MATH 332: Vector and Tensor Analysis

Vector Algebra

Scalar Product

$$\boxed{\mathbf{A}\cdot\mathbf{B} = |\mathbf{B}||\mathbf{A}|\cos\theta}$$

Commutativity

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

Magnitude

$$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$$

Vector Product

$$\boxed{\mathbf{A}\times\mathbf{B} = \mathbf{n}\,|\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\sin\theta}$$

Anti-commutativity

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\times\mathbf{A} = 0$$

Scalar Triple Product

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$$

Cyclic symmetry

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$

Vector Triple Product

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})}$$

Kronecker Symbol

$$\boxed{\delta_{ik} = \delta^i_k = \delta^k_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}}$$

Symmetry

$$\delta_{jk} = \delta_{kj}$$

Unity Matrix

$$I = (\delta_{ik}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformation of Rectangular Coordinates

$$x'_i = \alpha_{i'k} x_k + x_{(0)\,i}$$

Summation from 1 to 3 is assumed over all *repeated* indices

Vector Form

$$x' = (x'_i) = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}, \qquad x = (x_i) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Transformation matrix

$$\alpha = (\alpha_{i'k}) = \begin{pmatrix} \alpha_{1'1} & \alpha_{1'2} & \alpha_{1'3} \\ \alpha_{2'1} & \alpha_{2'2} & \alpha_{2'3} \\ \alpha_{3'1} & \alpha_{3'2} & \alpha_{3'3} \end{pmatrix}$$

Matrix Form of the Transformation

$$x' = \alpha x + x_0$$

Orthonormal Basis $\mathbf{i}_k$, $\quad (k = 1, 2, 3)$

$$\boxed{\mathbf{i}_k\cdot\mathbf{i}_j = \delta_{kj}}$$

Orientation of Orthonormal Basis:
right-handed if $(\mathbf{i}_1\times\mathbf{i}_2)\cdot\mathbf{i}_3 = +1$
left-handed if $(\mathbf{i}_1\times\mathbf{i}_2)\cdot\mathbf{i}_3 = -1$

Transformation of Orthonormal Basis

$$\mathbf{i}'_j = \alpha_{j'k}\,\mathbf{i}_k\,, \qquad \mathbf{i}_k = \alpha_{j'k}\,\mathbf{i}'_j$$

$$\alpha_{j'k} = \cos(\mathbf{i}'_j,\,\mathbf{i}_k)$$

Orthogonality condition

$$\boxed{\alpha_{i'k}\alpha_{j'k} = \delta_{ij}\,, \qquad \alpha_{k'i}\alpha_{k'j} = \delta_{ij}}$$

Matrix Form of Orthogonality Condition

$$\alpha\alpha^T = I$$

T means *transposition* of a matrix (replacement of rows by columns)

Proper transformation (no change of orientation)

$$\det(\alpha_{j'k}) = 1$$

Improper transformation (changes orientation)

$$\det(\alpha_{j'k}) = -1$$

Cartesian Vectors

$$\mathbf{A} = A_k\,\mathbf{i}_k\,, \qquad A_k = \mathbf{A}\cdot\,\mathbf{i}_k$$

Scalar Product in Cartesian Components

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_iB_i}$$

$$|\mathbf{A}|^2 = A_iA_i$$

Vector Product

$$\boxed{\mathbf{A}\times\mathbf{B} = \det\begin{pmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}}$$

Scalar Triple Product

$$\boxed{(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}}$$

Levi-Civita (Alternating) Symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3), \\ & (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3), \\ & (3,2,1), (1,3,2) \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{ijk} = \varepsilon^{ijk}$$

Anti-symmetry

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$$

Cyclic symmetry

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

Orthonormal basis

$$\boxed{\mathbf{i}_j\times\,\mathbf{i}_k = \varepsilon_{jkl}\,\mathbf{i}_l}$$

$$\boxed{(\,\mathbf{i}_j\times\,\mathbf{i}_k)\cdot\,\mathbf{i}_l = \varepsilon_{jkl}}$$

Vector Product in Cartesian Components

$$\boxed{(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}A_jB_k}$$

Scalar Triple Product

$$\boxed{(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \varepsilon_{ijk}A_iB_jC_k}$$

Tensor Notation

$$\delta_{ii} = \delta^i_i = 3$$

$$\delta^j_i A_j = A_i$$

$$\delta^{ij}A_iB_j = A_iB_i = \mathbf{A}\cdot\mathbf{B}$$

$$\varepsilon_{ijj} = \varepsilon_{jij} = \varepsilon_{jji} = 0$$

$$\varepsilon_{ijk}\delta_{ij} = \varepsilon_{ijk}\delta_{ik} = \varepsilon_{ijk}\delta_{jk} = 0$$

$$\varepsilon_{ijk}A_jA_k = \varepsilon_{ijk}A_iA_k = \varepsilon_{ijk}A_iA_j = 0$$

$$\begin{aligned}\varepsilon_{ijk}\varepsilon^{mnl} &= \delta^m_i\delta^n_j\delta^l_k + \delta^m_j\delta^n_k\delta^l_i + \delta^m_k\delta^n_i\delta^l_j \\ &\quad - \delta^m_i\delta^n_k\delta^l_j - \delta^m_j\delta^n_i\delta^l_k - \delta^m_k\delta^n_j\delta^l_i\end{aligned}$$

$$\boxed{\varepsilon_{ijk}\varepsilon^{mnk} = \delta^m_i\delta^n_j - \delta^m_j\delta^n_i}$$

$$\boxed{\varepsilon_{ijk}\varepsilon^{mjk} = 2\delta^m_i}$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6$$

Cartesian Tensors

(tensors in rectangular coordinates)

Scalar: 0-tensor φ
Vector: 1-tensor A_i
2-tensor A_{ik}
n-tensor $A_{i_1\dots i_n}$

Transformation Laws

$$\varphi' = \varphi$$

$$\boxed{A'_i = \alpha_{i'k}A_k}$$

$$\boxed{A'_{ij} = \alpha_{i'k}\alpha_{j'l}A_{kl}}$$

$$A'_{i_1\dots i_n} = \alpha_{i'_1 j_1}\cdots\alpha_{i'_n j_n}A_{j_1\dots j_n}$$

Matrix Form of the Transformation Laws
Vector

$$A = (A_i) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$A' = \alpha A$$

Matrix Form of a 2-tensor

$$A = (A_{ik}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A' = \alpha A \alpha^T$$

Stress Tensor p_{ik}
Stress

$$\boxed{p_i = p_{ik}n_k}$$

n_i unit normal
Moment of Inertia Tensor

$$I_{ik} = \sum_{j=1}^{N} m_j\left[\delta_{ik}x^{(j)}_l x^{(j)}_l - x^{(j)}_i x^{(j)}_k\right]$$

Angular Momentum

$$\boxed{L_i = I_{ik}\omega_k}$$

ω_k angular velocity
Deformation Tensor

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

u_i displacement vector

Rate of Deformation Tensor

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

v_i velocity vector field

Isotropic Tensors (built from δ_{ik} only; no preferred directions)
Isotropic 2-tensor

$$A_{ik} = p\delta_{ik}$$

Isotropic 4-tensor

$$A_{iklm} = p\delta_{ik}\delta_{lm} + \rho\delta_{il}\delta_{km} + \lambda\delta_{im}\delta_{kl}$$

Tensor Algebra

Tensor Product

$$C_{ik} = A_i B_k, \qquad C_{ijkl} = A_{ij} B_{kl}$$

Contraction

$$D_{jkl} = C_{iijkl}$$

Trace

$$\boxed{A = A_{ii}}$$

Inner Product

$$\boxed{C_i = A_{ik} B_k}, \qquad D_{ij} = C_{ijkl} B_{kl}$$

Symmetric Tensors

$$S_{ij} = S_{ji}$$

$$(S_{ik}) = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}$$

Anti-symmetric Tensors

$$A_{ij} = -A_{ji}$$

$$(A_{ik}) = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix}$$

Contraction of antisymmetric tensor

$$A_{ii} = 0$$

Symmetrization

$$\boxed{T_{(ij)} = \frac{1}{2}(T_{ik} + T_{ki})}$$

Antisymmetrization

$$\boxed{T_{[ij]} = \frac{1}{2}(T_{ik} - T_{ki})}$$

Decomposition of 2-Tensor

$$\boxed{T_{ik} = T_{(ik)} + T_{[ij]}}$$

Duality

(Equivalence of antisymmetric 2-tensor to an axial vector)

$$\boxed{\tilde{A}_i = \frac{1}{2}\varepsilon_{ijk} A_{jk}, \qquad A_{ij} = \varepsilon_{ijk} \tilde{A}_k}$$

$$\tilde{A}_1 = A_{23}, \qquad \tilde{A}_2 = A_{31}, \qquad \tilde{A}_3 = A_{12},$$

Principal Axes

Eigenvalues $\lambda_{(r)}$ (characteristic values) and **Eigenvectors** $\mathbf{n}_{(r)}$ (characteristic or principal directions)

$$T_{ik} n_k^{(r)} = \lambda_{(r)} n_i^{(r)}$$

$$|\mathbf{n}_{(r)}| = 1$$

Characteristic Equation

$$\boxed{\det(T_{ik} - \lambda\delta_{ik}) = 0}$$

$$\det \begin{pmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{pmatrix} = 0$$

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Invariants of a 2-Tensor: I_k

$$I_1 = T_{ii} = T_{11} + T_{22} + T_{33}$$

$$I_2 = \det \begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix} + \det \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$+\det\begin{pmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{pmatrix}$$

$$I_3 = \det(T_{ik}) = \det\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

The eigenvalues λ_r, $(r = 1, 2, 3)$, of a symmetric 2-tensor are *real*

A symmetric 2-tensor has three *orthogonal* principal axes $\mathbf{n}_{(r)}$, $(r = 1, 2, 3)$

$$\mathbf{n}_{(r)} \cdot \mathbf{n}_{(p)} = \delta_{rp}$$

In the principal axes a symmetric 2-tensor T_{ik} has *diagonal* matrix

$$\boxed{(T_{ik}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}$$

Decomposition in terms of orthonormal eigenvectors $\mathbf{n}^{(r)}$ and eigenvalues $\lambda_{(r)}$

$$\boxed{T_{ik} = \sum_{r=1}^{3} \lambda_{(r)} n_i^{(r)} n_k^{(r)}}$$

Traceless Tensors (Deviators)

$$\overline{T}_{ik} = T_{ik} - \frac{1}{3}\delta_{ik} T$$

where $T = T_{jj}$ (trace)

$$\overline{T} = \overline{T}_{ii} = 0$$

Decomposition

$$\boxed{T_{ik} = \overline{T}_{ik} + \frac{1}{3}\delta_{ik} T_{jj}}$$

Curvilinear Coordinates

$$q^i = q^i(\mathbf{r}), \qquad (i = 1, 2, 3)$$

Cartesian Coordinates

$$x_i = x_i(q)$$

Radius vector

$$\boxed{\mathbf{r}(q) = x_k(q)\,\mathbf{i}_k = x_1(q)\,\mathbf{i}_1 + x_2(q)\,\mathbf{i}_2 + x_3(q)\,\mathbf{i}_3}$$

Basis (tangent vectors to coordinate curves)

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial q^i}$$

Transformation of basis

$$\mathbf{e}'_j = \alpha^k{}_{j'}\,\mathbf{e}_k\,, \qquad \mathbf{e}_k = \alpha^{j'}{}_k\,\mathbf{e}'_j$$

$$\alpha^{i'}{}_k \alpha^k{}_{j'} = \delta^i_j\,, \qquad \alpha^i{}_{k'} \alpha^{k'}{}_j = \delta^i_j$$

Orientation:
right-handed if $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0$
and left-handed if $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 < 0$

Reciprocal Basis

$$\boxed{\mathbf{e}^j \cdot \mathbf{e}_k = \delta^j_k}$$

$$\mathbf{e}^j = \frac{\mathbf{e}_k \times \mathbf{e}_l}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3}$$

where $(j, k, l) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$

Contravariant Components

$$\mathbf{A} = A^i\,\mathbf{e}_i\,, \qquad \boxed{A^i = \mathbf{A} \cdot \mathbf{e}^i}$$

Covariant components

$$\mathbf{A} = A_i\,\mathbf{e}^i\,, \qquad \boxed{A_i = \mathbf{A} \cdot \mathbf{e}_i}$$

Transformation of components

$$A'^i = \alpha^{i'}{}_k A^k \,, \qquad A'_i = \alpha^k{}_{i'} A_k$$

Metric Tensor

$$\boxed{g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k = \frac{\partial \mathbf{r}}{\partial q^i} \cdot \frac{\partial \mathbf{r}}{\partial q^k} \,, \qquad g^{ik} = \mathbf{e}^i \cdot \mathbf{e}^k}$$

Symmetry

$$g_{ik} = g_{ki} \,, \qquad g^{ik} = g^{ki}$$

$$\boxed{g_{ik} g^{kn} = \delta_i^n}$$

Determinant of the Metric Tensor

$$G = \det(g_{ik}) \,, \qquad \det(g^{ik}) = \frac{1}{G}$$

The matrix g^{ik} is the inverse matrix of the matrix g_{ik} given by

$$g^{ik} = (-1)^{ik} \frac{\det M^{ik}}{G}$$

where M^{ik} is a 2×2 matrix obtained from the 3×3 matrix g_{ij} by removing the i-th row and the k-th column

Relations between components

$$\boxed{A_i = g_{ik} A^k \,, \qquad A^i = g^{ik} A_k}$$

Displacement

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i} dq^i$$

Line Element (Arc Length)

$$ds^2 = d\,\mathbf{r} \cdot d\,\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i} \cdot \frac{\partial \mathbf{r}}{\partial q^k}\, dq^i\, dq^k$$

$$\boxed{ds^2 = g_{ik}\, dq^i\, dq^k}$$

Volume Element

$$\boxed{dV = \sqrt{G}\, dq^1\, dq^2\, dq^3, \qquad G = \det(g_{ik})}$$

Tensors in Curvilinear Coordinate System

Contravariant components A^{ik}
Covariant components A_{ik}
Mixed components $A^i{}_k$, $A_i{}^k$
Relations

$$A_{ik} = g_{in} A^n{}_k = g_{kn} A_i{}^n = g_{in} g_{km} A^{nm}$$

$$A^{ik} = g^{in} A_n{}^k = g^{kn} A^i{}_n = g^{in} g^{km} A_{nm}$$

$$A^i{}_k = g^{in} A_{nk} = g_{kn} A^{in}$$

etc

Orthogonal Coordinate System

$$\frac{\partial \mathbf{r}}{\partial q^i} \cdot \frac{\partial \mathbf{r}}{\partial q^k} = 0, \qquad \text{if } i \neq k$$

Basis

$$\mathbf{e}_i \cdot \mathbf{e}_k = \mathbf{e}^i \cdot \mathbf{e}^k = 0 \quad \text{if } i \neq k$$

$$|\,\mathbf{e}_i| = h_i, \qquad |\,\mathbf{e}^i| = \frac{1}{h_i}$$

Metric

$$g_{ik} = 0, \qquad \text{if } i \neq k$$

$$g_{ii} = h_i^2 \quad \text{(no summation!)}$$

$$g^{ik} = 0, \qquad \text{if } i \neq k$$

$$g^{ii} = \frac{1}{h_i^2} \quad \text{(no summation!)}$$

Metric Coefficients

$$\boxed{h_i = \left| \frac{\partial \mathbf{r}}{\partial q^i} \right|}$$

$$\boxed{h_i = \sqrt{\left(\frac{\partial x_1}{\partial q^i}\right)^2 + \left(\frac{\partial x_2}{\partial q^i}\right)^2 + \left(\frac{\partial x_3}{\partial q^i}\right)^2}}$$

Relation between covariant and contravariant components

$$A_i = h_i^2 A^i \quad \text{(no summation!)}$$

Orthonormal Basis

$$\boxed{\mathbf{e}^i = \mathbf{e}_i = \frac{\dfrac{\partial \mathbf{r}}{\partial q^i}}{\left|\dfrac{\partial \mathbf{r}}{\partial q^i}\right|}}$$

Line Element

$$\boxed{ds^2 = h_1^2\,(dq^1)^2 + h_2^2\,(dq^2)^2 + h_3^2\,(dq^3)^2}$$

Volume Element

$$\boxed{dV = h_1 h_2 h_3\, dq^1\, dq^2\, dq^3}$$

Cartesian Coordinates

Line Element

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

Volume Element

$$dV = dx_1\, dx_2\, dx_3$$

Metric Coefficients

$$h_1 = h_2 = h_3 = 1$$

Cylindrical Coordinates

$$\rho \geq 0, \qquad 0 \leq \varphi < 2\pi, \qquad -\infty < z < \infty$$

$$\rho = \sqrt{x_1^2 + x_2^2}, \qquad \tan\varphi = \frac{x_2}{x_1}, \qquad z = x_3$$

$$x_1 = \rho\cos\varphi, \qquad x_2 = \rho\sin\varphi, \qquad x_3 = z$$

Radius Vector

$$\boxed{\mathbf{r} = \rho\cos\varphi\,\mathbf{i}_1 + \rho\sin\varphi\,\mathbf{i}_2 + z\,\mathbf{i}_3}$$

Orthonormal basis

$$\mathbf{e}_\rho = \cos\varphi\,\mathbf{i}_1 + \sin\varphi\,\mathbf{i}_2$$

$$\mathbf{e}_\varphi = -\sin\varphi\,\mathbf{i}_1 + \cos\varphi\,\mathbf{i}_2$$

$$\mathbf{e}_z = \mathbf{i}_3$$

Line Element

$$\boxed{ds^2 = (d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2}$$

Volume Element

$$\boxed{dV = \rho\, d\rho\, d\varphi\, dz}$$

Metric Coefficients

$$h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1$$

Spherical Coordinates

$$r \geq 0, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \varphi < 2\pi$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \tan\theta = \frac{\sqrt{x_1^2 + x_2^2}}{x_3},$$

$$\tan\varphi = \frac{x_2}{x_1}$$

$$x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi,$$

$$x_3 = r\cos\theta$$

Radius Vector

$$\boxed{\mathbf{r} = r\sin\theta\cos\varphi\,\mathbf{i}_1 + r\sin\theta\sin\varphi\,\mathbf{i}_2 + r\cos\theta\,\mathbf{i}_3}$$

Orthonormal Basis

$$\mathbf{e}_r = \sin\theta\cos\varphi\,\mathbf{i}_1 + \sin\theta\sin\varphi\,\mathbf{i}_2 + \cos\theta\,\mathbf{i}_3$$

$$\mathbf{e}_\theta = \cos\theta\cos\varphi\,\mathbf{i}_1 + \cos\theta\sin\varphi\,\mathbf{i}_2 - \sin\theta\,\mathbf{i}_3$$

$$\mathbf{e}_{\varphi} = -\sin\varphi\,\mathbf{i}_1 + \cos\varphi\,\mathbf{i}_2$$

Line Element

$$\boxed{ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\varphi)^2}$$

Volume Element

$$\boxed{dV = r^2\sin\theta\,dr\,d\theta\,d\varphi}$$

Metric Coefficients

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta$$

Vector And Tensor Analysis

Functions of Single variable

Product Rules

$$\frac{d}{dt}(\varphi\,\mathbf{A}) = \frac{d\varphi}{dt}\mathbf{A} + \varphi\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}(\mathbf{B}\cdot\mathbf{A}) = \frac{d\mathbf{B}}{dt}\cdot\mathbf{A} + \mathbf{B}\cdot\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}(\mathbf{B}\times\mathbf{A}) = \frac{d\mathbf{B}}{dt}\times\mathbf{A} + \mathbf{B}\times\frac{d\mathbf{A}}{dt}$$

Trajectory

$$\mathbf{r}(t) = x_k(t)\,\mathbf{i}_k, \qquad a \le t \le b$$

Velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

Unit Tangent Vector

$$\mathbf{u} = \frac{d\mathbf{r}}{ds} = \frac{\dfrac{d\mathbf{r}}{dt}}{\left|\dfrac{d\mathbf{r}}{dt}\right|}$$

Speed

$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right| = \frac{ds}{dt}$$

Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Arc Length (Line Element)

$$ds = |\mathbf{v}|dt$$

Fields in Cartesian Coordinates

Partial derivatives

$$\partial_i = \frac{\partial}{\partial x_i}$$

Nabla (Del) Operator

$$\boxed{\nabla = \mathbf{i}_k\,\partial_k = \mathbf{i}_1\partial_1 + \mathbf{i}_2\partial_2 + \mathbf{i}_3\partial_3}$$

Gradient

$$\boxed{\mathbf{grad}\ f = \nabla f = \mathbf{i_k}\,\partial_k f}$$

$$(\mathbf{grad}\ f)_i = \partial_i f$$

Directional Derivative

$$\boxed{\frac{df}{ds} = \frac{d\mathbf{r}}{ds}\cdot\mathbf{grad} f = \mathbf{u}\cdot\mathbf{grad} f}$$

Flow Lines

$$\frac{d\,\mathbf{r}}{dt} = \beta\,\mathbf{F}$$

$$\boxed{\frac{dx_1}{F_1} = \frac{dx_2}{F_2} = \frac{dx_3}{F_3}}$$

Divergence

$$\boxed{\mathrm{div}\ \mathbf{F} = \nabla\cdot\mathbf{F} = \partial_i F_i}$$

Curl

$$\boxed{\textbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

$$(\textbf{curl}\,\mathbf{F})_i = \varepsilon_{ijk}\partial_j F_k$$

$$\boxed{\textbf{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{i}_l\,\varepsilon_{ljk}\partial_j F_k}$$

Laplacian

$$\boxed{\Delta = \text{div}\,\textbf{grad} = \nabla^2 = \nabla\cdot\nabla}$$

$$\boxed{\Delta = \partial_i\partial_i = \partial_1^2 + \partial_2^2 + \partial_3^2}$$

Vector identities:

$$\nabla\times(\nabla\times\mathbf{F}) = -\Delta\mathbf{F} + \nabla(\nabla\cdot\mathbf{F})$$

$$\textbf{curl curl}\,\mathbf{F} = -\Delta\mathbf{F} + \textbf{grad}\,\text{div}\,\mathbf{F}$$

$$\nabla\times\nabla\varphi = 0,$$

$$\boxed{\textbf{curl grad} = 0}$$

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0,$$

$$\boxed{\text{div}\,\textbf{curl} = 0}$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla(f\mathbf{F}) = (\nabla f)\cdot\mathbf{F} + f\nabla\mathbf{F}$$

$$\nabla\times(f\mathbf{F}) = (\nabla f)\times\mathbf{F} + f(\nabla\times\mathbf{F})$$

$$\nabla(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$$

$$\nabla f(\varphi) = \frac{df}{d\varphi}\nabla\varphi$$

$$\nabla\cdot\mathbf{F}(\varphi) = \nabla\varphi\cdot\frac{d\mathbf{F}}{d\varphi}$$

$$\nabla\times\mathbf{F}(\varphi) = \nabla\varphi\times\frac{d\mathbf{F}}{d\varphi}$$

$$\boxed{\nabla\mathbf{r} = 3}$$

$$\boxed{\nabla\times\mathbf{r} = 0}$$

$$r = |\,\mathbf{r}\,| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\boxed{\nabla r = \frac{\mathbf{r}}{r}}, \qquad |\nabla r| = 1$$

$$\nabla f(r) = \frac{df}{dr}\frac{\mathbf{r}}{r}$$

$$\nabla\cdot\mathbf{F}(r) = \frac{\mathbf{r}}{r}\cdot\frac{d\mathbf{F}}{dr}$$

$$\nabla\times\mathbf{F}(r) = \frac{\mathbf{r}}{r}\times\frac{d\mathbf{F}}{dr}$$

$$(\mathbf{F}\cdot\nabla)\mathbf{r} = \mathbf{F}$$

Fields in Orthogonal Coordinate System (in orthonormal basis $\mathbf{e}_i$)

Vector Components

$$F_i = \mathbf{F}\cdot\mathbf{e}_i$$

$$\textbf{grad}\,f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial q^1}f + \mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial q^2}f + \mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial q^3}f$$

$$\text{div}\,\mathbf{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}(h_2h_3F_1) + \frac{\partial}{\partial q^2}(h_3h_1F_2) + \frac{\partial}{\partial q^3}(h_1h_2F_3)\right\}$$

$$\textbf{curl}\,\mathbf{F} = \mathbf{e}_1\frac{1}{h_2h_3}\left[\frac{\partial}{\partial q^2}(h_3F_3) - [\frac{\partial}{\partial q^3}(h_2F_2)\right]$$

$$+\mathbf{e}_2\frac{1}{h_3h_1}\left[\frac{\partial}{\partial q^3}(h_1F_1) - [\frac{\partial}{\partial q^1}(h_3F_3)\right]$$

$$+\mathbf{e}_3\frac{1}{h_1h_2}\left[\frac{\partial}{\partial q^1}(h_2F_2)-[\frac{\partial}{\partial q^2}(h_1F_1)\right]$$

$$\Delta f=\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)\right.$$

$$\left.+\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)+\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)\right\}$$

Cylindrical Coordinates:

$$\mathbf{grad}\, f=\mathbf{e}_\rho\partial_\rho f+\mathbf{e}_\varphi\frac{1}{\rho}\partial_\varphi f+\mathbf{e}_z\partial_z f$$

$$\operatorname{div}\mathbf{F}=\frac{1}{\rho}\partial_\rho(\rho F_\rho)+\frac{1}{\rho}\partial_\varphi F_\varphi+\partial_z F_z$$

$$\mathbf{curl}\,\mathbf{F}=\frac{1}{\rho}\begin{vmatrix}\mathbf{e}_\rho & \rho\mathbf{e}_\varphi & \mathbf{e}_z\\ \partial_\rho & \partial_\varphi & \partial_z\\ F_\rho & \rho F_\varphi & F_z\end{vmatrix}$$

$$\Delta f=\frac{1}{\rho}\,\partial_\rho\left(\rho\,\partial_\rho f\right)+\frac{1}{\rho^2}\partial_\varphi^2 f+\partial_z^2 f$$

Spherical Coordinates

$$\mathbf{grad}\, f=\mathbf{e}_r\partial_r f+\mathbf{e}_\theta\frac{1}{r}\partial_\theta f+\mathbf{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi f$$

$$\operatorname{div}\mathbf{F}=\frac{1}{r^2}\partial_r(r^2F_r)+\frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, F_\theta)$$
$$+\frac{1}{r\sin\theta}\partial_\varphi F_\varphi$$

$$\mathbf{curl}\,\mathbf{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\varphi\\ \partial_r & \partial_\theta & \partial_\varphi\\ F_r & rF_\theta & r\sin\theta F_\varphi\end{vmatrix}$$

$$\Delta f=\frac{1}{r^2}\partial_r(r^2\partial_r f)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f)$$
$$+\frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f$$

Integrals

Parametrization of a curve C

$$\mathbf{r}\;=\;\mathbf{r}\,(t),\qquad a\le t\le b$$

Line Integrals

$$\int_C\mathbf{F}\cdot d\mathbf{r}=\int_a^b\mathbf{F}(x(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

Circulation of vector field along a closed contour

$$\oint_C\mathbf{F}\cdot d\mathbf{r}$$

Parametrization of a surface S

$$\mathbf{r}=\mathbf{r}(u,v),$$

$$a\le u\le b,\qquad c\le v\le d$$

Unit Normal

$$\boxed{\mathbf{n}=\frac{\partial_u\mathbf{r}\times\partial_v\mathbf{r}}{|\partial_u\mathbf{r}\times\partial_v\mathbf{r}|}}$$

For a surface given by

$$f(\,\mathbf{r}\,)=C$$

the Unit Normal is

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}$$

Surface Element

$$\boxed{d\mathbf{S} = \mathbf{n}\, dS = \partial_u \mathbf{r} \times \partial_v \mathbf{r}\, du\, dv}$$

$$\boxed{d\,S = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}|\ du\, dv}$$

For a surface given by

$$z = f(x, y)$$

$$a \le x \le b,\ y_1(x) \le y \le y_2(x)$$

the Surface Element is

$$\boxed{dS = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dy\, dx}$$

Surface Integral of a scalar field

$$\iint_S \varphi\, dS = \int_c^d \int_a^b \varphi(\,\mathbf{r}\,(u, v))\ |\partial_u \mathbf{r} \times \partial_v \mathbf{r}|\ du\, dv$$

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} \varphi(x, y, z(x, y))$$

$$\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\ dy\, dx$$

Flux of a Vector Field $\mathbf{F}$ through the surface S

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\,S$$

Line Integral of a Gradient

$$\boxed{\int_P^Q \mathbf{grad}\, \varphi \cdot d\,\mathbf{r}\ = \varphi(Q) - \varphi(P)}$$

Circulation of a Gradient along a closed contour

$$\boxed{\oint_C \mathbf{grad}\, \varphi \cdot d\,\mathbf{r}\ = 0}$$

Gauss (Divergence) Theorem

$$\boxed{\iiint_D \operatorname{div} \mathbf{F}\, dV = \iint_{\partial D} \mathbf{F}\ \cdot d\mathbf{S}}$$

∂D is a closed surface, which is the boundary of the solid region D

Green's Theorem

$$\boxed{\iint_S (\partial_1 F_2 - \partial_2 F_1) dx_1 dx_2 = \oint_{\partial S} (F_1 dx_1 + F_2 dx_2)}$$

∂S is a closed plane curve, which is the boundary of the region S in the $x_1 x_2$-plane

Stokes' Theorem

$$\boxed{\iint_S \mathbf{curl}\ \mathbf{F}\ \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F}\ \cdot d\,\mathbf{r}}$$

∂S is a closed space curve, which is the boundary of the surface S

Flux of a curl through a *closed* surface S

$$\boxed{\iint_S \mathbf{curl}\ \mathbf{F}\ \cdot d\mathbf{S} = 0}$$

Tensor Fields

Flux of a Tensor Field

$$\iint_S T_{ik} n_k\, dS, \qquad \iint_S T_{ik} n_i\, dS$$

Divergence of a Tensor (in Cartesian Coordinates)

$$\partial_i T_{ik}, \qquad \partial_k T_{ik}$$

Directional Derivative (in Cartesian Coordinates)

$$\frac{dT_{ik}}{ds} = \frac{dx_j}{ds}\,\partial_j T_{ik}$$

Analog of curl of an antisymmetric 2-tensor

$$3\varepsilon_{ijk}\partial_i A_{jk} = \partial_1 A_{23} + \partial_2 A_{31} + \partial_3 A_{12}$$

$$= \partial_1 \tilde{A}_1 + \partial_2 \tilde{A}_2 + \partial_3 \tilde{A}_3 = \operatorname{div} \tilde{\mathbf{A}}$$