quad (3.39)$$

This equations take especially compact form by using the matrix-valued forms, namely,

$$\mathcal{L}_{L_a}C(X) = [C_a, C(X)], \quad (3.40)$$

$$\mathcal{L}_{R_a}C(Y) = -[C_a, C(Y)]. \quad (3.41)$$

Moreover, one can show that the vector fields L_a and R_b commute

$$[L_a, R_b] = 0, \quad (3.42)$$

therefore,

$$\mathcal{L}_{L_a}R_b = \mathcal{L}_{R_a}L_b = 0. \quad (3.43)$$

This can be used further to show that the one-forms Y^b are invariant under the vector fields L_a and the one-forms X^b are invariant under the the vector fields R_a , that is,

$$\mathcal{L}_{L_a}Y^b = \mathcal{L}_{R_a}X^b = \mathcal{L}_{L_a}C(Y) = \mathcal{L}_{R_a}C(X) = 0. \quad (3.44)$$

The properties (3.15) and (3.16) simply mean that the vector fields L_a are invariant under the left action of the group on the manifold, that is why they are called *left-invariant* vector fields, and the vector fields R_a are invariant under the right action of ther group on the manifold, and are therefore called *right-invariant*. The corresponding one-forms X^a are also left-invariant and the one-forms Y^a are right-invariant.

3.3 Canonical Coordinates

Let $x(t)$ be a one-parameter Abelian subgroup of G such that

$$x(0) = 0, \quad (3.45)$$

$$x(t + s) = F(x(t), x(s)) = F(x(s), x(t)), \quad (3.46)$$

and

$$x(-t) = \varphi(x(t)). \quad (3.47)$$

Let $\dot{x}(0) = \xi$. Then the subgroup satisfies the differential equations

$$\dot{x}^\mu(t) = L^\mu_a(x(t))\xi^a = R^\mu_a(x(t))\xi^a. \quad (3.48)$$

The set of all one-parameter subgroups completely covers the group, at least in the neighborhood of the identity. Therefore, one can introduce local coordinates such that

$$x^a(t) = t\xi^a, \quad \varphi^a(x) = -x^a. \quad (3.49)$$

Such coordinates are called the *canonical coordinates*.

The canonical coordinates enable one to find an explicit form of the matrices X and Y and, therefore, the matrices L and R . Notice that in canonical coordinates

$$x^a = L^a_b x^b = R^a_b x^b = X^a_b x^b = Y^a_b x^b. \quad (3.50)$$

Differentiating these equations and using the algebra for the one-forms X^a and Y^a one can obtain the following differential equation for the matrices X and Y

$$[x^a(\partial_a - C_a) + \mathbb{I}] X = \mathbb{I}, \quad (3.51)$$

$$[x^a(\partial_a + C_a) + \mathbb{I}] Y = \mathbb{I}. \quad (3.52)$$

Now, let us define a linear matrix-valued function by

$$C(x) = C_a x^a. \quad (3.53)$$

Then the solution of these equations is

$$X = \frac{\exp(C) - \mathbb{I}}{C} = \mathbb{I} + \frac{1}{2}C + \dots, \quad (3.54)$$

$$Y = \frac{\mathbb{I} - \exp(-C)}{C} = \mathbb{I} - \frac{1}{2}C + \dots, \quad (3.55)$$

where we denoted $C = C(x)$ for simplicity. The matrices L and R are now given by

$$L = \frac{C}{\exp(C) - \mathbb{I}} = \mathbb{I} - \frac{1}{2}C + \dots, \quad (3.56)$$

$$R = \frac{C}{\mathbb{I} - \exp(-C)} = \mathbb{I} + \frac{1}{2}C + \dots. \quad (3.57)$$

Obviously, all matrices L , R , X and Y commute. Also,

$$X(x) = Y(-x), \quad L(x) = R(-x). \quad (3.58)$$

Moreover,

$$X(x) = \exp[C(x)]Y(x), \quad L(x) = \exp[-C(x)]R(x). \quad (3.59)$$

It is easy to see also that

$$X = \exp[C/2] \frac{\sinh(C/2)}{C/2}, \quad (3.60)$$

$$Y = \exp[-C/2] \frac{\sinh(C/2)}{C/2}, \quad (3.61)$$

$$L = \exp[-C/2] \frac{C/2}{\sinh(C/2)}, \quad (3.62)$$

$$R = \exp[C/2] \frac{C/2}{\sinh(C/2)}. \quad (3.63)$$

Therefore,

$$XY = YX = \left(\frac{\sinh(C/2)}{C/2} \right)^2, \quad (3.64)$$

and

$$LR = RL = \left(\frac{C/2}{\sinh(C/2)} \right)^2, \quad (3.65)$$

as well as

$$XR = RX = \exp(C), \quad YL = LY = \exp(-C). \quad (3.66)$$

3.4 Group Actions

Let M be a manifold of dimension m . Let z^i , ($i = 1, \dots, m$) be local coordinates on M . Let $\text{Diff}(M)$ be the group of diffeomorphisms of the manifold M . A *left action* ψ of the group G on M is a map

$$\psi : G \rightarrow \text{Diff}(M), \quad (3.67)$$

such that for any $x \in G$ there is a diffeomorphism

$$\psi_x : M \rightarrow M, \quad (3.68)$$

or, in other words, there is a map

$$\Psi : G \times M \rightarrow M, \quad (3.69)$$

such that for any $z \in M$

$$\Psi(x, z) = \psi_x(z), \quad (3.70)$$

satisfying the conditions

$$\Psi(0, z) = z, \quad (3.71)$$

$$\Psi(F(x, y), z) = \Psi(x, \Psi(y, z)), \quad (3.72)$$

$$\Psi(\varphi(x), z) = \Psi^{-1}(x, z). \quad (3.73)$$

Then we say that the group G *acts* on the manifold M , or that the representation defines an *action* of the group G on the manifold M .

The *right action* is defined similarly. The only difference is the order of the successive actions of the product of two group elements, that is,

$$\Psi(F(x, y), z) = \Psi(y, \Psi(x, z)). \quad (3.74)$$

An action is *faithful* (or effective, or exact) if for any $x, y \in G$, if $x \neq y$ then there is $z \in M$ such that $\psi_x(z) \neq \psi_y(z)$.

An action is *transitive* if for any $z, z' \in M$ there is $x \in G$ such that $z' = \psi_x(z)$.

An action is *free* if for any $x, y \in G$, if $x \neq y$ then for all $z \in M$, $\varphi_x(z) \neq \psi_y(z)$.

An action is *regular* (or *simply transitive*) if it is both transitive and free; this is equivalent to saying that for any $z, z' \in M$ there exists precisely one $x \in G$ such that $z' = \psi_x(z)$.

Of course, the group G always acts on itself by left and right actions. In this case the map Ψ is just the group multiplication map $F : G \times G \rightarrow G$.

Let M be a manifold on which a Lie group G acts. Then a Riemannian metric on M is called *natural* if the group actions are isometries, that is, the group G is a subgroup of the group of isometries of the metric.

More generally, let T be a tensor and $\psi_y : G \rightarrow G$ be an action of the group on itself, $\psi_y(x) = F(y, x)$ for the left action and $\psi_y(x) = F(x, y)$ for the right action. Then T is *invariant* under this action if

$$T(\psi_y(x)) = \psi_y^* T(x), \quad (3.75)$$

where ψ_y^* is the pullback of the tensor.

3.5 Representations

In particular, an action of the group G on a m -dimensional vector space V defines a *representation* of the group

$$D : G \rightarrow \text{Aut}(V). \quad (3.76)$$

The $m \times m$ matrices $D(x) = (D^i_j(x)) \in \text{Aut}(V)$ are automorphisms (invertible matrices) of the vector space V satisfying the conditions

$$D(0) = \mathbb{I}, \quad (3.77)$$

$$D(x)D(y) = D(F(x, y)), \quad (3.78)$$

$$D(\varphi(x)) = D^{-1}(x). \quad (3.79)$$

In local coordinates the action of the group is defined by the functions

$$z^i(x) = \Psi^i(x, z). \quad (3.80)$$

For representations these functions are linear

$$\Psi^i(x, z) = D^i_j(x)z^j. \quad (3.81)$$

Every Lie group has a natural representation, called the *adjoint representation*,

$$\text{Ad} : G \rightarrow \text{GL}(n), \quad (3.82)$$

where $n = \dim G$, which is defined by

$$\text{Ad}(x) = X(x)R(x). \quad (3.83)$$

In particular, this means

$$R(x) = L(x)\text{Ad}(x), \quad X(x) = \text{Ad}(x)Y(x). \quad (3.84)$$

In canonical coordinates the adjoint representation is given simply by

$$\text{Ad}(x) = \exp[C(x)]. \quad (3.85)$$

Two representations D and D' are said to be *equivalent* if there is a non-degenerate $m \times m$ matrix $U = (U_{ij})$ such that for any $x \in G$

$$D'(x) = UD(x)U^{-1}. \quad (3.86)$$

A representation D' is called *dual* (or *contragredient*) to the representation D if

$$D'(x) = [D(x)]^{-1T}, \quad (3.87)$$

where D^{-1T} is the transpose of the inverse of D . If a representation D is equivalent to its dual, then, obviously,

$$\det D(x) = \pm 1. \quad (3.88)$$

Since $D(0) = \mathbb{I}$, then in the neighborhood of the identity

$$\det D(x) = 1. \quad (3.89)$$

The representation dual to the adjoint representation is called the *co-adjoint representation*. It is defined by

$$\tilde{\text{Ad}}(x) = [\text{Ad}(x)]^{-1T} = L^T(x)Y^T(x), \quad (3.90)$$

in canonical coordinates

$$\tilde{\text{Ad}}(x) = \exp[-C^T(x)]. \quad (3.91)$$

The determinant of the adjoint representation determines an important function, called the *modular function*,

$$\Delta(x) = \det \text{Ad}(x). \quad (3.92)$$

In the case when the adjoint and the co-adjoint representations are equivalent the modular function is equal to 1,

$$\Delta(x) = 1. \quad (3.93)$$

Such groups are called *uni-modular*.

3.6 Algebra Lie

Let D be a representation of the group G . The *generators*, $G_a = (G^i_{aj})$, of the group G in the representation D are $m \times m$ matrices defined by

$$G_a = \partial_a D(x) \Big|_{x=0}. \quad (3.94)$$

It is easy to show that they satisfy the commutation relations

$$[G_a, G_b] = C^c_{ab} G_c, \quad (3.95)$$

and form the *Lie algebra* of the group G . In canonical coordinates the representation matrices are given then by

$$D(x) = \exp[G(x)], \quad (3.96)$$

where $G(x) = G_a x^a$. Recall the useful formula for the determinant

$$\det D(x) = \exp[\text{tr } G(x)]. \quad (3.97)$$

For the adjoint representation we obviously have

$$C_a = \partial_a \text{Ad}(x) \Big|_{x=0}, \quad (3.98)$$

so that

$$\text{Ad}(x) = \exp[C(x)], \quad (3.99)$$

where $C(x) = C_a x^a$. The matrices C_a realize the *adjoint representation* of the Lie algebra

$$[C_a, C_b] = C^c_{ab} C_c, \quad (3.100)$$

which simply follows from the Jacobi identity. The matrix $C(x)$ is usually denoted by

$$\text{ad}(x) = C(x), \quad (3.101)$$

so that

$$\text{Ad}(x) = \exp[\text{ad}(x)]. \quad (3.102)$$

The determinant of the adjoint representation of the group (the modular function) is then determined by the trace of the adjoint representation of the Lie algebra

$$\Delta(x) = \det \text{Ad}(x) = \exp[\text{tr } \text{ad}(x)]. \quad (3.103)$$

One can express the group multiplication in adjoint representation entirely in Lie algebra terms as a Taylor series in canonical coordinates. It is given by the so-called Campbell-Hausdorff formula. Namely

$$\text{Ad}(x)\text{Ad}(y) = \text{Ad}(z), \quad (3.104)$$

where $z = F(x, y)$. For the sake of compactness of notation let us denote here $X = \text{ad}(x)$, $Y = \text{ad}(y)$, and $Z = \text{ad}(z)$, so that $\text{Ad}(x) = \exp X$, $\text{Ad}(y) = \exp Y$, and $\text{Ad}(z) = \exp Z$. These should not be confused with the matrices X and Y introduced above. Then

$$\begin{aligned} Z &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{r_i, s_i} \frac{X^{r_1} Y^{s_1} \dots X^{r_k} Y^{s_k}}{r_1! s_1! \dots r_k! s_k!} \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots, \end{aligned} \quad (3.105)$$

where the second summation goes over all non-negative integers r_i and s_i such that $r_i + s_i > 0$.

The generators of the representation dual to the representation generated by G_a are $-G_a^T$. Notice that if a representation is equivalent to its dual then there exists a non-degenerate $m \times m$ matrix $U = (U_{ij})$ such that

$$G_a^T = -UG_aU^{-1}, \quad (3.106)$$

and, therefore, the generators are traceless

$$\text{tr } G_a = 0, \quad (3.107)$$

and, as a consequence,

$$\det D(x) = 1. \quad (3.108)$$

The co-adjoint representation is generated by the matrices $-C_a^T$. If the adjoint representation is equivalent to the co-adjoint representation then there exists a nondegenerate matrix $\gamma = (\gamma_{ab})$ such that

$$C_a^T = -\gamma C_a \gamma^{-1}, \quad (3.109)$$

and, therefore,

$$\text{tr } C_a = C^b_{ac} = 0. \quad (3.110)$$

This means that

$$\text{tr } \text{ad}(x) = 0 \quad (3.111)$$

and, therefore, the group is uni-modular

$$\Delta(x) = \det \text{Ad}(x) = 1. \quad (3.112)$$

Notice that for unimodular groups

$$\det X(x) = \det Y(x) = [\det L(x)]^{-1} = [\det R(x)]^{-1}. \quad (3.113)$$

In canonical coordinates this determinant takes the form

$$\det X(x) = \det Y(x) = \det \left(\frac{\sinh[C(x)/2]}{C(x)/2} \right). \quad (3.114)$$

3.7 Invariant Metrics

Every group has a natural family of Riemannian metrics. Such metrics have the form

$$g_L = \gamma_{ab} Y^a \otimes Y^b, \quad g_L^{-1} = \gamma^{ab} R_a \otimes R_b, \quad (3.115)$$

$$g_R = \gamma_{ab} X^a \otimes X^b, \quad g_R^{-1} = \gamma^{ab} L_a \otimes L_b, \quad (3.116)$$

where γ_{ab} is an *arbitrary* constant symmetric non-degenerate matrix. The components of these metrics are

$$g_{\alpha\beta}^L = \gamma_{ab} Y^a_{\alpha} Y^b_{\beta}, \quad g_L^{\alpha\beta} = \gamma^{ab} R^{\alpha}_a R^{\beta}_b, \quad (3.117)$$

$$g_{\alpha\beta}^R = \gamma_{ab} X^a_{\alpha} X^b_{\beta}, \quad g_R^{\alpha\beta} = \gamma^{ab} L^{\alpha}_a L^{\beta}_b, \quad (3.118)$$

where γ^{ab} is the matrix inverse to γ_{ab} . Of course, for positive definite Riemannian metric g one needs a positive definite matrix γ_{ab} .

One can show that the metric g_L is invariant with respect to the vector fields L_a and the metric g_R is invariant with respect to the vector fields R_a , that is,

$$\mathcal{L}_{L_a} g^L = 0, \quad (3.119)$$

$$\mathcal{L}_{R_a} g^R = 0. \quad (3.120)$$

On another hand, it is easy to show that, in general, the right-invariant metric is not left-invariant and vice-versa. We compute

$$\mathcal{L}_{L_a} g^L = (\gamma_{cb} C^b_{ad} + \gamma_{db} C^b_{ac}) X^c \otimes X^d, \quad (3.121)$$

$$\mathcal{L}_{R_a} g^R = -(\gamma_{cb} C^b_{ad} + \gamma_{db} C^b_{ac}) Y^c \otimes Y^d. \quad (3.122)$$

These metrics will be invariant (in fact, bi-invariant) if the matrix γ is chosen so that

$$\gamma_{cb}C^b_{ad} + \gamma_{db}C^b_{ac} = 0, \quad (3.123)$$

or, in the matrix form

$$C_a^T = -\gamma C_a \gamma^{-1}, \quad (3.124)$$

which simply means that the adjoint representation is equivalent to the co-adjoint representation and the metric is defined with the matrix γ intertwining these representations. Such matrix γ always exists on uni-modular groups. Therefore, a bi-invariant metric exists on unimodular groups. However, such matrix may not exist at all, in which case a bi-invariant metric does not exist, or it may exist but be not positive definite. On such groups the bi-invariant metric exists but is not positive-definite.

Indeed, one can show that if the adjoint representation is equivalent to the co-adjoint representation then

$$X^T(x) = \gamma Y(x) \gamma^{-1}, \quad L^T(x) = \gamma R(x) \gamma^{-1}. \quad (3.125)$$

Therefore, in this case

$$X^T \gamma X = Y^T \gamma Y = \gamma Y X = \gamma Y X. \quad (3.126)$$

In canonical coordinates this matrix is equal to

$$g_{\mu\nu} = \left\{ \gamma \left(\frac{\sinh[C(x)/2]}{C(x)/2} \right)^2 \right\}_{\mu\nu}. \quad (3.127)$$

3.8 Volume Element

We recall a couple of useful formulas. Let $Z = Z^\mu \partial_\mu$ be a vector field. The *divergence* of the vector field Z is a scalar function defined by

$$\operatorname{div} Z = * \mathcal{L}_Z d\operatorname{vol}, \quad (3.128)$$

so that

$$\mathcal{L}_Z d\operatorname{vol} = (\operatorname{div} Z) d\operatorname{vol}. \quad (3.129)$$

Let $\alpha = \alpha_\mu dx^\mu = g_{\mu\nu} Z^\nu dx^\mu$ be a 1-form dual to the vector Z , such that for any vector field V , the value of the one-form α on the vector field V is equal to the inner product of the vector fields Z and V , that is, $\alpha(V) = \langle Z, V \rangle =$

$g(Z, V)$. Let $\delta = *^{-1}d*$ be the coderivative acting on differential forms. Then one can show that the divergence of the vector field Z is equal to

$$\operatorname{div} Z = \delta\alpha . \quad (3.130)$$

It is easy to show that in local coordinates

$$\operatorname{div} Z = \delta\alpha = g^{-1/2}\partial_\mu(g^{1/2}Z^\mu) . \quad (3.131)$$

Now, let f be a scalar function and $f d\operatorname{vol}$ be an n -form. Then

$$\mathcal{L}_Z(f d\operatorname{vol}) = \operatorname{div}(fZ)d\operatorname{vol} = \delta(f\alpha)d\operatorname{vol} , \quad (3.132)$$

which in local coordinates simply means

$$*\mathcal{L}_Z(f d\operatorname{vol}) = g^{-1/2}\partial_\mu(fg^{1/2}Z^\mu) . \quad (3.133)$$

The Riemannian volume elements of the metrics g_L and g_R are defined as usual

$$d_L\operatorname{vol} = g_L^{1/2}(x)dx^1 \wedge \cdots \wedge dx^n , \quad (3.134)$$

$$d_R\operatorname{vol} = g_R^{1/2}(x)dx^1 \wedge \cdots \wedge dx^n , \quad (3.135)$$

where

$$g_L^{1/2}(x) = \gamma^{1/2} \det Y(x) , \quad (3.136)$$

$$g_R^{1/2}(x) = \gamma^{1/2} \det X(x) , \quad (3.137)$$

and $g = \det g_{\mu\nu}$ and $\gamma = \det \gamma_{ab}$. These volume elements are left (or right)-invariant by construction. The invariance of the metric lead to the invariance of the volume elements,

$$\mathcal{L}_{L_a}d_L\operatorname{vol} = 0 , \quad (3.138)$$

$$\mathcal{L}_{R_a}d_R\operatorname{vol} = 0 . \quad (3.139)$$

These equations read in local coordinates

$$\partial_\mu(g_L^{1/2}L^\mu{}_a) = 0 , \quad (3.140)$$

$$\partial_\mu(g_R^{1/2}R^\mu{}_a) = 0 . \quad (3.141)$$

Also, it is easy to see that

$$d_L \text{vol}(\varphi(x)) = d_R \text{vol}(x). \quad (3.142)$$

Moreover, as we have seen above

$$X(x) = \text{Ad}(x)Y(x) \quad (3.143)$$

and $\text{Ad}(x) = \exp[\text{ad}(x)]$. Therefore,

$$\det X(x) = \Delta(x) \det Y(x), \quad (3.144)$$

where $\Delta(x) = \det \text{Ad}(x)$. Therefore,

$$d_R \text{vol}(x) = \Delta(x) d_L \text{vol}(x). \quad (3.145)$$

Now we compute the Lie derivative of the left-invariant volume element with respect to the right-invariant vector fields and vice versa. By using the above equation and the invariance properties of the volume elements we obtain

$$\mathcal{L}_{L_a} d_R \text{vol} = \mathcal{L}_{L_a} (\Delta d_L \text{vol}) = (\mathcal{L}_{L_a} \Delta) d_L \text{vol} = L_a(\Delta) d_L \text{vol}; \quad (3.146)$$

similarly,

$$\mathcal{L}_{R_a} d_L \text{vol} = \mathcal{L}_{R_a} (\Delta^{-1} d_R \text{vol}) = (\mathcal{L}_{R_a} \Delta^{-1}) d_R \text{vol} = R_a(\Delta^{-1}) d_R \text{vol}. \quad (3.147)$$

It is easy to compute

$$L_a \Delta = \Delta L_a \text{ad}(x). \quad (3.148)$$

In canonical coordinates this can be simplified further

$$L_a \text{ad}(x) = \text{tr} C_b L^b_a(x). \quad (3.149)$$

Similarly

$$R_a \Delta = \Delta R_a \text{ad}(x). \quad (3.150)$$

and

$$R_a \text{ad}(x) = \text{tr} C_b R^b_a(x). \quad (3.151)$$

This is obviously equal to zero when the generators of the adjoint representation of the Lie algebra are traceless, $\text{tr} \text{ad} = \text{tr} C_a = 0$. Recall that for $x = 0$

the matrices L and R are equal to the unit matrix $L(0) = R(0) = \mathbb{I}$. This means that this quantity can be equal to zero if and only if $\text{tr } C_a = 0$.

Summarizing, for unimodular groups (when the adjoint and the co-adjoint representations are equivalent, and, as a result, $\text{tr ad}(x) = 0$ and $\Delta(x) = 1$) the left and the right-invariant volume elements coincide

$$d_L \text{vol} = d_R \text{vol} . \quad (3.152)$$

In canonical coordinates this volume element is equal to

$$d\text{vol} = \gamma^{1/2} \det \left(\frac{\sinh[C(x)/2]}{C(x)/2} \right) dx^1 \wedge \cdots \wedge dx^n . \quad (3.153)$$

Notice that even if the matrix γ that intertwines the adjoint and the coadjoint representation is not positive definite, and, as a result, the bi-invariant metric is not positive definite, there is still a well-defined bi-invariant volume element.

3.9 Invariant Connections and Curvature

We compute the Levi-Civita connections of the left- and right-invariant metrics. We recall a couple of useful formulas. Let e_a be a basis of vector fields. Then the *commutation coefficients* θ^c_{ab} are defined by

$$[e_a, e_b] = \theta^c_{ab} e_c . \quad (3.154)$$

An affine connection ∇ is defined by

$$\nabla_{e_a} e_b = \omega^c_{ba} e_c . \quad (3.155)$$

where ω^c_{ba} are the *coefficients of the affine connection*. Let σ^a be the dual basis of one-forms. Then the coefficients of the affine connection are given by

$$\omega^c_{ba} = \sigma^c(\nabla_{e_a} e_b) . \quad (3.156)$$

The *torsion* of the affine connection is a tensor of type $(1, 2)$ defined for any one-form α and two vector fields X and Y by

$$T(\alpha, X, Y) = \alpha(\nabla_X Y - \nabla_Y X - [X, Y]) . \quad (3.157)$$

The components of the torsion and the curvature tensors are

$$T^a_{bc} = \omega^a_{cb} - \omega^a_{bc} - \theta^a_{bc} . \quad (3.158)$$

The connection is *torsion-free* (or symmetric) if for any two vector fields X and Y

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad (3.159)$$

that is,

$$\omega^a_{cb} - \omega^a_{bc} = -\theta^a_{cb}. \quad (3.160)$$

The *curvature* of the affine connection is a tensor of type $(1, 3)$ defined for any one-form α and three vector fields X, Y and Z by

$$\text{Riem}(\alpha, Z, X, Y) = \alpha \left[([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) Z \right]. \quad (3.161)$$

The components of the curvature tensor are

$$R^a_{bcd} = e_c(\omega^a_{bd}) - e_d(\omega^a_{bc}) + \omega^a_{fc}\omega^f_{bd} - \omega^a_{fd}\omega^f_{bc} - \omega^a_{bf}\theta^f_{dc}. \quad (3.162)$$

The connection is *flat* if for any three vector fields X, Y and Z

$$[\nabla_X, \nabla_Y]Z = \nabla_{[X, Y]}Z. \quad (3.163)$$

The curvature is *parallel* if for any vector fields X, Y, Z, V and a one-form α ,

$$\begin{aligned} V[\text{Riem}(\alpha, Z, X, Y)] &= \text{Riem}(\alpha, \nabla_V Z, X, Y) + \text{Riem}(\alpha, Z, \nabla_V X, Y) \\ &\quad + \text{Riem}(\alpha, Z, X, \nabla_V Y) + \text{Riem}(\nabla_V \alpha, Z, X, Y). \end{aligned} \quad (3.164)$$

Let g be a Riemannian metric. The affine connection ∇ is said to be *compatible with the metric g* if for any three vector fields X, Y and Z ,

$$Z[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \quad (3.165)$$

On any Riemannian manifold there is a unique connection, called the *Levi-Civita connection*, which is torsion-free and compatible with the metric. One can show that the coefficients of the Levi-Civita connection are

$$\begin{aligned} \omega^a_{bc} &= \frac{1}{2}g^{ad} [e_b(g_{cd}) + e_c(g_{bd}) - e_d(g_{bc})] \\ &\quad + \frac{1}{2} (g^{af}g_{ce}\theta^e_{fb} + g^{af}g_{be}\theta^e_{fc} - \theta^a_{bc}), \end{aligned} \quad (3.166)$$

where

$$g_{ab} = g(e_a, e_b) \quad (3.167)$$

are the components of the metric tensor and $(g^{ab}) = (g_{ab})^{-1}$ is the inverse matrix.

Now, we can apply these formulas to the calculation of the connection and the curvature on the Lie group. Let us define some group tensors by

$$\omega^a{}_{bc} = \frac{1}{2}\gamma^{af}(\gamma_{ce}C^e{}_{fb} + \gamma_{be}C^e{}_{fc}) - \frac{1}{2}C^a{}_{bc}, \quad (3.168)$$

$$R^a{}_{bcd} = \omega^a{}_{fc}\omega^f{}_{bd} - \omega^a{}_{fd}\omega^f{}_{bc} - \omega^a{}_{bf}C^f{}_{dc}, \quad (3.169)$$

and

$$S^a{}_{bcdg} = \omega^a{}_{pg}R^p{}_{bcd} - \omega^p{}_{bg}R^a{}_{pcd} - \omega^p{}_{cg}R^a{}_{bpd} - \omega^p{}_{dg}R^a{}_{bcp}. \quad (3.170)$$

Notice that the tensor S is not zero, in general.

Now, let us consider the left-invariant metric g_L first. It is defined with the help of the right-invariant basis of vector fields R_a and one-forms Y_a . These play the role of the basis e_a and σ_a above. The commutation coefficients for the right-invariant vector fields are

$$\theta^a{}_{Lbc} = C^a{}_{bc}. \quad (3.171)$$

The components of the metric are simply the matrix γ_{ab} , which is constant. Therefore, the left-invariant Levi-Civita connection reads

$$\omega^a{}_{Lbc} = \omega^a{}_{bc}. \quad (3.172)$$

The curvature of this connection reads

$$\text{Riem}_L = R^a{}_{bcd}R_a \otimes Y^b \otimes Y^c \otimes Y^d. \quad (3.173)$$

Notice that, in general case, even if the components of the curvature tensor are constant, the curvature tensor is not parallel. The covariant derivative of the curvature is

$$\nabla_{R_g}^L \text{Riem}_L = S^a{}_{bcdg}R_a \otimes Y^b \otimes Y^c \otimes Y^d, \quad (3.174)$$

which is not equal to zero, in general.

Similarly, we can compute the Levi-Civita of the right-invariant metric g_R . Now, the basis vector fields and one-forms are L_a and X^a . The commutation coefficients are now

$$\theta^a{}_{Rbc} = -C^a{}_{bc}. \quad (3.175)$$

Therefore, since the connection is linear in commutation coefficients the coefficients of the right-invariant Levi-Civita connection differ by just a sign from the left-invariant one, that is,

$$\omega_{Rbc}^a = -\omega_{bc}^a. \quad (3.176)$$

Also, since the curvature is quadratic in the commutation coefficients, its components with respect to the basis X^a and L_a are the same as the curvature of the left-invariant connection, that is,

$$\text{Riem}_R = R^a{}_{bcd} L_a \otimes X^b \otimes X^c \otimes X^d. \quad (3.177)$$

This curvature is also not parallel with respect to the right-invariant metric, the covariant derivative

$$\nabla_{L_g}^R \text{Riem}_R = -S^a{}_{bcdg} L_a \otimes X^b \otimes X^c \otimes X^d, \quad (3.178)$$

is not zero, in general.

Laplace operator acting on functions for the left-invariant and the right-invariant metrics is

$$\Delta^R = \gamma^{ab} L_a L_b + \gamma^{ab} C_{ac}^c L_b, \quad (3.179)$$

$$\Delta^L = \gamma^{ab} R_a R_b + \gamma^{ab} C_{ac}^c R_b, \quad (3.180)$$

which, for unimodular algebras, become simply

$$\Delta^R = \gamma^{ab} L_a L_b, \quad (3.181)$$

$$\Delta^L = \gamma^{ab} R_a R_b. \quad (3.182)$$

3.10 Bi-invariant Connection and Curvature

Now, let us consider the case when the adjoint and the co-adjoint representations are equivalent (which is true for unimodular groups, in particular, any semi-simple or compact groups) and the matrix γ is proportional to the matrix intertwining these representations, such that

$$C_a^T = -\gamma C_a \gamma^{-1}, \quad (3.183)$$

or, in components,

$$\gamma_{ac} C_{db}^c + \gamma_{bc} C_{da}^c = 0. \quad (3.184)$$

In this case the objects introduced above take an especially simple form. First, we have

$$\omega^a{}_{bc} = -\frac{1}{2}C^a{}_{bc}. \quad (3.185)$$

Now, by using Jacobi identity one can show that

$$R^a{}_{bcd} = -\frac{1}{4}C^f{}_{cd}C^a{}_{fb}. \quad (3.186)$$

By using the equation (3.184) this can be rewritten in a more symmetric form

$$R^a{}_{bcd} = \gamma^{ap}R_{pbcd}, \quad (3.187)$$

where

$$R_{pbcd} = \frac{1}{4}\gamma_{fg}C^g{}_{pb}C^f{}_{cd}. \quad (3.188)$$

Finally, one can also show that in this case

$$S^a{}_{bcdg} = 0. \quad (3.189)$$

Indeed, in this case we have

$$S^a{}_{bcdg} = \frac{1}{8} \left\{ (C^a{}_{gp}C^p{}_{bf} - C^a{}_{fp}C^p{}_{bg})C^f{}_{cd} + C^a{}_{bf} (C^f{}_{cp}C^p{}_{dg} - C^f{}_{dp}C^p{}_{cg}) \right\}. \quad (3.190)$$

Now, by using the Jacobi identity we get

$$S^a{}_{bcdg} = \frac{1}{8} \left\{ (C^a{}_{gp}C^p{}_{bf} - C^a{}_{fp}C^p{}_{bg})C^f{}_{cd} + C^a{}_{bf}C^q{}_{cd}C^f{}_{qg} \right\}. \quad (3.191)$$

Relabeling (and permuting) indices we get

$$S^a{}_{bcdg} = \frac{1}{8}C^f{}_{cd} \left\{ C^a{}_{gp}C^p{}_{bf} + C^a{}_{fp}C^p{}_{gb} + C^a{}_{bp}C^p{}_{fg} \right\}, \quad (3.192)$$

which is equal to zero by the Jacobi identity.

Now, the bi-invariant metric is given by

$$g = \gamma_{ab}X^a \otimes X^b = \gamma_{ab}Y^a \otimes Y^b. \quad (3.193)$$

That is why, one can use either the left-invariant or the right-invariant basis (but not both!). Although the coefficients of the left-invariant and the right-invariant connection differ by sign, they define the same connection. The

Christoffel coefficients of the bi-invariant metric are, of course, the same. To clarify this point, the bi-invariant connection is defined either by

$$\nabla_{R_a}^L R_b = \frac{1}{2} C_{ab}^c R_c = \frac{1}{2} [R_a, R_b], \quad (3.194)$$

or

$$\nabla_{L_a}^R L_b = -\frac{1}{2} C_{ab}^c L_c = \frac{1}{2} [L_a, L_b]. \quad (3.195)$$

These equations define a unique connection ∇ such that for any two vector fields X and Y

$$\nabla_X Y = \frac{1}{2} [X, Y]. \quad (3.196)$$

Therefore, the curvature of this connection is

$$\text{Riem} = R^a{}_{bcd} L_a \otimes X^b \otimes X^c \otimes X^d = R^a{}_{bcd} R_a \otimes Y^b \otimes Y^c \otimes Y^d. \quad (3.197)$$

Finally, this curvature tensor is parallel, that is,

$$\nabla_{R_g} \text{Riem} = \nabla_{L_g} \text{Riem} = 0. \quad (3.198)$$

The Ricci curvature tensor is

$$\text{Ric} = R_{ab} X^a \otimes X^b = R_{ab} Y^a \otimes Y^b, \quad (3.199)$$

where

$$R_{ab} = -\frac{1}{4} C_{ad}^c C_{bc}^d = -\frac{1}{4} \text{tr} C_a C_b, \quad (3.200)$$

and the scalar curvature is

$$R = \frac{1}{4} \gamma^{ab} C_{ad}^c C_{bc}^d. \quad (3.201)$$

Laplace operator acting on functions for the bi-invariant metric is

$$\Delta = \gamma^{ab} L_a L_b = \gamma^{ab} R_a R_b. \quad (3.202)$$

3.11 Semi-simple Groups

The *Killing form* is a bilinear form on the Lie algebra defined by

$$\langle A, B \rangle = \text{tr} (AB). \quad (3.203)$$

In adjoint representation $A = A^a C_a$ and $B = B^a C_a$, therefore,

$$\langle A, B \rangle = -\gamma_{ab} A^a B^b, \quad (3.204)$$

where the matrix $\gamma = (\gamma_{ab})$ is defined by

$$\gamma_{ab} = -\text{tr}(C_a C_b) = -C^d{}_{ac} C^c{}_{bd}. \quad (3.205)$$

A groups is called *semi-simple* if this matrix is non-degenerate. This is so called *Cartan metric* on the Lie algebra. One can show that if this matrix is non-degenerate, then it satisfies the equation (3.124),

$$C_a^T = -\gamma C_a \gamma^{-1}, \quad (3.206)$$

which means that the adjoint and the co-adjoint representations are equivalent. That is, the adjoint representation of a semi-simple algebra is traceless, $\text{tr ad} = \text{tr} C_a = 0$, which means that every semi-simple group is uni-modular.

Of course, Cartan metric is not necessarily positive-definite. However, for a compact semi-simple group it is positive definite. Moreover, for a compact simple group Cartan metric is proportional to the Euclidean metric, that is,

$$\gamma_{ab} = c^2 \delta_{ab}, \quad (3.207)$$

where c^2 is a positive constant. More generally, for an irreducible representation of a compact simple group

$$\text{tr} G_a G_b = -\frac{N}{n} G^2 \delta_{ab}, \quad (3.208)$$

where N is the dimension of the representation and G^2 is a positive number characterizing the representation. One can show that the adjoint representation of a compact simple group is always irreducible and has the dimension equal to the dimension of the group.

Thus, on semi-simple group there exists a bi-invariant metric (not necessarily positive definite),

$$g = \gamma_{ab} X^a \otimes X^b = \gamma_{ab} Y^a \otimes Y^b. \quad (3.209)$$

This bi-invariant metric is unique up to scaling and can be written in terms of the matrix-valued one-form $C(X)$ (or $C(Y)$) in a very compact form

$$g = -\text{tr}[C(X) \otimes C(X)] = -\text{tr}[C(Y) \otimes C(Y)]. \quad (3.210)$$

The Ricci curvature tensor of a bi-invariant metric for semi-simple group is proportional to the metric

$$\text{Ric} = \frac{1}{4}g, \quad (3.211)$$

and, therefore, the scalar curvature is

$$R = \frac{n}{4}. \quad (3.212)$$

3.12 $\text{SL}(2, \mathbb{C})$ and Its Subgroups: $\text{SL}(2, \mathbb{R})$, $\text{SU}(2)$, $\text{SU}(1, 1)$ and $\text{SO}(1, 2)$

The group $\text{SL}(2, \mathbb{C})$ is the group of non-degenerate complex matrices with unit determinant. Of course, it has the subgroup $\text{SL}(2, \mathbb{R})$ of real non-degenerate matrices with unit determinant, as well as the subgroup $\text{SU}(2)$ of complex unitary matrices with unit determinant. The Lie algebra of $\text{SL}(2, \mathbb{C})$ consists of traceless complex matrices, while the Lie algebra of $\text{SL}(2, \mathbb{R})$ consists of traceless real matrices and the Lie algebra of $\text{SU}(2)$ consists of traceless complex anti-Hermitian matrices. Obviously, the dimensions of these groups are

$$\dim \text{SL}(2, \mathbb{C}) = 6, \quad \dim \text{SL}(2, \mathbb{R}) = \dim \text{SU}(2) = 3. \quad (3.213)$$

In the Lie algebra of $\text{SL}(2, \mathbb{C})$ we can choose the following basis

$$\sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad (3.214)$$

$$\Sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.215)$$

so that the matrices σ_j are anti-Hermitian and the matrices Σ_k are Hermitian, that is,

$$\sigma_j^\dagger = -\sigma_j \quad \Sigma_k^\dagger = \Sigma_k. \quad (3.216)$$

The Lie algebra formed by these matrices has the form

$$[\sigma_i, \sigma_j] = \varepsilon_{ijk} \sigma_k, \quad (3.217)$$

$$[\sigma_i, \Sigma_j] = \varepsilon_{ijk} \Sigma_k, \quad (3.218)$$

$$[\Sigma_i, \Sigma_j] = -\varepsilon_{ijk} \sigma_k, \quad (3.219)$$

where ε_{ijk} is the Levi-Civita symbol. More explicitly,

$$[\sigma_1, \sigma_2] = \sigma_3, \quad [\sigma_2, \sigma_3] = \sigma_1, \quad [\sigma_3, \sigma_1] = \sigma_2, \quad (3.220)$$

$$[\sigma_1, \Sigma_2] = \Sigma_3, \quad [\sigma_2, \Sigma_3] = \Sigma_1, \quad [\sigma_3, \Sigma_1] = \Sigma_2, \quad (3.221)$$

$$[\Sigma_1, \sigma_2] = \Sigma_3, \quad [\Sigma_2, \sigma_3] = \Sigma_1, \quad [\Sigma_3, \sigma_1] = \Sigma_2, \quad (3.222)$$

$$[\Sigma_1, \Sigma_2] = -\sigma_3, \quad [\Sigma_2, \Sigma_3] = -\sigma_1, \quad [\Sigma_3, \Sigma_1] = -\sigma_2. \quad (3.223)$$

Now, we immediately see that this algebra has a subalgebra formed by the matrices σ_1 , σ_2 and σ_3 . This is the Lie algebra of $SU(2)$.

Another subalgebra is formed by the real traceless matrices

$$B_1 = \Sigma_1, \quad B_2 = \sigma_2, \quad B_3 = \Sigma_3. \quad (3.224)$$

It has the form

$$[B_1, B_2] = B_3, \quad [B_2, B_3] = B_1, \quad [B_3, B_1] = -B_2. \quad (3.225)$$

This is the Lie algebra of $SL(2, \mathbb{R})$. We see that it is very similar to the Lie algebra of $SU(2)$. The only difference is the sign of the last commutator.

The third subalgebra is formed by the matrices

$$A_1 = \sigma_3, \quad A_2 = \Sigma_1, \quad A_3 = \Sigma_2, \quad (3.226)$$

it has the form

$$[A_1, A_2] = A_3, \quad [A_2, A_3] = -A_1, \quad [A_3, A_1] = A_2. \quad (3.227)$$

This is the algebra of complex traceless matrices that satisfy the identity

$$A_i^\dagger = -\eta A_i \eta^{-1}, \quad (3.228)$$

where η is the matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.229)$$

The corresponding group $SU(1, 1)$ is the group of complex matrices with unit determinant that preserve the bilinear form η in the sense

$$U^\dagger = \eta U^{-1} \eta^{-1}. \quad (3.230)$$

This algebra is isomorphic to the Lie algebra of the group $SO(1, 2)$.

In the following we will consider these algebras simultaneously. We introduce parameters $(\lambda_i) = (\lambda_1, \lambda_2, \lambda_3)$ such that the algebra is given by

$$[G_1, G_2] = \lambda_3 G_3, \quad [G_2, G_3] = \lambda_1 G_1, \quad [G_3, G_1] = \lambda_2 G_2. \quad (3.231)$$

Then for $(\lambda_i) = (1, 1, 1)$ this is the algebra of $SU(2)$, while for $(\lambda_i) = (1, -1, 1)$ this is the algebra of $SL(2, \mathbb{R})$, and for $(\lambda_i) = (-1, 1, 1)$ this is the algebra of $SU(1, 1)$.

Notice also that if one of the parameters is equal to zero, this describes the *solvable* Lie algebra of the group $E(2)$ of motions of the \mathbb{R}^2 , for example, if $(\lambda_i) = (1, 0, 1)$ this is the Lie algebra of the group of motions of the xz -plane in \mathbb{R}^3 . In this case, G_2 generates rotations around the y -axis and G_1 and G_3 generate translations along the x -axis and the z -axis. If two parameters are equal to zero, then this is the *nilpotent* Heisenberg algebra $H_3(\mathbb{R})$. Finally, if all three parameters are equal to zero then this is just an Abelian algebra of \mathbb{R}^3 . We will only consider the case when all parameters are not equal to zero.

The structure constants of the group define the generators of the adjoint representation

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 \\ 0 & \lambda_3 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \\ -\lambda_3 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -\lambda_1 & 0 \\ \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.232)$$

We see that the matrices C_a are traceless, which follows, of course, from the fact that the group is uni-modular. Moreover, for (λ_i) all matrices C_a are anti-symmetric.

Let x_a be the canonical coordinates on the group. Then

$$C(x) = C_a x^a = \begin{pmatrix} 0 & -\lambda_1 x_3 & \lambda_1 x_2 \\ \lambda_2 x_3 & 0 & -\lambda_2 x_1 \\ -\lambda_3 x_2 & \lambda_3 x_1 & 0 \end{pmatrix}. \quad (3.233)$$

This enables us to compute the Cartan metric $\gamma = (\gamma_{ab})$. We obtain

$$\gamma = -\frac{1}{2}(\text{tr } C_a C_b) = \begin{pmatrix} \lambda_2 \lambda_3 & 0 & 0 \\ 0 & \lambda_1 \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \lambda_2 \end{pmatrix}. \quad (3.234)$$

We see that if all parameters λ_i are not equal to zero, then this metric is non-degenerate, and, therefore, the groups $\text{SL}(2, \mathbb{R})$, $\text{SU}(2)$ and $\text{SU}(1, 1)$ are semisimple.

We parametrize the parameters λ_i by complex parameters μ_i such that

$$\lambda_i = \mu_i^2, \quad (3.235)$$

and introduce the parameters

$$\omega_1 = \mu_2 \mu_3, \quad \omega_2 = \mu_1 \mu_3, \quad \omega_3 = \mu_1 \mu_2. \quad (3.236)$$

Then

$$\lambda_1 = \frac{\omega_2 \omega_3}{\omega_1}, \quad \lambda_2 = \frac{\omega_1 \omega_3}{\omega_2}, \quad \lambda_3 = \frac{\omega_1 \omega_2}{\omega_3}, \quad (3.237)$$

and the Cartan metric takes the form

$$\gamma = \begin{pmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}. \quad (3.238)$$

This suggest rescaling the coordinates by

$$y_i = \omega_i x_i, \quad (\text{no summation!}) \quad (3.239)$$

Notice that these coordinates are complex, in general. Then

$$C(x) = \begin{pmatrix} 0 & -\frac{\omega_2}{\omega_1} y_3 & \frac{\omega_3}{\omega_1} y_2 \\ \frac{\omega_1}{\omega_2} y_3 & 0 & -\frac{\omega_3}{\omega_2} y_1 \\ -\frac{\omega_1}{\omega_3} y_2 & \frac{\omega_2}{\omega_3} y_1 & 0 \end{pmatrix}. \quad (3.240)$$

Now, let Ω be a square root of the Cartan metric defined by

$$\Omega = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}, \quad (3.241)$$

so that

$$\Omega^2 = \gamma. \quad (3.242)$$

Then it is easy to check that

$$C(x) = \Omega S(x) \Omega^{-1}, \quad (3.243)$$

where $S(x)$ is anti-symmetric matrix

$$S(x) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}. \quad (3.244)$$

To rewrite it in a more compact and more familiar form, we introduce the generators T_a of the group $SU(2)$ in adjoint representation defined by

$$(T_a)_{bc} = \varepsilon_{bac}, \quad (3.245)$$

or, more explicitly,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.246)$$

Then the matrix $S(x)$ takes the form

$$S(x) = T_a y_a, \quad (3.247)$$

or

$$S_{ab}(x) = -\varepsilon_{abc} y_c. \quad (3.248)$$

Therefore, the matrix C is now

$$C_{ab}(x) = -\Omega_{ad}\varepsilon_{dfc}\Omega_{fb}^{-1}y^c. \quad (3.249)$$

One can show that the products of the matrices T_a form the Lie algebra

$$[T_a, T_b] = \varepsilon_{abc}T_c, \quad (3.250)$$

and, moreover, their products are

$$(T_a T_b)_{cd} = -\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}. \quad (3.251)$$

This form of the matrix $S(x)$ and the matrix $C(x)$ greatly simplifies the calculations. Indeed, one can easily compute the square of the matrix S

$$S^2 = -y^2 P, \quad (3.252)$$

where

$$y^2 = y_1^2 + y_2^2 + y_3^2 = \gamma_{ab}x_a x_b = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2, \quad (3.253)$$

and P is a projection on the plane orthogonal to vector y^a defined by

$$P_{ab}(x) = \delta_{ab} - \frac{y_a y_b}{y^2}, \quad (3.254)$$

and satisfying the equations

$$P^2 = P, \quad PS = SP = S, \quad \text{tr } P = 2. \quad (3.255)$$

Therefore,

$$S^{2n} = (-y^2)^n P, \quad S^{2n+1} = (-y^2)^n S. \quad (3.256)$$

Thus, for any analytic function of S one can compute

$$f(S) = f(0)\mathbb{I} + \left\{ \frac{1}{2} [f(r) + f(-r)] - f(0) \right\} P + \frac{1}{2r} [f(r) - f(-r)] S, \quad (3.257)$$

where $r = \sqrt{-y^2}$. By using this equation one can compute now any analytic function of the matrix $C(x)$; we get

$$f(C) = f(0)\mathbb{I} + \left\{ \frac{1}{2} [f(r) + f(-r)] - f(0) \right\} \Pi + \frac{1}{2r} [f(r) - f(-r)] C, \quad (3.258)$$

where Π is another projection defined by

$$\Pi = \Omega P \Omega^{-1}. \quad (3.259)$$

It satisfies the identities

$$\Pi^2 = \Pi, \quad C\Pi = \Pi C = C, \quad \text{tr } \Pi = 2. \quad (3.260)$$

Notice that the projection Π could be determined by the square of the matrix C ,

$$C^2 = -y^2 \Pi. \quad (3.261)$$

Notice that the matrix $C(x)$ and the invariant y^2 are real; therefore, the projection Π is also real.

Let us introduce rescale coordinates according to

$$\tilde{x}_a = \omega_a^2 x_a = \lambda_a x_a, \quad (\text{no summation!}), \quad (3.262)$$

so that

$$y^2 = \tilde{x}_a x_a. \quad (3.263)$$

Then the projection Π has the form

$$\Pi_{ab} = \delta_{ab} - \frac{1}{y^2} \tilde{x}_a x_b. \quad (3.264)$$

Now, we can compute everything in canonical coordinates. The matrix X determining the left-invariant one-forms has the form

$$X = \frac{\exp C - \mathbb{I}}{C} = \mathbb{I} + \left(\frac{\sinh r}{r} - 1 \right) \Pi + \frac{\cosh r - 1}{r^2} C, \quad (3.265)$$

and its inverse, $L = X^{-1}$, determining the left-invariant vector fields is

$$L = \frac{C}{\exp C - \mathbb{I}} = \mathbb{I} + \left[\frac{r}{2} \coth \left(\frac{r}{2} \right) - 1 \right] \Pi - \frac{1}{2} C. \quad (3.266)$$

The matrix Y determining the one-forms and the matrix $R = Y^{-1}$ determining the right-invariant vector fields are obtained by just changing the sign of x , which is equivalent to changing the sign of C , that is,

$$Y = \frac{\mathbb{I} - \exp(-C)}{C} = \mathbb{I} + \left(\frac{\sinh r}{r} - 1 \right) \Pi - \frac{\cosh r - 1}{r^2} C, \quad (3.267)$$

$$R = \frac{C}{\mathbb{I} - \exp(-C)} = \mathbb{I} + \left[\frac{r}{2} \coth \left(\frac{r}{2} \right) - 1 \right] \Pi + \frac{1}{2} C. \quad (3.268)$$

Now, one can obtain the metric, the connection, and the curvature.

3.13 Example: $\text{SO}(n)$

Let us consider a specific example of a compact simple group $\text{SO}(4)$. The Betti numbers for this group are

$$B_0 = B_6 = 1, \quad B_3 = 2. \quad (3.269)$$

The algebra of this group consists of real 4×4 traceless anti-symmetric matrices and the dimension of this groups is $\dim \text{SO}(4) = 6$. We consider the group $\text{SO}(n)$ for generality and will specify $n = 6$ later. The generators of this algebra T_{ij} can be labeled by two indices, ij such that $i, j = 1, \dots, n$ and $i < j$. The Lie algebra has the form

$$[T^{ij}, T_{kl}] = -4\delta^{[i}_{[k} T^{j]}_{l]}, \quad (3.270)$$

which can be written in the form

$$[T_{ij}, T_{kl}] = \frac{1}{2} C^{rs}_{ij,kl} T_{rs}. \quad (3.271)$$

where the structure constants are

$$C^{rs}_{ij,kl} = 8\delta^{[r}_{[i} \delta^s_{j]} \delta^{[k}_{l]}. \quad (3.272)$$

The factor $\frac{1}{2}$ appears because we formally sum over all indices, whereas we should only sum over $r < s$. Such factors will appear elsewhere too. The Cartan metric defined by

$$(\alpha, \beta) = \frac{1}{4} \gamma_{ij,kl} \alpha^{ij} \beta^{kl}, \quad (3.273)$$

has the form

$$\gamma_{ij,kl} = -\frac{1}{4} C^{rs}_{ij,pq} C^{pq}_{kl,rs} \quad (3.274)$$

which can be computed now easily

$$\gamma^{ij}_{kl} = 16(n-2)\delta^{[i}_{[k} \delta^j_{l]}. \quad (3.275)$$

This defines a positive definite quadratic form (for $n > 2$)

$$(\alpha, \beta) = \frac{1}{4} \gamma_{ij,kl} \alpha^{ij} \beta^{kl} = 4(n-2)\alpha_{ij}\beta^{ij}. \quad (3.276)$$

Notice that

$$\delta^{[i}_{[k}\delta^j]_{l]}\delta^{[k}_{[r}\delta^l]_{s]} = \delta^{[i}_{[r}\delta^j]_{s]} \quad (3.277)$$

therefore, the tensor $2\delta^{[i}_{[k}\delta^j]_{l]}$ plays the role of the identity in the sense that

$$\frac{1}{2} (2\delta^{[i}_{[k}\delta^j]_{l]}) (2\delta^{[k}_{[r}\delta^l]_{s]}) = 2\delta^{[i}_{[r}\delta^j]_{s]}. \quad (3.278)$$

Therefore, the inverse of this metric, which will be denoted by $\tilde{\gamma}^{ijkl}$, should be defined so that

$$\frac{1}{2} \tilde{\gamma}^{ij,rs} \gamma_{rs,kl} = 2\delta^{[i}_{[k}\delta^j]_{l]}, \quad (3.279)$$

it reads

$$\tilde{\gamma}^{ij}_{kl} = \frac{1}{4(n-2)} \delta^{[i}_{[k}\delta^j]_{l]}. \quad (3.280)$$

Thus, this group is semi-simple (for $n > 2$) and has a bi-invariant metric. The curvature of this metric is

$$R_{ij,kl,rs,pq} = \frac{1}{16} \gamma_{ab,cd} C^{ab}_{ij,kl} C^{cd}_{rs,pq}. \quad (3.281)$$

This can be computed now; we obtain

$$R_{ij,kl,rs,pq} = 32(n-2) \{ \delta^{[i}_{[r}\delta^j]_{[k}\delta^q]_{l]}\delta^{p]}_{s]} - \delta^{[i}_{[q}\delta^j]_{[k}\delta^r]_{l]}\delta^{s]}_{p]} \}. \quad (3.282)$$

Here the position of indices on the right-hand side should not mean anything; it is just written in this way for readability (they all should be down indices). Also, there is anti-symmetrization over pairs of the pairs of indices ij , kl , rs and pq separately. The Ricci tensor is

$$R_{ij,rs} = \frac{1}{64} \tilde{\gamma}^{kl,pq} \gamma_{ab,cd} C^{ab}_{ij,kl} C^{cd}_{rs,pq}. \quad (3.283)$$

We compute

$$R_{ij,rs} = \frac{(n-2)}{16} \delta^{[i}_{[r}\delta^j]_{s]}, \quad (3.284)$$

which should be equal to $\frac{1}{4}\gamma_{ij,rs}$.

Now, for any vector X^{ij} on the group we have

$$\text{Ric}(X, X) = \frac{1}{4} R_{ij,rs} X^{ij} X^{rs} = \frac{(n-2)}{64} X_{ij} X^{ij}. \quad (3.285)$$

This is strictly positive for any non-zero X . Also, for any two-form (or an anti-symmetric tensor $X^{ij,kl}$, that is, $X^{ij,kl} = -X^{kl,ij}$) we have

$$\text{Riem}(X, X) = \frac{1}{64} R_{ij,kl,rs,pq} X^{ij,kl} X^{rs,pq} = (n-2) X^{ij}_{jl} X_{ip}{}^{pl}. \quad (3.286)$$

This is also strictly positive for any non-zero X .

3.14 Particular case: $SO(4)$

References