Notes on Special Relativity and Quantum Fields

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January 2000[1]

1 Fields and Particles.

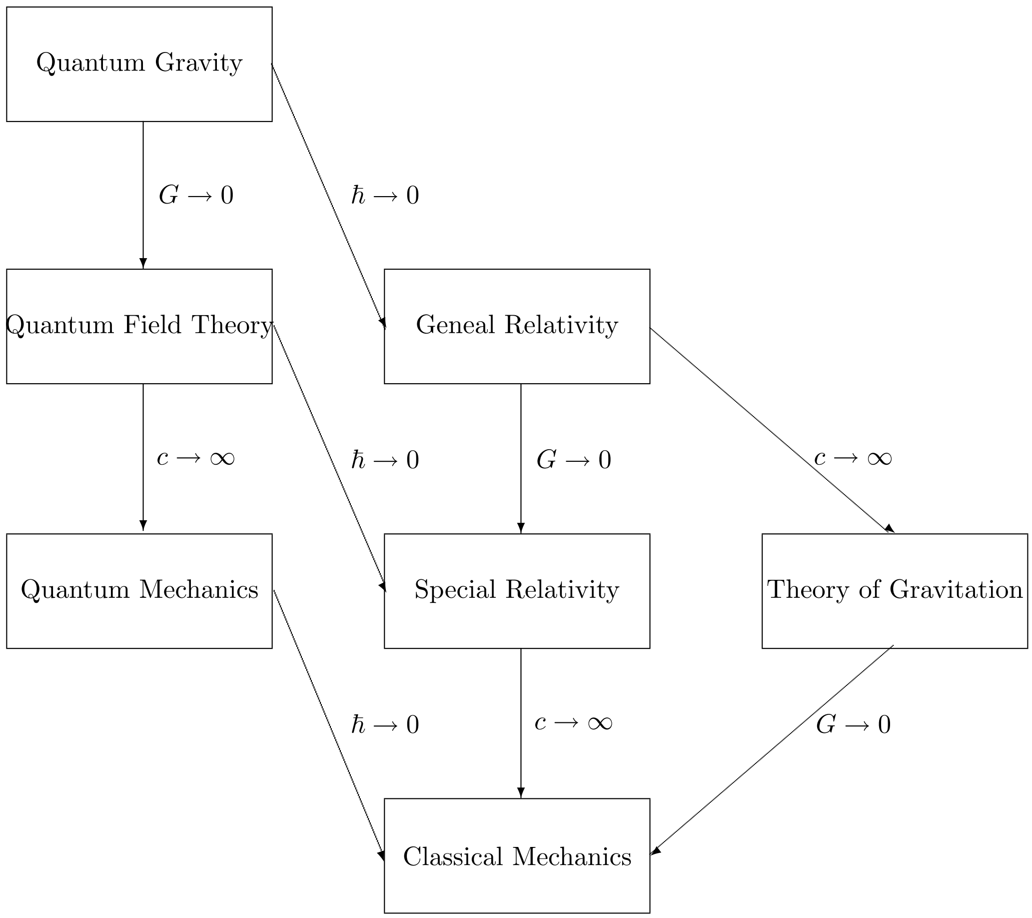
Quantum field theory (QFT) is a theory of *relativistic fields* for describing the properties of *elementary particles* and their interaction.

Quantum fields enable to describe such physical phenomena as *creation* and *annihilation* of elementary particles, and their transformation in each other.

The relativistic field is, in fact, a typical example of a *continuum system*. However, it can be described by a discrete mechanical system with *infinite many degrees of freedom*, so called *field oscillators*. This enables one to *quantize* the classical fields by associativing to the field some discrete *quanta of energy*, which corresponds to different energetic states of the field oscillators. The elementary particles have some specific *spins*, internal angular momentum, electric *charge* and other characteristics. They are identified with the quanta of corresponding relativistic fields.

The *kinematic* properties of elementary particles are described by quantum theory of *free (noninteracting)* fields. The quantum theory of *interacting fields* is the theory of the *interaction of elementary particles*. The QFT is a *quantum relativistic* theory. It contains two fundamental physical constants, reflecting these properties: the *speed of light* c and the *Plank constant* $\hbar$. The more ambitious *quantum gravity* is a further generalization (not completed yet) and contains in addition the *Newton gravitational constant* G. The relations between different theories can be illustrated on the following diagram

[1] corrected on September 16,2003



Therefore, the Quantum Field Theory and the Quantum Gravity can be viewed on as *deformations* of the classical theories corresponding to the fundamental constants c, $\hbar$, G. In the following we always put $\hbar = c = G = 1$, except for some explicit expressions where they are left for convenience.

2 Spacetime.

The classical mechanical systems have finite degrees of freedom, i.e. they are described by a finite set of configuration variables (e.g. coordinates of the particles in the space), $q^i(i=1,\cdots,N)$, which are functions of time: $q^i = q^i(t)$. The field, in contrary, is described by a finite set of functions over the space $\varphi^A_{\mathbf{x}}, (A = 1,\cdots,D)$, $\mathbf{x}$ denoting a point in the space, which are functions of time $\varphi^A_{\mathbf{x}} = \varphi^A(t,\mathbf{x})$. The number of degrees of freedom of a field is, therefore, proportional to the number of the points $\mathbf{x}$ of the space, which is of course infinite. Thus the field has inifinitely many degrees of freedom and is an example of an infinitely-dimensional system. It is sometimes helpful to view on the space argument of the field just as on a continuous label, i.e. to replace $i = (A,\mathbf{x})$ and $q^i = \varphi^A(t,\mathbf{x})$.

So, the fields are just functions of the time and space coordinates. The collection of the time t together with space coordinates $\mathbf{x} = (x,y,z)$ define an *event* with coordinates $x = (x^0,x^1,x^2,x^3)$, where $x^0 = t, x^1 = x, x^2 = y, x^3 = z$. The set of all events determines the *spacetime* M, one of the basic object of any physical theory. Although there are only three physical space coordinates, sometimes there is a need to consider physical models in spaces of lower or higher dimension. That is why we will assume that there are $d-1$ space coordinates so that $\mathbf{x} = (x^1,\dots,x^{d-1})$ and $x = (x^0,x^1,\dots,x^{d-1})$. To enumerate the space coordinates we will always use the small Latin index, x^i, $(i = 1,2,\dots,d-1)$ and for the spacetime coordinates we use the small Greek one, x^μ, $(\mu = 0,1,2,\dots,d-1)$.

The spacetime should be endowed with a *metric*, i.e. a rule for calculating the *spacetime interval*, i.e. the distance, between two close points x and $x+dx$,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{1}$$

where $\eta = (\eta_{\mu\nu})$ is a symmetric $d\times d$ matrix. Here and everywhere below we use the usual convention that one should perform a summation over repeated (dummy) indices. In the special theory of relativity it is postulated to be the Minkowski metric. If the coordinates $\mathbf{x}$ are Cartesian, then it has the form

$$\eta = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2}$$

or, in short, $\eta = \mathrm{diag}\,(-1,1,\cdots,1)$. The metric of this kind is called *pseudo-Euclidean* metric. The spacetime interval can be positive, negative or zero. The corresponding vector $dx = (dt, d\mathbf{x})$ is called *space-like*, *time-like* or *light* (also *null* or *isotropic*) vector. The light vectors form the *light cone*, so that the time-like vectors lie inside and the spacelike outside it.

3 Relativistic Invariance

The principle of relativistic invariance states that all systems of coordinates are physically equivalent. This means that all physical observables should be invariant under the Poincaré group.

3.1 Poincaré Group

Let us remind the definition of the Poincaré group. The set of inhomogeneous linear transformations g of the coordinates

$$x'^{\mu} = \Lambda^{\mu}{}_{\alpha} x^{\alpha} + a^{\mu} \tag{3}$$

or in matrix form

$$g\ :\ x \to x' = \Lambda x + a \tag{4}$$

leaving the interval (1) invariant is called the *general Poincaré group* (or the *general inhomogeneous Lorentz group*) and will be denoted by $\mathcal{P}$. The invariance of the spacetime interval means

$$\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta} \quad \text{or} \quad \Lambda^{T}\eta\Lambda = \eta, \tag{5}$$

where T denote the transposition.

So, an element of the Poincaré group g is determined by a matrix $\Lambda = (\Lambda^{\mu}{}_{\nu})$, satisfying the condition (5), and a vector $a = (a^{\mu})$

$$g = (\Lambda, a). \tag{6}$$

The identity transformation is given by

$$e = (I, 0), \tag{7}$$

where $\mathbb{I}$ is the identity matrix. The product of two transformations is easy to show to be

$$g_1 g_2 = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2) \tag{8}$$

and an inverse

$$g^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \tag{9}$$

so that

$$g g^{-1} = g^{-1} g = e. \tag{10}$$

3.2 Group of Translations

It is obvious that the transformations of the coordinates

$$x' = x + a \tag{11}$$

i.e. the elements of the Poincaré group of the form

$$\tau = (1, a) \tag{12}$$

form a subgroup of the Poincaré group called the *group of translations* and denoted by $\mathcal{T}$. From (8) it is also clear that this group is commutative, or *Abelian*,

$$\tau_1\tau_2 = \tau_2\tau_1 = (1, a_1 + a_2). \tag{13}$$

3.3 Lorentz Group

The set of homogeneous linear transformations

$$x'^\mu = \Lambda^\mu{}_\alpha x^\alpha, \qquad \text{or} \qquad x' = \Lambda x, \tag{14}$$

with Λ satisfying the condition (5), consists of the elements of the Poincaré group of the form

$$l = (\Lambda, 0) \tag{15}$$

and forms another subgroup of the Poincaré group, called the *general Lorentz group* $\mathcal{L}$. It is isomorphic to the group $O(1, d-1)$ of matrices satisfying the condition (5): $\mathcal{L} \simeq O(1, d-1)$. This group is non-commutative (or non-Abelian).

3.4 Structure of the Poincaré Group

So, the Poincaré group has two subgroups – the Abelian subgroup of translations $\mathcal{T}$ and the non-Abelian Lorentz subgroup $\mathcal{L}$. Besides, there is a natural action of the non-Abelian subgroup on the Abelian one

$$l\tau = (1, \Lambda a). \tag{16}$$

A group $\mathcal{P}$ with elements $g = (\Lambda, a)$ and the product rule (8) is called the *semi-direct product* of the groups $\mathcal{T}$ and $\mathcal{L}$

$$\mathcal{P} = \mathcal{T} \ltimes \mathcal{L}. \tag{17}$$

Thus the Poincaré group is the semidirect product of the group of translations and the Lorentz group.

The elements of the Poincaré group can be presented also in form of a $(d+1) \times (d+1)$ matrix

$$g = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \tag{18}$$

which acts on the $(d+1)$-column-vectors

$$\tilde{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \tag{19}$$

$$g\tilde{x} = \begin{pmatrix} \Lambda x + a \\ 1 \end{pmatrix}. \tag{20}$$

3.5 Structure of the general Lorentz Group

From the condition (14) we have

$$(\det \Lambda)^2 = 1 \tag{21}$$

and, therefore,

$$\det \Lambda = \pm 1. \tag{22}$$

Besides

$$-(\Lambda^0{}_0)^2 + \delta_{ik}\Lambda^i{}_0\Lambda^k{}_0 = -1. \tag{23}$$

Hence

$$\Lambda^0{}_0 = \pm\sqrt{1 + \delta_{ik}\Lambda^i{}_0\Lambda^k{}_0}. \tag{24}$$

Therefore, the general Lorentz group has *four connected components*

$$\mathcal{L} = \{\mathcal{L}_{\rm I}, \mathcal{L}_{\rm II}, \mathcal{L}_{\rm III}, \mathcal{L}_{\rm IV}\}. \tag{25}$$

Connected component means that all the transformations from this component can be transformed into each other by a *continuous* transformation.

I. $\mathcal{L}_{\rm I}$.

$$\det \Lambda = +1, \qquad \Lambda^0{}_0 > 0. \tag{26}$$

This component is called the *proper Lorentz group*. It contains obviously the identity transformation

$$\Lambda = I \tag{27}$$

and all the *pseudoorthogonal rotations*. Therefore, it is sometimes denoted by $\mathcal{L}_1$.

II. $\mathcal{L}_{\rm II}$

$$\det \Lambda = -1, \qquad \Lambda^0{}_0 < 0. \tag{28}$$

This component contains the products of proper Lorentz transformations and the reflection of the time coordinate

$$\mathrm{T} : x^0 \to x'^0 = -x^0. \tag{29}$$

given by the matrix

$$\Lambda(T) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{30}$$

Thus

$$\mathcal{L}_{\mathrm{II}} = \mathrm{T} \times \mathcal{L}_1. \tag{31}$$

III. $\mathcal{L}_{\mathrm{III}}$

$$\det \Lambda = -1, \qquad \Lambda^0{}_0 > 0. \tag{32}$$

These transformations are the products of the pseudoorthogonal rotations from the proper Lorentz group and the reflection of one space coordinate, say x^1,

$$P : x^1 \to x'^1 = -x^1 \tag{33}$$

given by the matrix

$$\Lambda(\mathrm{P}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{34}$$

Therefore,

$$\mathcal{L}_{\mathrm{III}} = \mathrm{P} \times \mathcal{L}_1. \tag{35}$$

IV. $\mathcal{L}_{\mathrm{IV}}$

$$\det \Lambda = +1, \qquad \Lambda^0{}_0 < 0. \tag{36}$$

These are the products of the proper Lorentz transformations with the time reflection T and one reflection P of the space coordinate given by the matrix

$$\Lambda(\mathrm{TP}) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} . \tag{37}$$

Hence,

$$\mathcal{L}_{\mathrm{IV}} = \mathrm{T} \times \mathrm{P} \times \mathcal{L}_1 . \tag{38}$$

The proper Lorentz subgroup $\mathcal{L}_1$ together with the component $\mathcal{L}_{\mathrm{IV}}$ form a subgroup of the general Lorentz group

$$SO(1, d-1) = \{\mathcal{L}_1, \mathcal{L}_{\mathrm{IV}}\} \tag{39}$$

with the property

$$\det \Lambda = 1. \tag{40}$$

The proper Lorentz subgroup $\mathcal{L}_1$ together with the component $\mathcal{L}_{\mathrm{III}}$ form another subgroup of the general Lorentz group

$$\mathcal{L}_+ = \{\mathcal{L}_1, \mathcal{L}_{\mathrm{III}}\} \tag{41}$$

with the property

$$\Lambda^0{}_0 > 0. \tag{42}$$

This subgroup is called the *full (complete) orthochronous Lorentz group*. Correspondingly, the semi-direct product of the orthochronous Lorentz group with the group of translations defines the *orthochronous Poincaré group*

$$\mathcal{P}_+ = \mathcal{T} \ltimes \mathcal{L}_+ . \tag{43}$$

It is clear that the proper component of all subgroups of the Lorentz group is the same:

$$\mathcal{L}_1 \simeq O_1(1, d-1) \simeq SO_1(1, d-1) \simeq \mathcal{L}_{+1} . \tag{44}$$

The general Lorentz group $\mathcal{L} \simeq O(1, d-1)$ has four connected components. The role of the component of the identity G_1 plays the proper orthochronous Lorentz group $\mathcal{L}_1 \simeq SO_1(1, d-1)$. The discrete subgroup Γ is the finite group of reflections of the time and one space coordinate

$$\Gamma = \{1, T, P, TP\} \qquad TP = PT. \tag{45}$$

$$T^2 = P^2 = (TP)^2 = 1. \tag{46}$$

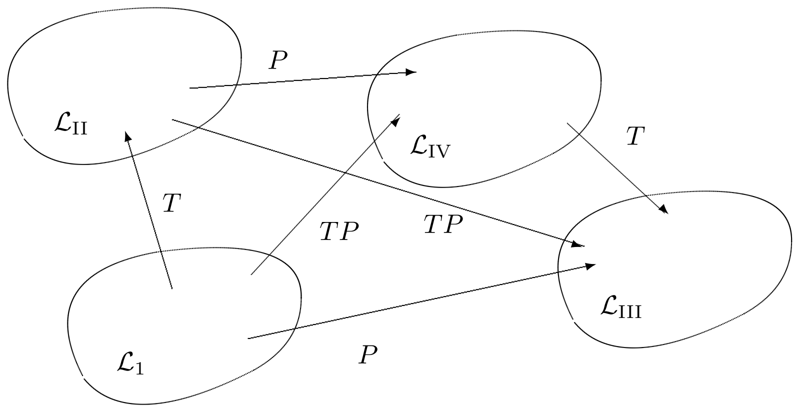


Figure 1: General Lorentz group

4 Poincarè and Lorentz Algebras.

The infinitesimal Poincaré transformations read

$$g = e + \omega, \tag{47}$$

where

$$\omega = (\epsilon, a), \tag{48}$$

a is an infinitesimal vector and $\epsilon = (\epsilon^\mu{}_\nu)$ is an infinitesimal matrix satisfying the condition

$$\epsilon^\mu{}_\alpha \eta_{\mu\beta} + \eta_{\alpha\mu}\epsilon^\mu{}_\beta = 0, \tag{49}$$

or

$$\epsilon^\tau \eta + \eta\epsilon = 0, \tag{50}$$

meaning

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \tag{51}$$

for

$$\epsilon_{\mu\nu} = \eta_{\mu\alpha}\epsilon^\alpha{}_\nu. \tag{52}$$

The number of independent components of an infitisemal vector a is, obviously, equal d. Thus the dimension of the group of translation is equal to d

$$\dim \mathcal{T} = d. \tag{53}$$

The Lorentz transformations are determined by the antisymmetric matrix $\epsilon_{\mu\nu}$ which has $d(d-1)/2$ independent parameters. Therefore, the dimension of the Lorentz group is

$$\dim \mathcal{L} = \frac{d(d-1)}{2}. \tag{54}$$

Since the Poincaré group is the semi-direct product of the group of translations and the Lorentz group, its dimension is just the sum of both

$$\dim \mathcal{P} = d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}. \tag{55}$$

Any infinitesimal Poincaré transformation can be presented, hence, in form

$$\omega = \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta} + a^\gamma P_\gamma = \lambda^a X_a, \tag{56}$$

where

$$\lambda^a = (\epsilon^{\alpha\beta}, a^\gamma) \tag{57}$$

are the infinitesimal parameters and

$$X_a = (M_{\alpha\beta}, P_\gamma) \tag{58}$$

are the generators, which have the form

$$M_{\alpha\beta} = -M_{\beta\alpha} = (M^\mu{}_{\nu\alpha\beta}, 0) \tag{59}$$

with

$$M^\mu{}_{\nu\alpha\beta} = \delta^\mu_\alpha \eta_{\beta\nu} - \delta^\mu_\beta \eta_{\alpha\nu} = 2\delta^\mu_{[\alpha}\eta_{\beta]\nu}. \tag{60}$$

and

$$P_\gamma = (0, P^\mu_\gamma) \tag{61}$$

with

$$P^\mu_\gamma = \delta^\mu_\gamma. \tag{62}$$

The generators $M_{\alpha\beta}$ and P_α of the Poincaré group satisfy the commutation relations

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\beta\delta}M_{\alpha\gamma} + \eta_{\alpha\delta}M_{\beta\gamma} + \eta_{\beta\gamma}M_{\alpha\delta}, \tag{63}$$

$$[M_{\alpha\beta}, P_\mu] = \eta_{\mu\alpha}P_\beta - \eta_{\mu\beta}P_\alpha, \tag{64}$$

$$[P_\alpha, P_\beta] = 0, \tag{65}$$

and form the Lie algebra of the Poincaré group, called the *Poincaré algebra*.

It is imediately seen that the generators of translations P_α form an Abelian subalgebra (65) and the generators of Lorentz transformations form the non-Abelian *Lorentz algebra* (63). The infinitesimal form of pure Lorentz transformations is

$$\Lambda = 1 + \epsilon = 1 + \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta}. \tag{66}$$

Exercise. Obtain the structure constants of the Lorentz group.

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \frac{1}{2} C^{\mu\nu}{}_{\alpha\beta\gamma\delta} M_{\mu\nu}. \tag{67}$$

Answer:

$$C^{\mu\nu}{}_{\alpha\beta\gamma\delta} = 8\delta^{[\mu}_{[\delta}\delta^{\mu]}_{[\alpha}\eta_{\beta]\gamma]} = \delta^{\nu}_{\delta}\delta^{\mu}_{\alpha}\eta_{\beta\gamma} + \cdots \tag{68}$$

5 Tensor and Spinor Fields

5.1 Representations of the Lorentz group

Let us consider now a set of some differentiable functions over the spacetime

$$\varphi(x) = \varphi^A(x) \tag{69}$$

This set of functions defines a *field* if it transforms according to some specific rule under transformation of coordinates from the Poincaré group

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{70}$$

Namely, to the transformation of the coordinates (70) it is assigned a *homogeneous linear* transformation of the field components

$$\varphi(x) \to \varphi'(x') = D(\Lambda)\varphi(x) \tag{71}$$

where the operator $D(\Lambda)$ is completely determined by the matrix Λ.

1. Thus to each Lorentz transformation Λ it is assigned a linear transformation $D(\Lambda)$

$$\Lambda \to D(\Lambda) \tag{72}$$

2. Besides, the identity element of the Lorentz group corresponds to the identity transformation

$$D(1) = 1. \tag{73}$$

3. The product of two elements of Lorentz group is represented by the product of corresponding transformations

$$D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2). \tag{74}$$

A system of operators (matrices) with such properties is called the *representation* of the group. The operators $D(\Lambda)$ are the matrices with the rank equal to the number of the field components. If the number of the field components is finite, it is said that the transformations $D(\Lambda)$ form a *finite-dimensional* *representation* of the Lorentz group. Otherwise we have infinite-dimensional representation of the Lorentz group.

Usually one restricts oneself to the finite-dimensional representations. Thus one can treat $D(\Lambda)$ as operators acting in finite-dimensional linear space V_D of field components and take them as $p \times p$ square matrices

$$D(\Lambda) = (D^A{}_B(\Lambda)), \qquad A = 1, \dots, p, \tag{75}$$

with $p = \dim V_D$ being the number of the field component.

Sometimes it is possible to divide the space of field components V_D , where the representation D acts, into subspaces $V_{D_{(i)}}$ that are invariant under all transformations of the representation, (i.e., into subspaces that are mapped onto oneself under D). Such representations are called *reducible.* Otherwise the representation is called *irreducible.*

Any reducible representation is a *direct sum* of irreducible ones,

$$D = D_{(1)} \oplus D_{(2)} \oplus \cdots \oplus D_{(r)}. \tag{76}$$

The matrix of a reducible representation can be always put in block form by the linear transformations of the basis in the space of the field components.

$$D = \left(\begin{array}{c|c|c|c|c}
D_{(1)} & 0 & 0 & \cdots & 0 \\ \hline
0 & D_{(2)} & 0 & \cdots & 0 \\ \hline
0 & 0 & D_{(3)} & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0 & 0 & \cdots & D_{(r)}
\end{array}\right) \tag{77}$$

where the blocks on the diagonal are given by the irreducible representations.

Thus the study of any field is reduced to the study of finite-dimensional irreducible representations of the Lorentz group. All such representations can be classified. In general they can be *single-valued* and *two-valued.* This is connected with the point that the relation

$$\Lambda \to D(\Lambda) \tag{78}$$

does not need to be single-valued, because the fields themselves are not directly observable variables in the experiments. The observables are, however, always constructed from the *bilinear* combinations of the fields. Thus the non-single-valued representations $D(\Lambda)$ must lead to single-valued observables given by bilinear combinatorious of the fields.

Besides, there is a need that the operators $D(\Lambda)$ are *continuous* functions of the Lorentz transformations parameters $\Lambda^\mu{}_\nu$, i.e., that an *infinitesimal transformation* of the coordinates causes an infinitesimal transformation of the field components.

That means

$$D\left(1 + \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta}\right) = 1 + \frac{1}{2}\epsilon^{\alpha\beta}T_{\alpha\beta} \tag{79}$$

where

$$T_{\alpha\beta} = D(M_{\alpha\beta}) \tag{80}$$

are the *generators of the Lorentz group in the representation* D. Obviously they form a representation of the Lie algebra of the Lorentz group

$$[T_{\alpha\beta}, T_{\gamma\delta}] = -\eta_{\alpha\gamma}T_{\beta\delta} - \eta_{\beta\delta}T_{\alpha\gamma} + \eta_{\alpha\delta}T_{\beta\gamma} + \eta_{\beta\gamma}T_{\alpha\delta} \tag{81}$$

and form a basis of $d(d-1)/2$ matrices of the same dimension as $D(\Lambda)$.

Thus the finite-dimensional representations of the Lorentz group are splitted in two categoties. The first one is characterized by the *single-valued* relation $\Lambda \to D(\Lambda)$ and contains so called *tensor* (and *pseudotensor*) representations. The fields transforming with respect to the tensor (pseudotensor) representations are called *tensor (pseudotensor) fields*. Sometimes they can be observed directly (electromagnetic field). The second category contains *two-valued* representations

$$\Lambda \to \pm D(\Lambda). \tag{82}$$

and describes the spinor fields.

5.2 Tensor Representations

The transformation law of a *contravariant tensor of rank* p under the proper Lorentz group reads

$$\varphi'^{\mu_1\cdots\mu_p}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \varphi^{\alpha_1\cdots\alpha_p}(x) \tag{83}$$

or, remembering that

$$\frac{\partial x'^{\mu_1}}{\partial x^{\alpha}} = \Lambda^{\mu}{}_{\alpha} \tag{84}$$

$$\varphi'^{\mu_1\cdots\mu_p}(x') = \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_p}{}_{\alpha_p} \varphi^{\alpha_1\cdots\alpha_p}(x). \tag{85}$$

The representation matrices $D^{(p)}(\Lambda)$ are given by the so called *tensor power*, i.e. the *tensor product* with itself, of the matrix Λ

$$D^{(p)}(\Lambda) = \otimes^p \Lambda = \underbrace{\Lambda \otimes \Lambda \otimes \cdots \otimes \Lambda}_{p}. \tag{86}$$

The infinitesimal form of the tensor representation is

$$D^{(p)}(1 + \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta}) = 1 + \frac{1}{2}\epsilon^{\alpha\beta}T^{(p)}{}_{\alpha\beta}, \tag{87}$$

where $T^{(p)}_{\alpha\beta} = D^{(p)}(M_{\alpha\beta})$ are the generators of the representation given by the matrices

$$\begin{aligned} T^{(p)}{}_{\alpha\beta} &= \left(T^{\mu_1\dots\mu_p}{}_{\alpha_1\dots\alpha_p,\alpha\beta}\right) && (88)\\ &= \sum_{n=1}^{p} \delta^{\mu_1}{}_{\alpha_1}\cdots\delta^{\mu_{n-1}}{}_{\alpha_{n-1}} M^{\mu_n}{}_{\alpha_n\alpha\beta}\delta^{\mu_{n+1}}{}_{\alpha_{n+1}}\cdots\delta^{\mu_p}{}_{\alpha_p} && (89)\end{aligned}$$

where $M^\mu{}_{\nu\alpha\beta}$ are the Lorentz generators given by (60). These matrices satisfy the commutation relations of the Lorentz group (63).

Excercise Prove the above statement!

$$[T_{\alpha\beta}, T_{\gamma\delta}] = \frac{1}{2} C^{\mu\nu}{}_{\alpha\beta\gamma\delta} T_{\mu\nu}. \tag{90}$$

The transformation law of *covariant tensor of rank* q under the proper Lorentz group is

$$\begin{aligned} \varphi'_{\nu_1\dots\nu_q}(x') &= \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}}\varphi_{\beta_1\dots\beta_q}(x) && (91)\\ &= \tilde\Lambda_{\nu_1}{}^{\beta_1}\cdots\tilde\Lambda_{\nu_q}{}^{\beta_q}\varphi_{\beta_1\dots\beta_q}(x) && (92)\end{aligned}$$

where

$$\tilde\Lambda = \Lambda^{-1T} = \eta\Lambda\eta^{-1}; \qquad \tilde\Lambda_\mu{}^\nu = \eta_{\mu\alpha}\Lambda^\alpha{}_\beta\eta^{\beta\nu}. \tag{93}$$

Thus the covariant representation of the rank q is given by the tensor power of the transpositioned matrix of the contravariant representation

$$D_{(q)}(\Lambda) = \otimes^q\tilde\Lambda = \underbrace{\tilde\Lambda\otimes\tilde\Lambda\otimes\cdots\otimes\tilde\Lambda}_{q}. \tag{94}$$

In infinitesimal form

$$D_{(q)}\left(1 + \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta}\right) = 1 + \frac{1}{2}\epsilon^{\alpha\beta}\tilde T_{(q)\alpha\beta} \tag{95}$$

where $\tilde T_{(q)\alpha\beta} = D_{(q)}(M_{\alpha\beta})$

$$\tilde T_{\nu_1\dots\nu_q}{}^{\beta_1\dots\beta_q}{}_{,\alpha\beta} = -\eta_{\nu_1\mu_1}\cdots\eta_{\nu_q\mu_q}\eta^{\beta_1\alpha_1}\cdots\eta^{\beta_q\alpha_q}T^{\mu_1\dots\mu_q}{}_{\alpha_1\dots\alpha_q,\alpha\beta}. \tag{96}$$

The generators $\tilde T_{(q)\alpha\beta}$ satisfy the same commutation relations and form another representation of the Lorentz algebra.

The general tensor field of rank (p,q) transforms according to a *tensor product* of contravariant and covariant representations

$$\varphi'^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}(x') = \Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda^{\mu_p}{}_{\alpha_p}\tilde\Lambda_{\nu_1}{}^{\beta_1}\cdots\tilde\Lambda_{\nu_q}{}^{\beta_q}\varphi^{\alpha_1\dots\alpha_p}{}_{\beta_1\dots\beta_q}(x). \tag{97}$$

That means

$$D^{(p)}_{(q)}(\Lambda) = \otimes^p\Lambda\otimes^q\tilde\Lambda = \underbrace{\Lambda\otimes\Lambda\otimes\cdots\otimes\Lambda}_{p}\otimes\underbrace{\tilde\Lambda\otimes\tilde\Lambda\otimes\cdots\otimes\tilde\Lambda}_{q}. \tag{98}$$

5.2.1 Examples

The simplest tensor fields are:

1. tensor of rank zero, so called *scalar* field that is invariant under proper Lorentz transformations
$$\varphi'(x') = \varphi(x), \tag{99}$$

2. *contravariant* φ^μ and *covariant* φ_μ *vector* fields that transform according to
$$\varphi'^\mu(x') = \Lambda^\mu{}_\nu \varphi^\nu(x), \qquad \varphi'_\mu(x') = \tilde\Lambda_\mu{}^\nu \varphi_\nu(x), \tag{100}$$

3. thensors of rank 2 $\varphi^\mu{}_\nu, \varphi^{\mu\nu}$
$$\varphi'^\mu{}_\nu(x') = \Lambda^\mu{}_\alpha \tilde\Lambda_\nu{}^\beta \varphi^\alpha{}_\beta(x), \qquad \varphi'^{\mu\nu}(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \varphi^{\alpha\beta}(x), \tag{101}$$

   etc.

The metric itself $\eta_{\mu\nu}$, the inverse matrix $\eta^{-1} = (\eta^{\mu\nu})$, determined by
$$\eta^{\mu\nu}\eta_{\nu\alpha} = \delta^\mu{}_\alpha, \tag{102}$$
and the Kronecker symbol $\delta^\mu{}_\nu$, the unit matrix, are covariant, contravariant and mixed tensors of rank 2. The metric enables one to state a correspondence between contravariant and covariant tensors by stifting the indices up and down
$$\varphi_\mu = \eta_{\mu\nu}\varphi^\nu, \qquad \varphi^\mu = \eta^{\mu\nu}\varphi_\nu, \tag{103}$$
etc.

5.2.2 Reflections. Pseudo-tensors

So far we considered only the transformation laws of tensor fields under the *proper* Lorentz group $\mathcal{L}_1$, which contains only continuous transformations of the spacetime coordinates.

The general Lorentz group $\mathcal{L}$ contains in addition discrete transformations of the time and space coordinates. Thus, the fields can transform, in general, under the discrete transformations (reflections) too. Since any reflection, say P, repeated twice is equal to the identical transformation
$$P^2 = 1 \tag{104}$$
and the tensor field representations are single-valued, i.e.
$$D(1) = 1, \tag{105}$$
one has
$$D(P)D(P) = D(P^2) = D(1) = 1. \tag{106}$$
Therefore, there are only two possibilities
$$D(P) = \pm 1. \tag{107}$$

The tensor fields that are invariant under reflections are called simply tensor fields, whereas the fields that change sign are caled *pseudo-tensors*, *pseudo-scalar, pseudo-vector*, etc.).

The transformation law of the fields under the reflections determines a specific physical property of the corresponding particles, called *parity*. It plays an essential role in determining the possible forms of interaction of various fields.

5.3 Spinor Fields

The two-valued representations of the Lorentz group are caled *spinor representations*. The corresponding fields are called *spinor fields*, or simply spinors. The transformation law of the spinor fields is more complicated. A spinor field is given by a set of 2^m, $(m = [d/2])$ complex, in general, functions

$$\psi(x) = (\psi^A(x)), \qquad A = 1, 2, \dots, 2^m. \tag{108}$$

Sometimes, they can be taken to be real functions, for example in four-dimensional Minkowski spacetime. Real spinor fields are called *Majorana* spinors, the complex ones are caled *Dirac spinors*. The Dirac spinors are generic, they exist in any spacetime, whereas the Majorana ones exist only in spacetimes of *distinguished* dimension. Of course, any Dirac spinor can be presented also by $2 \times 2^m = 2^{m+1}$ real functions. The important fact is, however, that the Majorana spinor is such one that can be presented by real functions, the number of which is *one half* of the number of components of the Dirac spinor.

Under the Lorentz tranformations

$$x \to x' = \Lambda x \tag{109}$$

the spinor fields transform also *linerly* and *homogeneously*

$$\psi(x) \to \psi'(x') = D(\Lambda)\psi(x). \tag{110}$$

However, now the relation

$$\Lambda \to D(\Lambda) \tag{111}$$

is more complicated.

5.3.1 Dirac Matrices

To describe this relation one has to introduce a set of d complex square $2^m \times 2^m$ matrices $(m \equiv [d/2])$, i.e. elements of $M(2^m, \mathbb{C})$,

$$\gamma_\mu = (\gamma^A{}_{B\mu}), \qquad A = 1, 2, \dots, 2^m, \quad \mu = 0, 1, 2, \dots, d-1, \tag{112}$$

that satisfy the anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I, \tag{113}$$

where I is the identity matrix. A set of elements satisfying such anticommutation relations is called also *Clifford algebra*.

Even Dimension $d = 2m$. Let, first, the dimension d of the spacetime be *even* $d = 2m$. Let us further define new matrices $\gamma_{\mu_1\dots\mu_k}$ by *antisymmetrized* products of Dirac matrices

$$\gamma_{\mu_1\dots\mu_k} = \gamma_{[\mu_1} \cdots \gamma_{\mu_k]}. \tag{114}$$

These matrices have a number of important properties. We list below some of them without the proof.

1. The square of all matrices $\gamma_{\mu_1\dots\mu_k}$ is proportional to the identity matrix

$$\gamma_{\mu_1\dots\mu_k}\gamma^{\mu_1\dots\mu_k} = (-1)^{k(k-1)/2} I, \qquad \text{(no summation over } \mu_1, \dots, \mu_k!). \tag{115}$$

2. All matrices $\gamma_{\mu_1\dots\mu_k}$ have the trace equal to zero

$$\mathrm{tr}\, \gamma_{\mu_1\dots\mu_k} = 0. \tag{116}$$

3. From the property (114) it folows that the eigenvalues of the matrices $\gamma_{\mu_1\dots\mu_k}$ are either $\pm i$ or ± 1. From the property (114) it follows then that the number of $+i$'s is equal to the number of $-i$'s and the number of $+1$'s is equal to the number of -1's. Besides, the number of -1's is always even (for $m \geq 2$). Thus the determinants of all matrices $\gamma_{\mu_1\dots\mu_k}$ are equal to unity

$$\det \gamma_{\mu_1\dots\mu_k} = 1. \tag{117}$$

4. The matrices $\gamma_{\mu_1\dots\mu_k}$ are orthonormal in the scalar product on the algebra $M(2^m, \mathbb{C})$ defined by the trace of the product of matrices

$$< A, B >= \mathrm{tr}\,(AB), \tag{118}$$

i.e. there hold

$$\mathrm{tr}\,(\gamma_{\mu_1\dots\mu_k}\gamma^{\nu_1\dots\nu_j}) = 0, \qquad \text{for } k \neq j, \tag{119}$$

$$\mathrm{tr}\,(\gamma_{\mu_1\dots\mu_k}\gamma^{\nu_1\dots\nu_k}) = (-1)^{k(k-1)/2} k! 2^m \delta^{\nu_1}_{[\mu_1} \cdots \delta^{\nu_k}_{\mu_k]}. \tag{120}$$

5. The system of the matrices

$$I,\ \gamma_{\mu_1},\ \gamma_{\mu_1\mu_2},\ \dots\ , \gamma_{\mu_1\dots\mu_j}, \qquad (\mu_1 < \mu_2 < \cdots < \mu_j) \tag{121}$$

is *linearly independent* and forms a *basis* in the full algebra $M(2^m, \mathbb{C})$ of complex matrices of order 2^m. The number of the matrices in the set (121) is equal to dimension of the algebra $M(2^m, \mathbb{C})$ and is equal to

$$\dim M(2^m, \mathbb{C}) = 2^{2m} = 2^d. \tag{122}$$

This means that any complex matrix $\psi = (\psi^A{}_B) \in M(2^m, \mathbb{C})$ can be presented in form

$$\psi = F \cdot I + \sum_{k=1}^{2m} \frac{1}{k!} F^{\mu_1 \dots \mu_k} \gamma_{\mu_1 \dots \mu_k} \tag{123}$$

where

$$F = 2^{-m} \mathrm{tr}\, \psi \tag{124}$$
$$F^{\mu_1 \dots \mu_k} = (-1)^{k(k-1)/2} 2^{-m} \mathrm{tr}\ (\psi \gamma^{\mu_1 \dots \mu_k}) \tag{125}$$

6. **Schur's Lemma.** If a matrix A commutes with all matrices γ_μ

$$A\gamma_\mu = \gamma_\mu A, \tag{126}$$

then A is proportional to the identity matrix

$$A = \lambda I, \tag{127}$$

where λ is a complex number.

7. **Pauli Theorem.** Any two systems of Dirac matrices γ_μ and $\tilde{\gamma}_\mu$ are equivalent, i.e., there exists a non-degenerate matrix T such that

$$\tilde{\gamma}_\mu = T^{-1} \gamma_\mu T. \tag{128}$$

The matrix T is determined up to an *arbitrary* non-zero complex number. The same can be written also in another form

$$\tilde{\gamma}_\mu = -S^{-1} \gamma_\mu S, \tag{129}$$

where

$$S = \gamma_{\mu_1 \cdots \mu_{2m}} T. \tag{130}$$

8. Any two systems of Dirac matrices γ_μ and $\tilde{\gamma}_\mu$ are equivalent, i.e. there exists essentially only one representation of the anticommutation relations (113) in form of matrices of order 2^m.

9. The system of transposed matrices γ_μ^T also satisfies the anticommutation relations (113) and, therefore, there exists a unique (up to arbitrary complex factor) non-degenerate matrix $E = (E_{AB})$ such that

$$\gamma_\mu^T = -E \gamma_\mu E^{-1}. \tag{131}$$

The matrix E can be only symmetric or antisymmetric (depending on the dimension)

$$E^T = (-1)^{m(m+1)/2} E. \tag{132}$$

It is also clear that

$$(\gamma_{\mu_1\dots\mu_k})^T = (-1)^{k(k+1)/2} E\gamma_{\mu_1\dots\mu_k} E^{-1}. \tag{133}$$

The matrix E plays the role of the spinor metric and is used to shift the spinor indices, e.g.

$$\gamma_{AB\mu} = E_{AC}\gamma^C{}_{B\mu}, \tag{134}$$

etc. (One should be carefull in doing this when E is antisymmetric, as in four-dimensional Minkowski spacetime!)

10. The Hermitian conjugate matrices $\gamma_\mu^\dagger$ also satisfy the anticommutation relations (113) and are equivalent to γ_μ, i.e., there exists a unique (up to a complex number) non-degenerate matrix β such that

$$\gamma_\mu^\dagger = -\beta\gamma_\mu\beta^{-1}. \tag{135}$$

Hence

$$\gamma_{\mu_1\dots\mu_k}^\dagger = (-1)^{k(k+1)/2}\beta\gamma_{\mu_1\dots\mu_k}\beta^{-1}. \tag{136}$$

The matrix β can be choosen, for example, to be Hermitian $\beta^\dagger = \beta$ or anti-Hermitian $\beta^\dagger = -\beta$.

11. Using the behavior of the Dirac matrices under transposition and Hermitian conjugation one can calculate the complex conjugate matrices γ^*

$$\gamma^* = (\gamma^\dagger)^T = -(\beta\gamma_\mu\beta^{-1})^T = -\beta^{-1\,T} E\gamma_\mu E^{-1}\beta^T. \tag{137}$$

Therefore, the complex conjugate matrices γ^* are also equivalent to γ_μ:

$$\gamma_\mu^* = \Pi^{-1}\gamma_\mu\Pi, \tag{138}$$

where

$$\Pi = E^{-1}\beta^T. \tag{139}$$

After the normalization for the matrices E and β is choosen the matrix Π is unique.

To distinguish between the complex conjugate objects there is a rule to denote the components of the complex conjugate objects by dot indices,

$$\gamma_\mu^* = (\gamma^{*\dot{A}}{}_{\dot{B}\mu}). \tag{140}$$

Then the matrices β and Π have the following structure of the indices

$$\beta = (\beta_{\dot{A}B}), \qquad \Pi = (\Pi^A{}_{\dot{B}}) \tag{141}$$

Odd Dimension $d = 2m + 1$.

1. Let us consider a set of Dirac matrices γ_a, $(a = 0, 1, \dots, 2m - 1)$, in $2m = d - 1$ dimensions and let us introduce a matrix

$$\gamma_{2m} = \frac{i^m}{(2m)!}\varepsilon^{a_1 \dots a_{2m}} \gamma_{a_1 \dots a_{2m}}, \tag{142}$$

where $\varepsilon^{a_1 \dots a_{2m}}$ is the completely antisymmetric Levi-Civita tensor in $d - 1 = 2m$ dimensions. (In four-dimensions, $m = 2$, the matrix (142) is usually denoted by γ_5.)

The matrix γ_{2m} anticommutes with all matrices γ_a and has the square equal to the unit matrix

$$\gamma_{2m}\gamma_a + \gamma_a \gamma_{2m} = 0, \tag{143}$$

$$(\gamma_{2m})^2 = I. \tag{144}$$

Therefore, the system of matrices $\{\gamma_\mu\} = \{\gamma_a, \gamma_{2m}\}$, where the index μ runs now over $0, 1, \dots, 2m$, satisfies the anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I. \tag{145}$$

Thus the matrices γ_μ form the set of Dirac matrices in odd dimensions $d = 2m + 1$. Note that we could equally choose the matrix γ_{2m} with an extra minus sign

$$\tilde{\gamma}_{2m} = -\frac{i^m}{(2m)!}\varepsilon^{a_1 \dots a_{2m}} \gamma_{a_1 \dots a_{2m}}. \tag{146}$$

This would give us *another* system of Dirac matrices in odd dimensions $d = 2m + 1$. It can be shown that, in contrary to the case of even dimensions, in odd dimensions there are *two non-equivalent* systems of Dirac matrices, $\{\gamma_\mu\} = \{\gamma_a, \gamma_{2m}\}$ and $\{\tilde{\gamma}_\mu\} = \{\gamma_a, \tilde{\gamma}_{2m}\}$, defined by eqs. (144) and (146) respectively.

2. The system of matrices with the *even* number of indices

$$I,\ \gamma_{\mu_1\mu_2},\ \dots\ , \gamma_{\mu_1 \dots \mu_{2m}}, \qquad (\mu_1 < \mu_2 < \cdots < \mu_{2m}) \tag{147}$$

and the system of matrices with *odd* number of indices

$$\gamma_{\mu_1},\ \gamma_{\mu_1\mu_2\mu_3},\ \dots\ , \gamma_{\mu_1 \dots \mu_{2m+1}}, \qquad (\mu_1 < \mu_2 < \cdots < \mu_{2m+1}) \tag{148}$$

where μ_k run over $0, 1, \dots, 2m = d - 1$ are linearly independent and form two different basises in the full algebra of complex matrices $M(2^m, \mathbb{C})$. The number of matrices in both systems is, of course, equal to 2^{2m} which is the dimension of $M(2^m, \mathbb{C})$.

Thus, for any complex matrix $\psi \in M(2^m, \mathbb{C})$, similarly to (123), we have

$$\psi = F \cdot I + \sum_{k=1}^{m} \frac{1}{(2k)!} F^{\mu_1 \cdots \mu_{2k}} \gamma_{\mu_1 \dots \mu_{2k}} \tag{149}$$

where

$$F = 2^{-m} \mathrm{tr}\, \psi \tag{150}$$

$$F^{\mu_1 \cdots \mu_{2k}} = (-1)^k 2^{-m} \mathrm{tr}\, (\psi \gamma^{\mu_1 \cdots \mu_{2k}}) \tag{151}$$

and

$$\psi = \sum_{k=0}^{m} \frac{1}{(2k+1)!} F^{\mu_1 \cdots \mu_{2k+1}} \gamma_{\mu_1 \dots \mu_{2k+1}} \tag{152}$$

where

$$F^{\mu_1 \cdots \mu_{2k+1}} = (-1)^k 2^{-m} \mathrm{tr}\, (\psi \gamma^{\mu_1 \cdots \mu_{2k+1}}) \,. \tag{153}$$

3. The spinor metric E is now defined by

$$\gamma_\mu^T = (-1)^m E \gamma_\mu E^{-1}. \tag{154}$$

This *differs* from the metric in even dimensions! It posseses the following symmetry property

$$E^T = (-1)^{m(m+1)/2} E. \tag{155}$$

Analogously,

$$\gamma^T_{\mu_1 \dots \mu_k} = (-1)^{km + k(k-1)/2} E \gamma_{\mu_1 \dots \mu_k} E^{-1}. \tag{156}$$

4. The matrix β is defined in odd dimensions by the same rule

$$\gamma_\mu^\dagger = -\beta \gamma_\mu \beta^{-1}. \tag{157}$$

5.4 Spinor Representation

Let us consider a Lorentz transformation from the proper Lorentz group $\mathcal{L}_1$. It is easy to see that the matrices

$$\gamma'^\mu = \Lambda^\mu{}_\nu \gamma^\nu, \tag{158}$$

where Λ is the matrix of a Lorentz transformation satisfy the same anticommutation relations

$$\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = 2\eta^{\mu\nu} I \tag{159}$$

and, therefore, should be equivalent to γ_μ. This means that there exists a nondegenerate matrix $D(\Lambda)$ such that

$$\Lambda^\mu{}_\nu \gamma^\nu = D^{-1}(\Lambda) \gamma^\mu D(\Lambda). \tag{160}$$

The matrix $D(\Lambda)$ is determined only up to an arbitrary complex number. This equation together with a normalization condition

$$D^T(\Lambda)ED(\Lambda) = E \tag{161}$$

defines the matrices $D(\Lambda)$ for any proper Lorentz transformation Λ, which form a *spinor representation* of the proper Lorentz group $\mathcal{L}_1$.

For the general Lorentz group (including reflections) there are several spinor representations which differ by the behavior of spinors under the reflections. We will not consider this subject here.

Let us instead calculate the form of the matrices $D(\Lambda)$ for infinitesimal Lorentz transformations,

$$\Lambda = 1 + \frac{1}{2}\varepsilon^{\alpha\beta}M_{\alpha\beta}, \tag{162}$$

i.e., we are going to find the *generators of the Lorentz group in spinor representation*

$$G_{\alpha\beta} = D(M_{\alpha\beta}). \tag{163}$$

As usual we have

$$D\left(1 + \frac{1}{2}\epsilon^{\alpha\beta}M_{\alpha\beta}\right) = 1 + \frac{1}{2}\epsilon^{\alpha\beta}G_{\alpha\beta} \tag{164}$$

To find the explicit form of $G_{\mu\nu}$ we substitute (164) in (160). Taking into account only linear terms we obtain

$$\varepsilon^{\mu}{}_{\nu}\gamma^{\nu} = \frac{1}{2}\varepsilon^{\alpha\beta}[\gamma^{\mu}, G_{\alpha\beta}]. \tag{165}$$

Therefore,

$$[\gamma^{\mu}, G_{\alpha\beta}] = \delta^{\mu}_{\alpha}\gamma_{\beta} - \delta^{\mu}_{\beta}\gamma_{\alpha}. \tag{166}$$

The solution of this equation with the condition

$$G^T_{\alpha\beta} = -EG_{\alpha\beta}E^{-1} \tag{167}$$

which follows from the normalization condition (161) is

$$G_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta} = \frac{1}{4}[\gamma_{\alpha}, \gamma_{\beta}]. \tag{168}$$

Let us now obtain the matrix $D(\Lambda)$ for finite Lorentz transformations. Let

$$X \equiv \frac{1}{2}\epsilon^{\alpha\beta}G_{\alpha\beta}. \tag{169}$$

Define an operator Ad_X according to

$$\mathrm{Ad}_X\,\gamma^{\alpha} \equiv [X, \gamma^{\alpha}] = -\varepsilon^{\alpha}{}_{\beta}\gamma^{\beta}. \tag{170}$$

Taking more commutators we find similarly

$$\mathrm{Ad}_X^k\,\gamma^\alpha = \underbrace{[X[\cdots[X}_{k},\gamma^\alpha] = (-1)^k(\varepsilon^k)^\alpha{}_\beta\gamma^\beta \tag{171}$$

Therefore,

$$\sum_{k\ge 0}\frac{(-1)^k}{k!}\mathrm{Ad}_X^k\,\gamma^\alpha = \sum_{k\ge 0}\frac{1}{k!}(\varepsilon^k)^\alpha{}_\beta\gamma^\beta. \tag{172}$$

Thus

$$\exp(-\mathrm{Ad}_X\,)\gamma^\alpha = (e^\varepsilon)^\alpha{}_\beta\gamma^\beta. \tag{173}$$

The exponential of the operator $(-\mathrm{Ad}_X)$ is known to be

$$\exp(-\mathrm{Ad}_X)\gamma^\alpha = e^{-X}\gamma^\alpha e^X. \tag{174}$$

This is easy to prove by solving the differential equation

$$\frac{df^\alpha(t)}{dt} = [X, f^\alpha] \tag{175}$$

for the function

$$f^\alpha(t) \equiv \exp(t\mathrm{Ad}_X)\gamma^\alpha \tag{176}$$

with the initial condition

$$f^\alpha(0) = \gamma^\alpha. \tag{177}$$

Thus we find finally

$$e^{-X}\gamma^\alpha e^X = (e^\varepsilon)^\alpha{}_\beta\gamma^\beta. \tag{178}$$

This is nothing but the definition of the matrices $D(\Lambda)$: if the Lorentz transformation is parametrized by

$$\Lambda = e^\varepsilon \tag{179}$$

then

$$D(\exp\varepsilon) = \exp\left(\frac{1}{2}\varepsilon^{\alpha\beta}G_{\alpha\beta}\right). \tag{180}$$

Two-valuedness of Spinor Representation. As we already mentioned above the spinor representations are two-valued. This can be demonstrated by sonsidering a finite space rotation in the plane (x^1, x^2) on the angle θ. From (180) we get

$$D(\theta) = \exp(G_{12}\theta). \tag{181}$$

By using the fact that the square of the generator G_{12} is proportional to the identity matrix

$$G_{12}^2 = -\frac{1}{4}I, \tag{182}$$

we obtain

$$G_{12}^{2k} = \left(\frac{i}{2}\right)^{2k} I, \tag{183}$$

$$G_{12}^{2k+1} = \left(\frac{i}{2}\right)^{2k} G_{12}. \tag{184}$$

Therefore,

$$\exp(G_{12}\theta) = I\cos\left(\frac{\theta}{2}\right) + 2G_{12}\sin\left(\frac{\theta}{2}\right). \tag{185}$$

Thus when we do a pure space rotation on the angle 2π, which is, of course, identical tranformation, we find

$$D(2\pi) = -I. \tag{186}$$

But we also had

$$D(1) = I. \tag{187}$$

Thus one can define $D(\Lambda)$ *only up to the sign.* This means that the spinor fields $\psi(x)$ also change the sign under the space rotation on 2π

$$\psi(x) \to \psi'(x') = D(2\pi)\psi(x) = -\psi(x). \tag{188}$$

Only after the rotation on the angle 4π one returns to the identity. Therefore, one has to do not with the proper Lorentz group $\mathcal{L}_1$ but with its universal covering group Spin_1, so that

$$\mathcal{L}_1 = \mathrm{Spin}_1/\mathbb{Z}_2, \tag{189}$$

where $\mathbb{Z}_2 = \{+1, -1\}$ is the cyclic group of order 2.

Covariant Spinor Representation. In the same manner as for tensor fields (contravariant and covariant) there is another spinor representation $\bar\psi = (\bar\psi_A)$, so called *covariant spinor representation*, which transforms under Lorentz transformations according to

$$\bar\psi'(x') = \bar\psi(x)D^{-1}(\Lambda). \tag{190}$$

They are connected with the spinor representations by

$$\bar\psi = \psi^\dagger\beta, \qquad \text{or} \qquad \bar\psi_A = \psi^{*\dot B}\beta_{\dot B A}. \tag{191}$$

where the matrix $\beta = (\beta_{\dot A B})$ is defined by (135).

One can show that the bilinear combinations of the spinor fields like

$$\varphi_{\mu_1\dots\mu_k} = \bar\psi\gamma_{\mu_1\dots\mu_k}\psi \tag{192}$$

transform according to single-valued tensor representations.

Excercise. Prove the above statement!

Thus each spinor is equivalent to a set of complex tensors $\varphi_{\mu_1\dots\mu_k}$, $(k = 0, 1, 2, \dots, d-1)$, which are, of course not independent algebraically.

Reflections. The behavior of spinor fields under the reflection is a bit more complicated because the square of a reflection can be represented by *both* the identical transformation $+1$ *and* -1. Thus for spinor representation one has

$$D(P)D(P) = D(P^2) = \pm 1 \tag{193}$$

and, therefore, there are two possibilities

$$(D(P))^2 = +1 \qquad \text{and} \qquad (D(P))^2 = -1. \tag{194}$$

(This splits the group Spin in two inequivalent components: one for which $D^2(P) = 1$, which is the component containing identity, Spin_1, and another component, for which $D^2(P) = -1$. We will not consider the precise form of the matrix D for the reflections. They can be found in the literature.

Spinors of Higher Rank. Taking the tensor product of spinor representations one can obtain spinor fields of any rank

$$\psi^{A_1\dots A_p}{}_{B_1\dots B_q} \tag{195}$$

which transform according to the rule

$$\psi'(x')^{A_1\dots A_p}{}_{B_1\dots B_q} = D^{A_1}{}_{C_1}\dots D^{A_p}{}_{C_p}\psi^{C_1\dots C_p}{}_{F_1\dots F_q}D^{-1F_1}{}_{B_1}\dots D^{-1F_p}{}_{B_q} \tag{196}$$

One should note that many of these representations are equivalent to the tensor ones, namely, those which have the equal number of contravariant and covariant spinor indices

$$\psi^{A_1\dots A_p}{}_{B_1\dots B_p}. \tag{197}$$

Any such spinor field is converted in corresponding tensor field with the help of Dirac matrices

$$\psi^{A_1\dots A_p}{}_{B_1\dots B_p} = \gamma^{A_1}{}_{B_1\mu_1}\cdots\gamma^{A_p}{}_{B_p\mu_p}\varphi^{\mu_1\dots\mu_p} \tag{198}$$

$$\varphi_{\mu_1\dots\mu_p} = \gamma^{A_1}{}_{B_1\mu_1}\cdots\gamma^{A_p}{}_{B_p\mu_p}\psi^{B_1\dots B_p}{}_{A_1\dots A_p}. \tag{199}$$

This is a simple consequence of the basic relation

$$\gamma^A{}_{B\mu}\gamma^B{}_{A\nu} = \mathrm{tr}\,(\gamma_\mu\gamma_\nu) = \eta_{\mu\nu}. \tag{200}$$

Besides, the spinor indices can be shifted (up and down) by the matrix E and the dot indices (of complex conjugate spinors) can be converted into the normal (undotted) indices by the matrix β

$$\psi^{A_1\dots A_p}{}_{B_1\dots B_q} = E_{B_1F_1}\cdots E_{B_qF_q}\psi^{A_1\dots A_pF_1\dots F_q}. \tag{201}$$

Therefore, all spinor representations with *even* number of spinor indices are equivalent to tensor representations and the spinor field with *odd* number of spinor indices are equivalent to the *spin-tensor representations* having only one spinor index and a number of tensor indices

$$\psi^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x) = (\psi^{A\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x)) \tag{202}$$

Such spin-tensor fields are transformed like the tensor product of tensor and spinor representations

$$\psi'^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x') = \Lambda^{\mu_1}{}_{\alpha_1} \dots \Lambda^{\mu_p}{}_{\alpha_p} \tilde{\Lambda}_{\nu_1}{}^{\beta_1} \dots \tilde{\Lambda}_{\nu_q}{}^{\beta_q} D(\Lambda) \psi^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}(x), \tag{203}$$

where $D(\Lambda)$ are the matrices of spinor representation.

The Dirac matrices $\gamma_\mu = (\gamma^A{}_{B\mu})$ themselves are spin-tensors, more precisely the product of a vector and spinor of second rank. According to general rule they transform as follows

$$\gamma'^A{}_B{}^\mu = \Lambda^\mu{}_\nu D^A{}_C \gamma^C{}_F{}^\nu D^{-1F}{}_B \tag{204}$$

or in matrix form

$$\gamma'^\mu = \Lambda^\mu{}_\nu D(\Lambda) \gamma^\nu D^{-1}(\Lambda). \tag{205}$$

Therefore, from (160) we see that, similarly to the spacetime metric $\eta_{\mu\nu}$ and the Kronecker symbol δ^μ_ν, the spintensors defined by the Dirac matrices are *invariant* under the Lorentz transformation

$$\gamma'^\mu = \gamma^\mu. \tag{206}$$

One can take this fact as the *definition* of the spinor representation.

References

[1] N. N. Bogolyobov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, (Wiley, New York, 19??)

[2] N. N. Bogolyobov and D. V. Shirkov, *Quantum Fields*, (Nauka, Moskow, 1980)

[3] C. Itzukson and J.-B. Zuber, *Quantum Field Theory*, (McGraw-Hill, New York, 1980)

[4] P. Ramond, *Field Theory: a Modern Primer*, (Benjamin, Reading, 1981)

[5] S. Weinberg, *The Quantum Theory of Fields*, Vol I, Foundations, (Cambridge University Press, Cambridge, 1995)

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, (McGraw-Hill, New York, 1965)

[7] A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Theory of Gauge Fields*, (Nauka, Moskow, 1978)

[8] J. Schwinger, *Particles, Sources and Fields*, (Addison-Wesley, Reading, 1973)

[9] J. Glimm and A. Jaffe, *Quantum Physics: a Functional Integral Point of View*, (Springer, New York, 1981)

[10] A. N. Vasiliev, *Functional Methods in Quantum Field Theory and Statistics*, (Leningrad University, Leningrad, 1976)

[11] R. Rajaraman, *Solitons and Instantons: Introduction to Solitons and Instantons in Quantum Field Theory*, (North-Holland, Amsterdam, 1982)

[12] B. S. De Witt, *Dynamical Theory of Groups and Fields*, (Gordon & Breach, New York, 1965).

[13] B. S. De Witt, *The Spacetime Approach to Quantum Field Theory*, Relativity, Groups & Topology II, ed. by B. S. De Witt and R. Stora, (North-Holland, Amsterdam, 1984) pp. 381-738

[14] S. Fulling, *Aspects of Quantum Field Theory in Curved Space-time*, (Cambridge University Press, Cambridge, 1989)

[15] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Spacetime*, (Cambridge University Press, Cambridge, 1982)

[16] A. A. Grib, S. G. Mamaev, V. M. Mostepanenko, *Quantum Effects in Strong External Fields*, (Atomizdat, Moskow, 1980)

[17] C. J. Isham, *Lectures on Groups and Vector Spaces for physicists*, (World Scientific, Singapore, 1989)

[18] C. J. Isham, *Modern Differential Geometry for physicists*, (World Scientific, Singapore, 1989)

[19] M. Hamermesh, *Group Theory and its Applications to Physical Problems*, (Dover Publications, New York, 1989)

[20] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry — Methods and Applications*, (Springer, New York, 1989)

[21] V. A. Zhelnorovich, *Spinor Theory and its Application in Physics and Mechanics*, (Nauka, Moscow, 1982)

[22] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, (PWN — Polish Scientific Publishers, Warszawa, 1977)