3.2 Banach Spaces

- **Cauchy Sequence.** A sequence of vectors \((x_n)\) in a normed space is a **Cauchy sequence** if for every \(\varepsilon > 0\) there exists \(M \in \mathbb{N}\) such that for all \(n,m \geq M\),

\[\|x_m - x_n\| < \varepsilon.\]

- **Theorem 3.2.1** A sequence \((x_n)\) is a Cauchy sequence if and only if for any two subsequences \((x_{p_n})\) and \((x_{q_n})\)

\[\|x_{p_n} - x_{q_n}\| \to 0\] as \(n \to \infty\).

- Every convergent sequence is Cauchy.

- Not every Cauchy sequence in a normed space \(E\) converges to a vector in \(E\).

- **Example.** Incompleteness.

- **Lemma 3.2.1** Let \((x_n)\) be a Cauchy sequence of vectors in a normed space. Then the sequence \((\|x_n\|)\) of real numbers converges.

**Proof:**

1. We have \(|\|x_m\| - \|x_n\|| \leq \|x_m - x_n\|\). Thus \((\|x_n\|)\) is Cauchy.

- Every Cauchy sequence is bounded.

Proof: Exercise.

- **Banach Space.** A normed space \(E\) is **complete** (or **Banach space**) if every Cauchy sequence in \(E\) converges to an element in \(E\).

- **Examples.** \(\mathbb{R}^n\) and \(\mathbb{C}^n\) (with any norm) are complete.

- **Theorem 3.2.2** The spaces \(L^p(\Omega)\) with \(p \geq 1\) and \(l^p\) are complete.

- **Theorem 3.2.3** Completeness of \(l^2\). The space of complex sequences with the norm \(\|\cdot\|_2\) is complete.

**Proof:** No proof.
• **Theorem 3.2.4 Completeness of** $C(\Omega)$. The space of complex-valued continuous functions on a closed bounded set $\Omega \subset \mathbb{R}^n$ with the norm

\[ \| f \|_\infty = \sup_{x \in \Omega} |f(x)| \]

is complete.

**Proof:** No proof.

• **Convergent and Absolutely Convergent Series.** A series $\sum_{n=1}^{\infty} x_n$ converges in a normed space $E$ if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges in $E$.

• If $\sum_{n=1}^{\infty} \| x_n \| < \infty$, then the series converges absolutely.

• An absolutely convergent series does not need to converge.

• **Theorem 3.2.5** A normed space is complete if and only if every absolutely convergent series converges.

**Proof:** in functional analysis books.

• **Theorem 3.2.6** A closed vector subspace of a Banach space is a Banach space.

**Proof:** no proof.

• **Completion of Normed Spaces**

• Let $(E, \| \cdot \|)$ be a normed space. A normed space $(\tilde{E}, \| \cdot \|_1)$ is a completion of $(E, \| \cdot \|)$ if there exists a linear injection $\Phi : E \rightarrow \tilde{E}$, such that

1. for every $x \in E$

\[ \| x \|_1 = \| \Phi(x) \|_1, \]

2. $\Phi(E)$ is dense in $\tilde{E}$,

3. $\tilde{E}$ is complete.

• The space $\tilde{E}$ is defined as the set of equivalence classes of Cauchy sequences in $E$.

• Recall that if $(x_n)$ is Cauchy then the sequence $\| x_n \|$ converges.
• We define an **equivalence relation** of Cauchy sequences as follows.

• Two Cauchy sequences \((x_n)\) and \((y_n)\) in \(E\) are **equivalent**, \((x_n) \sim (y_n)\), if

\[
\lim_{n \to \infty} \| x_n - y_n \| = 0.
\]

• The set of Cauchy sequences \((y_n)\) equivalent to a Cauchy sequence \((x_n)\) is the **equivalence class** of \((x_n)\)

\[
[(x_n)] = \{(y_n) \in E \mid (y_n) \sim (x_n)\}
\]

• Obviously we can identify the space \(E\) with the set of constant Cauchy sequences.

• Then the set of all equivalent classes is

\[
\tilde{E} = E / \sim = \{[(x_n)] \mid (x_n) \text{ is Cauchy sequence in } E\}
\]

• The addition and multiplication by scalars in \(\tilde{E}\) are defined by

\[
[(x_n)] + [(y_n)] = [(x_n + y_n)], \quad \lambda[(x_n)] = [(\lambda x_n)]
\]

• The norm in \(\tilde{E}\) is defined by the limit

\[
\| [(x_n)] \|_1 = \lim_{n \to \infty} \| x_n \|,
\]

which exists for every Cauchy sequence.

• This definition is consistent since for any two equivalent Cauchy sequences \((x_n)\) and \((y_n)\)

\[
\| [(x_n)] \|_1 = \| [(y_n)] \|_1
\]

Proof: Exercise.

• The linear injection \(\Phi : E \to \tilde{E}\) is defined by a constant sequence

\[
\Phi(x) = [(x_n)] \text{ such that } x_n = x, \forall n \in \mathbb{N}.
\]

• Then \(\Phi\) is one-to-one.

Proof: Exercise.
• Obviously, \( \forall x \in E, \| x \| = \| \Phi(x) \|_1 \).

• Claim: \( \Phi(E) \) is dense in \( \tilde{E} \).

  Proof: Since every element \([(x_n)]\) of \( \tilde{E} \) is the limit of a sequence \( (\Phi(x_n)) \).

• Claim: \( \tilde{E} \) is complete.

  Proof:

  Let \((\tilde{x}_n)\) be a Cauchy sequence in \( \tilde{E} \).

  Then \( \exists (x_n) \) in \( E \) such that

  \[
  \| \Phi(x_n) - \tilde{x}_n \|_1 < \frac{1}{n}.
  \]

• Claim: \((x_n)\) is Cauchy sequence in \( E \).

  Proof:

  \[
  \| x_n - x_m \| = \| \Phi(x_n) - \Phi(x_m) \|_1 \\
  \leq \| \Phi(x_n) - \tilde{x}_n \|_1 + \| \tilde{x}_n - \tilde{x}_m \|_1 + \| \tilde{x}_m - \Phi(x_m) \|_1 \\
  \leq \| \tilde{x}_n - \tilde{x}_m \|_1 + \frac{1}{n} + \frac{1}{m}.
  \]

• Next, let \( \tilde{x} = [(x_n)] \).

• Claim:

  \( \| \tilde{x}_n - \tilde{x} \|_1 \to 0. \)

  Proof:

  \[
  \| \tilde{x}_n - \tilde{x} \|_1 \leq \| \tilde{x}_n - \Phi(x_n) \|_1 + \| \Phi(x_n) - \tilde{x} \|_1 \\
  < \frac{1}{n} + \| \Phi(x_n) - \tilde{x} \|_1 \to 0.
  \]

• **Homeomorphism.** Two topological spaces \( E_1 \) and \( E_2 \) are **homeomorphic** if there exists a bijection \( \Psi : E_1 \to E_2 \) from \( E_1 \) onto \( E_2 \) such that both \( \Psi \) and \( \Psi^{-1} \) are continuous.
• **Isomorphism of Normed Spaces.** Two normed spaces \((E_1, \| \cdot \|_1)\) and \((E_2, \| \cdot \|_2)\) are **isomorphic** if there exists a linear homeomorphism \(\Psi : E_1 \to E_2\) from \(E_1\) onto \(E_2\).

• **Theorem 3.2.7** Any two completions of a normed space are isomorphic.

  Proof: Read elsewhere.
3.3 Linear Mappings

- Let $L : E_1 \to E_2$ be a mapping from a vector space $E_1$ into a vector space $E_2$.
- If $x \in E_1$, then $L(x)$ is the image of the vector $x$.
- If $A \subset E_1$ is a subset of $E_1$, then the set
  \[ L(A) = \{ y \in E_2 \mid y = L(x) \text{ for some } x \in A \} \]
is the image of the set $A$.
- If $B \subset E_2$ is a subset of $E_2$, then the set
  \[ L^{-1}(B) = \{ x \in E_1 \mid L(x) \in B \} \]
is the inverse image of the set $B$.
- A mapping $L : D(L) \to E_2$ may be defined on a proper subset (called the domain) $D(L) \subset E_1$ of the vector space $E_1$.
- The image of the domain, $L(D(L))$, of a mapping $L$ is the range of $L$. That is the range of $L$ is
  \[ R(L) = \{ y \in E_2 \mid y = L(x) \text{ for some } x \in D(L) \} \, . \]
- The null space $N(L)$ (or the kernel $\text{Ker}(L)$) of a mapping $L$ is the set of all vectors in the domain $D(L)$ which are mapped to zero, that is
  \[ N(L) = \{ x \in D(L) \mid L(x) = 0 \} \, . \]
- The graph $\Gamma(L)$ of a mapping $L$ is the set of ordered pairs $(x, L(x))$, that is
  \[ \Gamma(L) = \{ (x, y) \subset E_1 \times E_2 \mid x \in D(L) \text{ and } y = L(x) \} \, . \]
- **Continuous Mappings.** A mapping $f : E_1 \to E_2$ from a normed space $E_1$ into a normed space $E_2$ is continuous at $x_0 \in E_1$ if any sequence $(x_n)$ in $E_1$ converging to $x_0$ is mapped to a sequence $f(x_n)$ in $E_2$ that converges to $f(x_0)$. 
That is \( f \) is continuous at \( x_0 \) if
\[
\| x_n - x_0 \| \to 0 \text{ implies } \| f(x_n) - f(x_0) \| \to 0.
\]

A mapping \( f : E_1 \to E_2 \) is **continuous** if it is continuous at every \( x \in E_1 \).

**Theorem 3.3.1** *The norm \( \| \cdot \| : E \to \mathbb{R} \) in a normed space \( E \) is a continuous mapping from \( E \) into \( \mathbb{R} \).*

**Proof:** If \( \| x_n - x \| \to 0 \), then
\[
| \| x_n \| - \| x \| | \leq \| x_n - x \| \to 0
\]

**Theorem 3.3.2** Let \( f : E_1 \to E_2 \) be a mapping from a normed space \( E_1 \) into a normed space \( E_2 \). The following conditions are equivalent:

1. \( f \) is continuous.
2. The inverse image of any open set of \( E_2 \) is open in \( E_1 \).
3. The inverse image of any closed set of \( E_2 \) is closed in \( E_1 \).

**Proof:** *Exercise.*

**Linear Mappings.** Let \( S \subseteq E_1 \) be a subset of a vector space \( E_1 \). A mapping \( L : S \to E_2 \) is **linear** if \( \forall x, y \in S \) and \( \forall \alpha, \beta \in \mathbb{F} \) such that \( \alpha x + \beta y \in S \),
\[
L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).
\]

**Theorem 3.3.3** If \( S \) is not a vector subspace of \( E_1 \), then there is a unique extension of \( L : S \to E_2 \) to a linear mapping \( \tilde{L} : \text{span} S \to E_2 \) from the vector subspace \( \text{span} S \) to \( E_2 \).

**Proof:** The extension \( \tilde{L} \) is defined by linearity.

Thus, one can always assume that the domain of a linear mapping is a vector space.

**Theorem 3.3.4** *The range, the null space and the graph of a linear mapping are vector spaces.*

**Proof:** *Exercise.*
3.3. LINEAR MAPPINGS

• For any linear mapping $L$, $L(0) = 0$. Thus, $0 \in N(L)$ and the null space $N(L)$ is always nonempty.

• **Theorem 3.3.5** A linear mapping $L : E_1 \rightarrow E_2$ from a normed space $E_1$ into a normed space $E_2$ is continuous if and only if it is continuous at a point.

  **Proof:**
  1. Assume $L$ is continuous at $x_0 \in E_1$.
  2. Let $x \in E_1$ and $(x_n) \rightarrow x$.
  3. Then $(x_n - x + x_0) \rightarrow x_0$.
  4. Thus
     \[ \| L(x_n) - L(x) \| = \| L(x_n - x + x_0) - L(x_0) \| \rightarrow 0 \]

• **Bounded Linear Mappings.** A linear mapping $L : E_1 \rightarrow E_2$ from a normed space $E_1$ into a normed space $E_2$ is **bounded** if there is a real number $K \in \mathbb{R}$ such that for all $x \in E_1$,

  \[ \| L(x) \| \leq K \| x \| . \]

• **Theorem 3.3.6** A linear mapping $L : E_1 \rightarrow E_2$ from a normed space $E_1$ into a normed space $E_2$ is continuous if and only if it is bounded.

  **Proof:**
  1. (I). Assume that $L$ is bounded.
  2. Claim: $L$ is continuous at 0.
  3. Indeed, $x_n \rightarrow 0$ implies
     \[ \| L(x_n) \| \leq K \| x_n \| \rightarrow 0 \]
  4. Hence, $L$ is continuous.
  5. (II). Assume that $L$ is continuous.
  6. By contradiction, assume that $L$ is unbounded.
  7. Then, there is a sequence $(x_n)$ in $E_1$ such that
     \[ \| L(x_n) \| > n \| x_n \| \]
8. Let
\[ y_n = \frac{x_n}{n \| x_n \|}, \quad n \in \mathbb{N} \]

9. Then
\[ \| y_n \| = \frac{1}{n}, \quad \text{and} \quad \| L(y_n) \| > 1 \]

10. Then \( y_n \to 0 \) but \( L(y_n) \not\to 0 \).

11. Thus, \( L \) is not continuous at zero.

\[ \blacksquare \]

**Remark.** For linear mappings, continuity and uniform continuity are equivalent.

- The set \( L(E_1, E_2) \) of all linear mappings from a vector space \( E_1 \) into a vector space \( E_2 \) is a vector space with the addition and multiplication by scalars defined by
  \[ (L_1 + L_2)(x) = L_1(x) + L_2(x), \quad \text{and} \quad (\alpha L)(x) = \alpha L(x) \]

- The set \( B(E_1, E_2) \) of all bounded linear mappings from a normed space \( E_1 \) into a normed space \( E_2 \) is a vector subspace of the space \( L(E_1, E_2) \).

- **Theorem 3.3.7** The space \( B(E_1, E_2) \) of all bounded linear mappings \( L : E_1 \to E_2 \) from a normed space \( E_1 \) into a normed space \( E_2 \) is a normed space with norm defined by
  \[ \| L \| = \sup_{x \in E_1, x \neq 0} \frac{\| L(x) \|}{\| x \|} = \sup_{x \in E_1, \| x \| = 1} \| L(x) \| \]

**Proof:**

1. Obviously, \( \| L \| \geq 0 \).
2. \( \| L \| = 0 \) if and only if \( L = 0 \).
3. Claim: \( \| L \| \) satisfies triangle inequality.
4. Let \( L_1, L_2 \in B(E_1, E_2) \).
5. Then
\[
\| L_1 + L_2 \| = \sup_{\|x\| = 1} \| L_1(x) + L_2(x) \| \\
\leq \sup_{\|x\| = 1} \| L_1(x) \| + \sup_{\|x\| = 1} \| L_2(x) \| \\
= \| L_1 \| + \| L_2 \| 
\]

- For any bounded linear mapping \( L : E_1 \to E_2 \)
  \[\| L(x) \| \leq \| L \| \| x \| , \forall x \in E_1.\]
- \( \| L \| \) is the least real number \( K \) such that
  \[\| L(x) \| \leq K \| x \| \text{ for all } x \in E_1.\]
- The norm defined by \( \| L \| = \sup_{x \in E_1 : \| x \| = 1} \| L(x) \| \) is called the operator norm.
- Convergence with respect to the operator norm is called the uniform convergence of operators.
- **Strong convergence.** A sequence of bounded linear mappings \( L_n \in B(E_1, E_2) \) converges strongly to \( L \in B(E_1, E_2) \) if for every \( x \in E_1 \) we have
  \[\| L_n(x) - L(x) \| \to 0 \text{ as } n \to \infty.\]
- **Theorem 3.3.8** Uniform convergence implies strong convergence.
  Proof: Follows from
  \[\| L_n(x) - L(x) \| \leq \| L_n - L \| \| x \| \]
- Converse is not true.
- **Theorem 3.3.9** Let \( E_1 \) be a normed space and \( E_2 \) be a Banach space. Then \( B(E_1, E_2) \) is a Banach space.
  **Proof:** see functional analysis books.
• **Theorem 3.3.10** Let $E_1$ be a normed space and $E_2$ be a Banach space. Let $S \subset E_1$ be a subspace of $E_1$ and $L : S \to E_2$ be a continuous linear mapping from $S$ into $E_2$. Then:

1. $L$ has a unique extension to a continuous linear mapping $\tilde{L} : \bar{S} \to E_2$ defined on the closure of the domain of the mapping $L$.
2. If $S$ is dense in $E_1$, then $L$ has a unique extension to a continuous linear mapping $\tilde{L} : E_1 \to E_2$.

**Proof:** see functional analysis.

• **Theorem 3.3.11** Let $E_1$ and $E_2$ be normed spaces, $S \subset E_1$ be a subspace of $E_1$ and $L : S \to E_2$ be a continuous linear mapping from $S$ into $E_2$. Then:

1. the null space $N(L)$ is a closed subspace of $E_1$.
2. If the domain $S$ is a closed subspace of $E_1$, then the graph $\Gamma(L)$ of $L$ is a closed subspace of $E_1 \times E_2$.

**Proof:** Exercise.

• A bounded linear mapping $L : E \to \mathbb{F}$ from a normed space $E$ into the scalar field $\mathbb{F}$ is called a **functional**.

• The space $B(E, \mathbb{F})$ of functionals is called the **dual space** and denoted by $E'$ or $E^*$.

• The dual space is always a Banach space.

Proof: since $\mathbb{F}$ is complete.
3.4 Banach Fixed Point Theorem

- **Contraction Mapping.** Let $E$ be a normed space and $A \subset E$ be a subset of $E$. A mapping $f : A \to E$ from $A$ into $E$ is a **contraction mapping** if there exists a real number $\alpha$, such that $0 < \alpha < 1$ and $\forall x, y \in A$

  $|| f(x) - f(y) || \leq \alpha || x - y ||$.

- A contraction mapping is continuous.
  
  Proof: Exercise.

- If $\forall x, y \in A$

  $|| f(x) - f(y) || \leq || x - y ||$,

  then it is not necessarily a contraction since the contraction constant $\alpha$ may not exist.

- **Theorem 3.4.1 Banach Fixed Point Theorem.** Let $E$ be a Banach space and $A \subset E$ be a closed subset of $E$. Let $f : A \to A$ be a contraction mapping from $A$ into $A$. Then there exists a unique $z \in A$ such that $f(z) = z$.

- **Examples.**

**Homework**

- Exercises: DM[9,10,11,12,14,21,23,26,31,34,36,37,38,39,42,44]