Notes on Lie Groups

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The Poincaré and Lorentz groups are typical examples of continuous groups. That is why we give below a very brief description of the theory of continuous groups.

1 Abstract Group

An abstract *group* G is a set of elements g for which

i. an *associative* composition law, called the *multiplication*, is given so that for each *ordered pair* of elements (g_1, g_2) another element $g_1 g_2$, called their *product*, is associated

$$(g_1, g_2) \to g_1 g_2, \tag{1}$$

and

$$g_1(g_2 g_3) = (g_1 g_2) g_3, \tag{2}$$

ii. there exists an element e, called the *unit element* (or identity element), such that for any g

$$ge = eg = g, \tag{3}$$

iii. an operation, called the *inversion* is given, i.e., with each element g its *inverse* g^{-1} is associated,

$$g \to g^{-1}, \tag{4}$$

such that

$$g^{-1} g = g g^{-1} = e. \tag{5}$$

[1]corrected on September 16,2003

Basic Notions

1. If for any elements g_1 and g_2 of the group G

$$g_1 g_2 = g_2 g_1, \tag{6}$$

then the group is called *Abelian* or commutative. Otherwise it is non-Abelian.

2. The number of elements of the group G is called the *order* of G and denoted by $|G|$.

3. The group of finite order is called the *finite group*. Otherwise it is infinite.

4. Infinite group can be *discrete* and *continuous*. If the elements g of a group G can be enumerated with the help of a discrete index, g_i, $(i = 1, 2, \ldots)$, the group G is called *discrete*. Otherwise it is *continuous*. Finite groups are obviously always discrete.

5. A subset H of elements of the group G is called a *subgroup* if it itself is a group with the same composition law. This means that the identity element e of G, the products of any elements of H as well as their inverses belong to H. One says that H is *closed* under the multiplication and the inversion laws.

6. A subgroup H is called *proper subgroup* if it consists of more than just the unit element but does not coincide with the whole group itself.

2 Continuous Groups

1. The elements of a general continuous group can be parametrized by a set of continuous real parameters

$$g = g(\lambda), \qquad (\lambda = (\lambda^a),\ a = 1, 2, \ldots). \tag{7}$$

If the set of continuous parameters is finite, i.e., $a = 1, 2, \ldots, p$, the group is called *finite dimensional*, the number of the parameters being the *dimension of the group* $\dim G = p$. Otherwise, the group is infinitely-dimensional.

2. If the parameters $f^a(\lambda, \mu)$ of the product of two elements

$$g(\lambda) g(\mu) = g(f(\lambda, \mu) \tag{8}$$

are *analytic* functions of the parameters of the factors, i.e. the functions $f^a(\lambda, \mu)$ possess derivatives of all orders with respect to all arguments, and, similarly, the parameters $\bar\lambda^a(\lambda)$ of the inverse element $g(\bar\lambda) = g^{-1}(\lambda)$ are analytic functions of the parameters λ then the continuous group is called *Lie group*.

3. The continuous parameters λ^a are called *coordinates* on the Lie group. For a finite-dimensional Lie group G the coordinates λ^a vary in some region of the Euclidean space $\mathbb{R}^p$, p being the dimension of the group. If the domain of variation of the coordinates is finite, or compact, i.e. $|\lambda^a| < \infty$, the group is said to be *compact* (for more precise definition see the bibliography).

4. A *curve* (or *path*) $g = g(\tau)$, $0 \le \tau \le 1$, on a Lie group G is a mapping

$$\tau \in [0,1] \to g(\tau) \in G, \tag{9}$$

where τ is a real parameter. The one-parameter subset $\{g(\tau)\}$ of the group G itself is called the curve too. A curve $g(\tau)$ is *continuous* if the coordinates $\lambda^a(\tau)$ of the element $g(\tau)$ are continuous functions of the parameter τ. We will call the continuous curves just curves.

5. One says that two elements g_0 and g_1 are *connected* by a curve $g(\tau)$ if

$$g(0) = g_0, \qquad g(1) = g_1. \tag{10}$$

6. If $g(0) = g(1) = g$ the curve is called *closed* curve, or the *loop*, going thfough the element g. The loop consisting only from one element g is called the *null loop* at g.

7. A subset H of the group G is called *arcwise connected* (or connected) if every two elements of H can be connected by a *continuous* curve.

8. A *component* of an element g of a Lie group G is the union of all *connected subsets* of G containing the element g.

9. The component G_1 of the identity element of the group G is called the *proper connected component* of the group G.

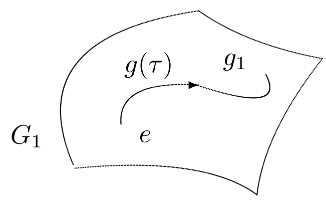

Figure 1: Proper connected component of the group

10. A general Lie group G consists of many connected components G_i, which are disconnected from each other. Each conected component G_i is obtained from the proper subgroup G_1 by applying some discrete transformation γ_i of a discrete subgroup Γ. Thus a Lie group is a direct product of the proper subgroup and some discrete subgroup

$$G = G_1 \times \Gamma, \tag{11}$$

where G_1 is the proper group and Γ the discrete subgroup.

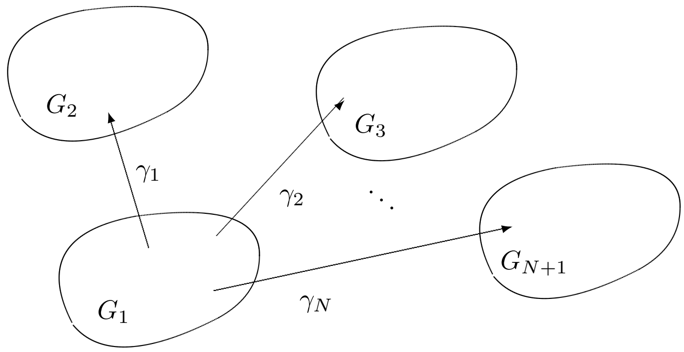

Figure 2: General continuous group

Lorentz Group as Example The general Lorentz group $\mathcal{L} \simeq O(1, d-1)$ has four connected components. The role of the component of the identity G_1 plays the proper orthochronous Lorentz group $\mathcal{L}_1 \simeq SO_1(1, d-1)$. The discrete subgroup Γ is the finite group of reflections of the time and one space coordinate

$$\Gamma = \{1, T, P, TP\} \qquad TP = PT. \tag{12}$$

$$T^2 = P^2 = (TP)^2 = 1. \tag{13}$$

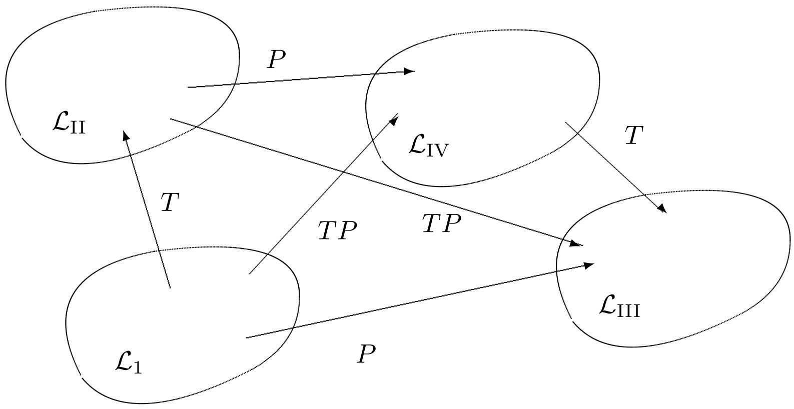

Figure 3: General Lorentz group

11. Two curves $g(\tau)$ and $g'(\tau)$ connecting the elements g_0 and g_1 are said to be *homotopic* if there exists a continuous deformation of one curve into another, which leaves the end points g_0 and g_1 unaltered, i.e., there exist a *continuous* function $h(\tau, s)$ of *two* parameters τ and s such that

$$h(0, s) = g_0, \qquad h(1, s) = g_1 \tag{14}$$

$$h(\tau, 0) = g(\tau), \qquad h(\tau, 1) = g'(\tau). \tag{15}$$

12. A Lie group is said to be *simply connected* if every loop is homotopic to the null loop, i.e., every loop is contractable to one point.

13. All loops at an element g are classified into the so called *homotopy classes.*

14. A Lie group is said to be n*-connected* if it has n homotopy classes at each element.

3 Invariant Subgroups

1. A parametrized curve $g = g(\tau)$, $(a \le \tau \le b)$, on the Lie group G is called the *one-parameter subgroup* of the Lie group G if

$$g(0) = e, \tag{16}$$

$$g(\tau_1 + \tau_2) = g(\tau_1)g(\tau_2), \tag{17}$$

$$g(-\tau) = (g(\tau))^{-1}. \tag{18}$$

2. Let H be a subgroup of G. The *orbit* gH of H through an element g in G is the set of all elements gh with h in H

$$gH = \{gh : \quad h \in H\}. \tag{19}$$

3. If H is a proper subgroup, the orbits of H in G are called the *left cosets* of H in G. The set of all left cosets is denoted by G/H. Analogously it is defined the set $H \setminus G$ of all *right cosets*

$$Hg = \{hg : \ h \in H\}. \tag{20}$$

4. A subgroup H of a group G is said to be *normal* or *invariant subgroup* of G if for any g in G and any h in H the element ghg^{-1}, called the *conjugate element*, is again in H

$$gH = Hg, \qquad \text{or} \qquad gHg^{-1} \subset H. \tag{21}$$

In other words the normal subgroup is closed under the conjugation and the left and right cosets of H in G coincide with each other.

5. **Theorem.** The set of left cosets G/H of a subgroup H in G is itself a group if H is normal subgroup.

6. **Theorem.** The component G_1 of the identity e of a Lie group G, i.e. the connected component containing the identity element, is a closed invariant subgroup of G.

7. A set $C(G)$ of *all* elements of a group G which commute with all elements of G is called the *center* of the group
$$C(G) = \{g \in G : \ g'g = gg' \quad \forall g' \in G\}. \tag{22}$$

8. **Theorem.** The center $C(G)$ is an Abelian normal subgroup of G. Thus $G/C(G)$ is itself a group.

9. **Theorem.** If G is a connected Lie group and H is an invariant discrete subgroup, then H is *central*, i.e. it is a subgroup of the center.

10. A Lie group is said to be *simple* if it has no proper, connected invariant Lie subgroup. It might, however, contain a discrete invariant subgroup.

11. A Lie group is said to be *semisimple* if it contains no proper invariant connected Abelian Lie subgroup.

4 Homomorphisms

1. A mapping
$$\varphi : \quad G \to G' \tag{23}$$
which preserves the group multiplication, i.e.
$$\varphi(gg') = \varphi(g)\varphi(g') \tag{24}$$
is called *homomorphism*.

2. Note that several elements of G may have the same image in G'! The set of all elements of G which are mapped to the identity element of G' is called the *kernel* of the homomorphism φ:
$$\operatorname{Ker}\varphi = \{g \in G : \ \varphi(g) = e' \in G'\}. \tag{25}$$

3. **Theorem.** The kernel of a homomorphism φ is a *normal subgroup* of G, i.e. for any g in G and any h in $\operatorname{Ker}\varphi$ the element ghg^{-1} is again in $\operatorname{Ker}\varphi$; in other words
$$g(\operatorname{Ker}\varphi)g^{-1} = \operatorname{Ker}\varphi. \tag{26}$$

4. **Theorem.** Thus $G/\operatorname{Ker}\varphi$ is a group.

5. Two groups G and G' are said to be *isomorphic*
$$G \simeq G' \tag{27}$$
if their elements can be put into *one-to-one* correspondence which is preserved under multiplication.

6. An isomorphism $\varphi: \; G \to G'$ is simply a one-to-one homomorphism, i.e. $\mathrm{Ker}\,\varphi = \{e\}$, so that the inverse map $\varphi^{-1}: \; G' \to G$ is also a homomorphism.

7. An isomorphism $\varphi: \; G \to G$ of a group with itself is called *automorphism* of the group.

8. **Theorem.** The set $\mathrm{Aut}\,(G)$ of all automorphisms of a group G is itself a group.

9. A homomorphism

$$\varphi: \; G' \to G \tag{28}$$

is called *surjective mapping onto* G if for any g in G there exists at least one element g' in G' such that $\varphi(g') = g$.

10. **Theorem.** If homomorphism $\varphi: \; \tilde{G} \to G$ is a *surjective mapping onto* G, then the group G is isomorphic to $\tilde{G}/\mathrm{Ker}\,\varphi$

$$G \simeq \tilde{G}/\mathrm{Ker}\,\varphi. \tag{29}$$

The group $\tilde{G}$ is called the *universal covering group* of G.

5 Direct and Semi-direct Products

1. The set of all ordered pairs

$$(g, g'), \tag{30}$$

where g is an element of a group G and g' an element of another one G', with the product rule

$$(g_1, g_1')(g_2, g_2') = (g_1 g_2, g_1' g_2') \tag{31}$$

is called the *direct product* (or *Cartesian product*, or *outer product*, or simply *product*) $G \times G'$ of the groups G and G'. The unit element of $G \times G'$ is (e, e') and the inverse of (g, g') is $(g, g')^{-1} = (g^{-1}, g'^{-1})$.

2. Let $G = \{g\}$ be a subgroup of the group of automorphisms $\mathrm{Aut}\,(H)$ of another group $H = \{h\}$, i.e., the group G acts isomorphically on the group H

$$g: \; h \in H \to g(h) \in H. \tag{32}$$

The set of all ordered pairs (g, h) with the product rule

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 g_1(h_2)) \tag{33}$$

defines the *semi-direct product* $G \ltimes H$ of the groups G and H. The unit element of $G \ltimes H$ is $(e, 1)$ and the inverse of (g, h) is

$$(g, h)^{-1} = (g^{-1}, g^{-1}(h^{-1})). \tag{34}$$

3. If the group $\mathcal{T} = \{a\}$ is Abelian with the group multiplication denoted by $+$ and the group $\mathcal{L} = \{\Lambda\}$ acts linearly on $\mathcal{T}$, then the semidirect product $\mathcal{L} \ltimes \mathcal{T}$ has the multiplication rule

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2). \tag{35}$$

4. **Theorem.** The semidirect product $G \ltimes H$ has the following properties:

   (a) H is a normal subgroup of $G \ltimes H$,

   (b) $G \ltimes H/H$ is isomorphic to G.

6 Group Representations

1. If for each element g of a group G it is given an invertible operator $D(g)$ in a vector space V

$$D(g):\ V \to V \tag{36}$$

and for all g_1 and g_2 in G

$$D(g_1 g_2) = D(g_1)D(g_2) \tag{37}$$

then the set of the operators $D(g)$ is said to form a *linear representation* of the group G on a vector space V.

2. Note that for any element g in G

$$D(g^{-1}) = D^{-1}(g) \tag{38}$$

and

$$D(e) = I, \tag{39}$$

where I is the identical operator in V.

3. In general, there are several elements in G which are mapped on I.

4. An invertible operator in the vector space is called *automorphism* of the vector space V.

5. **Theorem.** The set of all automorphisms of a vector space forms a group, denoted by $\text{Aut}\,(V)$.

6. Thus, a linear representation of a group G on a vector space V is a homomorphism

$$D:\ G \to \text{Aut}\,(V). \tag{40}$$

7. If D is isomorphism, i.e. the correspondence (40) is one-to-one, the representation D is said to be *faithful* (or *exact*).

8. If the dimension of the vector space $p = \dim V$ is finite then the operators $D(g)$ are described by $p \times p$ matrices and we have a *matrix representation* of the group G, I being the unit matrix.

9. If $\mathrm{Aut}\,(V) = G$ then the representation $D : G \to \mathrm{Aut}\,(V)$ is called the *defining* representation (or *fundamental* representation.

10. Every Lie group has the *adjoint* representation such that $\dim V = \dim G$, which is determined by its structure constants (defined later).

11. For compact simple groups the adjoint representation is irreducible.

12. If the operators $D(g)$ are unitary, i.e. $D^\dagger(g)D(g) = \mathbb{I}$, then the representation D is called *unitary*.

13. Not every Lie group has a faithful finite-dimensional (matrix) representation.

14. If two representations D_1 and D_2 of a group G on the vector space V are related by an invertible operator A on V, i.e. an automorphism of the vector space V,
$$D_1(g) = A^{-1} D_2(g) A \tag{41}$$
then the representations D_1 and D_2 are said to be *equivalent*.

15. With any matrix representation D of a group G it is associated a map
$$\chi_D : \; G \to \mathbb{C}, \tag{42}$$
defined by the trace of the representation matrices
$$\chi_D(g) = \mathrm{tr}\, D(g), \tag{43}$$
which is called the *character* of the representation D. The value $\chi_D(g)$ is called the character of the the element g in the representation D.

16. The equivalent representations have the same characters.

17. A representation D of a Lie group G is called *reducible* if there is a proper invariant subspace $V_1 \subset V$, i.e. $D : V_1 \to V_1$, so V_1 is closed under D. Otherwise the representation is called *irreducible*.

18. Every reducible unitary representation D of a Lie group G is a direct sum of irreducible ones, i.e. $D = D_1 \oplus \cdots \oplus D_n$.

19. For an Abelian Lie group all irreducible representations are one-dimensional.

7 Multiple-valued Representations. Universal Covering Group.

1. The matrix elements $D(g)$ of the representation D of a Lie group G are required to be continuous functions on the group G. Among continuous functions on the group G there may be some functions which are *multi-valued.* Thus the representation can be, in principle, *multiple-valued.*

2. We say that a representation D of G is *m-valued representation* if with each element g of the group G there are associated m diferent operators $D_1(g), \dots, D_m(g)$.

3. Let us consider a continuous function $D(g)$ on the group G and let us look at the values $D(g(\tau))$ along a continuous closed curve (loop) $g(\tau)$ on G, so that $g(0) = g(1) = g$. It could hapen, in principle, that

$$D(g(0)) \neq D(g(1)). \tag{44}$$

   Let us fix an initial value $D_0 = D(g(0))$ and take all possible loops in G starting at g.

   If the *maximal* number of different values $D(g(1))$ is m, then the function $D(g)$ is *m-valued.* This number is a property of the group and reflects the connectedness of the group itself.

4. If the loop $g(\tau)$ on the group G can be varied *continuously* so that it contracts to the initial point g, i.e., it is homotopic to the null loop, the continuous function $D(g)$ must return to its original value D_0. If this is the case for *all* loops on the group, i.e., if all loops are homotopic to the null loop, the group is called to be *simply conected*, and every continuous function on the group must be *single-valued.*

5. If there are m different loops which cannot be deformed into each other, i.e., if there are m homotopy classes, the group is said to be *m-connected*, and m-valued continuous functions can exist.

   If the group is m-conected, we may expect that some of the representations will be m-valued. These multiple-valued representations cannot be simply ignored.

6. It can be shown that for any multiply-connected group G there exists a simply connected group $\tilde{G}$, called the *universal covering group* of G, such that $\tilde{G}$ can be mapped homomorphically on G.

7. The group $\tilde{G}$ contains a discrete invariant subgroup Γ such that G is isomorphic to $\tilde{G}/\Gamma$

$$G \simeq \tilde{G}/\Gamma. \tag{45}$$

8. Every representation of the group G (whether single-valued or multiple-valued) is a single-valued representation of $\tilde{G}$.

   Thus, one can study instead of the group G its universal covering group $\tilde{G}$ which has only single-valued representtions.

8 Matrix Lie Groups

1. The set $M(n,\mathbb{R})$ of all real square $n \times n$ matrices forms an Abelian Lie group under the law of matrix addition. It is not a group under the law of matrix multiplication since not all matrices have inverses. The dimension of the group $M(n,\mathbb{R})$ is equal to the number of matrix elements, $\dim M(n,\mathbb{R}) = n^2$.

2. The set of all *invertible* real $n \times n$ matrices

   $$GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) : \ \det A \neq 0\} \tag{46}$$

   forms a *general linear* Lie group under the law of matrix multiplication. The set of all *invertible* real $n\times n$ matrices with *positive* determinant forms a subgroup of $GL(n,\mathbb{R})$ denoted by $GL_+(n)$

   $$GL_+(n) = \{A \in GL(n,\mathbb{R}) : \ \det A > 0\}. \tag{47}$$

   The dimension of both this groups is also equal to the number of the matrix elements: $\dim GL_+(n) = \dim GL(n,\mathbb{R}) = n^2$.

3. The *special linear group* $SL(n,\mathbb{R})$ is a subgroup of $GL(n,\mathbb{R})$ which is formed by all invertible matrices of order n with *unit* determinant

   $$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \ \det A = 1\}. \tag{48}$$

   $SL(n,\mathbb{R})$ is a Lie group of dimension $n^2 - 1$.

4. The real *orthogonal* group $O(n)$ is the subgroup of $GL(n,\mathbb{R})$ of all real *orthogonal* matrices of order n

   $$O(n) = \{A \in GL(n,\mathbb{R}) : \ A^T A = 1\}. \tag{49}$$

   $O(n)$ is a Lie group of dimension $n(n-1)/2$.

5. The *special orthogonal group* $SO(n)$ is a subgroup of $O(n)$ of orthogonal matrices with *unit determinant*

   $$SO(n) = \{A \in O(n) : \ \det A = 1\}, \tag{50}$$

   $\dim SO(n) = \dim O(n) = n(n-1)/2$.

6. The *pseudo-orthogonal group* $O(p, q)$, $0 < p \le q$, is a subgroup of $GL(n, \mathbb{R})$ of all pseudo-orthogonal matrices of type (p, q)

$$O(p, q) = \{A \in GL(n, \mathbb{R}) : \ A^T \eta A = \eta\}, \tag{51}$$

where η is the diagonal matrix of the form

$$\eta = \mathrm{diag}\,(\underbrace{-1, \cdots, -1}_{p}, \underbrace{+1, \cdots, +1}_{q}). \tag{52}$$

$\dim O(p, q) = \dim O(n) = n(n-1)/2$.

7. The *special pseudo-orthogonal group* $SO(p, q)$, $0 < p \le q$, is a subgroup of $O(p, q)$ of all pseudo-orthogonal matrices of type (p, q) with unit determinant

$$SO(p, q) = \{A \in O(p, q) : \ \det A = 1\}, \tag{53}$$

$\dim SO(p, q) = \dim SO(n) = n(n-1)/2$.

8. Similarly, one defines the groups of complex matrices $M(n, \mathbb{C})$, $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$. Obviously,

$$\dim M(n, \mathbb{C}) = 2 \cdot \dim M(n, \mathbb{R}) = \dim GL(n, \mathbb{C}) = 2 \cdot \dim GL(n, \mathbb{R}) = 2n^2 \tag{54}$$

and

$$\dim SL(n, \mathbb{C}) = 2 \cdot \dim SL(n, \mathbb{R}) = 2(n^2 - 1). \tag{55}$$

9. Analogously to the real orthogonal group $O(n)$, the *unitary group* $U(n)$ is a subgroup of $GL(n, \mathbb{C})$ of *unitary* matrices

$$U(n) = \{A \in GL(n, \mathbb{C}) : \ A^\dagger A = 1\}. \tag{56}$$

where $\dagger$ means the Hermitian conjugate: $A^\dagger = A^{T*}$. $U(n)$ is a Lie group of dimension n^2.

10. The *special uitary group* $SU(n)$ is defined as a subgroup of $U(n)$ of unitary matrices with unit determinant

$$SU(n) = \{A \in U(n) : \ \det A = 1\}. \tag{57}$$

$SU(n)$ is a Lie group of dimension $n^2 - 1$.

Theorem.

i.) The groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $SO(n)$, $SU(n)$ and $U(n)$ are connected.

ii.) The groups $SL(n, \mathbb{C})$ and $SU(n)$ are simply connected.

iii.) The groups $GL(n, \mathbb{R})$ and $SO(p, q)$ $(0 < p \le q)$ have two connected components.

For convenience of further references we present the information about these matrix groups in form of a table

group	dimension	connectedness	compactness
$M(n, \mathbb{C})$	$2n^2$	simply connected	non-compact
$M(n, \mathbb{R})$	n^2	simply connected	non-compact
$GL(n, \mathbb{C})$	$2n^2$	connected	non-compact
$GL(n, \mathbb{R})$	n^2	two connected components	non-compact
$GL_+(n)$	n^2	connected	non-compact
$SL(n, \mathbb{C})$	$2(n^2-1)$	simply connected	compact
$SL(n, \mathbb{R})$	n^2-1	connected	compact
$O(n)$	$n(n-1)/2$	two connected components	compact
$O(p, q)$	$n(n-1)/2$	four connected components	non-compact
$SO(n)$	$n(n-1)/2$	connected	compact
$SO(p, q)$	$n(n-1)/2$	two connected components	non-compact
$U(n)$	n^2	connected	compact
$SU(n)$	n^2-1	simply conected	compact

Figure 4: Matrix Groups

9 The Structure Constants of a Lie Group.

Let us consider a finite-dimensional Lie group G of dimension p, i.e. the group elements are parametrized by p real parameters ($\lambda = (\lambda^a),\ a = 1, 2, \dots, p$). One can always choose the coordinates λ^a so that the identity element e is at the origin, i.e. $e = g(0)$. If it is not so we just multiply all the elements of the group by $g^{-1}(0)$.

Let $f^a(\lambda, \mu)$, $(a = 1, 2, \dots, p)$, be the coordinates of the product $g(\lambda)g(\mu)$ of two elements of a Lie group, $g(\lambda)$ and $g(\mu)$, and $\bar{\lambda}^a(\lambda)$ be the coordinates of the inverse element $(g(\lambda))^{-1} = g(\bar{\lambda})$. By definition of the Lie group the functions $f^a(\lambda, \mu)$ and $\bar{\lambda}^a(\lambda)$ are analytic functions. These functions are not arbitrary but satisfy very important functional identities:

$$f(\lambda, 0) = f(0, \lambda) = \lambda, \tag{58}$$

$$f(\lambda, \bar{\lambda}) = 0, \tag{59}$$

$$f(\lambda, f(\mu, \nu)) = f(f(\lambda, \mu), \nu). \tag{60}$$

These are highly nontrivial identities that determine the functions $f(\lambda, \mu)$ and, hence, the group, up to a change of coordinates on the group. By differentiating these identities one can obtain a lot of other identities of higher order. Let us consider the neighbourhood of the identity element. The functions $f(\lambda, \mu)$

can be expanded then in the Taylor series in λ and μ:

$$f^a(\lambda,\mu) = \lambda^a + \mu^a + B^a{}_{bc}\lambda^b\mu^c + O_3(\lambda,\mu) \tag{61}$$

where

$$B^a_{\ bc} = \left.\frac{\partial^2 f^a(\lambda,\mu)}{\partial\lambda^b\partial\mu^c}\right|_{\lambda=\mu=0} \tag{62}$$

and $O_3(\lambda,\mu)$ denotes the terms of order higher than 3 in λ and μ. The numbers

$$C^a_{\ bc} = B^a_{\ bc} - B^a_{\ bc} \tag{63}$$

are called the *structure constants* of the Lie group. They can be also defined by

$$C^a{}_{bc} = \left.\frac{\partial^2}{\partial\lambda^b\partial\mu^c}\left[f^a(f(\lambda,\mu), f(\bar\lambda,\bar\mu))\right]\right|_{\lambda=\mu=0} \tag{64}$$

The structure constants are obviously antisymmetric in lower indices

$$C^a_{\ bc} = -C^a_{\ cb} \tag{65}$$

and satisfy the *Jacobi identities*

$$C^d_{\ ea}C^e_{\ bc} + C^d_{\ eb}C^e_{\ ca} + C^d_{\ ec}C^e_{\ ab} = 0, \tag{66}$$

or, in short,

$$C^d_{\ e[a}C^e_{\ bc]} = 0. \tag{67}$$

This identity, for example, can be obtained by differentiating the eq. (60) with respect to λ^a, μ^b and ν^c, putting $\lambda = \mu = \nu = 0$ and antisymmetrizing over a, b and c.

Let us consider a continuous curve $g(\tau) = g(\lambda(\tau))$ going through the unit element, so that $g(0) = e$. The components

$$X = \left.\frac{dg(\tau)}{d\tau}\right|_{\tau=0}, \tag{68}$$

define a vector X, called the *tangent vector to the curve* $g(\tau)$ at e. The set of tangent vectors to all curves going through identity element forms a linear vector space L, caled the *tangent space*, T_eG, at e.

Note that if $g(\lambda)$ is unitary, i.e. $g^\dagger = g^{-1}$, then X is anti-Hermitian, i.e. $X^\dagger = -X$.

Let X_a be the basis vectors, called *generators*, in the tangent space L. For example, one can always define the generators by

$$X_a = \left.\frac{\partial g}{\partial\lambda^a}\right|_{\lambda=0}, \tag{69}$$

so that

$$g(\lambda) = e + \lambda^a X_a + O(\lambda^2)\,. \tag{70}$$

Then one can introduce the structure of a *Lie algebra* by defining for each ordered pair (X_a, X_b) of tangent vectors X_a and X_b a composition rule, called the *Lie multiplication* (or *Lie bracket*, or simply *comutator*),

$$(X_a, X_b) \in L \times L \to [X_a, X_b] \in L, \tag{71}$$

so that

$$[X_a, X_b] = C^c{}_{ab} X_c. \tag{72}$$

This Lie algebra is called the *Lie algebra of the Lie group* G. The Jacobi identity (66) can be rewritten in terms of double commutators in form

$$[X_a[X_b, X_c]] + [X_b[X_c, X_a]] + [X_c[X_a, X_b]] = 0. \tag{73}$$

For Abelian groups all structure constants vanish $C^a{}_{bc} = 0$ and we have so called *Abelian Lie algebra*

$$[X_a, X_b] = 0. \tag{74}$$

If $C^a{}_{bc}$ are the structure constants of a Lie group, then the matrices T_a defined by $(T_a)^b{}_c = C^b{}_{ac}$ form a representation of the Lie algebra called the *adjoint representation* under the standard matrix multiplication. The commutation relations

$$[T_a, T_b] = C^c{}_{ab} T_c \tag{75}$$

are then nothing but the Jacobi identities.

Theorem. In the class Γ of all connected Lie groups having isomorphic Lie algebras there exists one and only one simply connected group $\tilde{G}$, called the *universal covering group* of the class Γ. Any group of the class Γ is a factor group $\tilde{G}/N$, where N is a discrete central invariant subgroup. Note that the members of the class Γ, i.e. the groups having isomorphic Lie algebras, although locally isomorphic, may be totally different globally.

Theorem. A compact connected Lie group G is a direct product of its connected center G_0 and of its simple compact connected Lie subgroups $G_k, k = 1, 2, \dots, n$,

$$G = G_0 \times G_1 \times \cdots \times G_n \tag{76}$$

10 Exponential Mapping

The exponential map exp is a homomorphism of the Lie algebra L into the Lie group G

$$\exp : \ X \in L \to \exp(X) \in G. \tag{77}$$

Theorem. Let G be a Lie group and L its Lie algebra. Then for every tangent vector $X \in L$ there exists a one-parameter subgroup $\exp(\tau X)$ of G, i.e. a unique analytic homomorphism $g(\tau) = \exp(\tau X)$ of $\mathbb{R}$ into G, such that

$$g(\tau_1)g(\tau_2) = g(\tau_1 + \tau_2), \tag{78}$$

$$\left.\frac{dg(\tau)}{d\tau}\right|_{\tau=0} = X, \tag{79}$$

$$g(0) = e. \tag{80}$$

In some cases, (but not generically!), the exponential map $X \to \exp(X)$, $X \in L$, covers the whole group G.

If T_a are the generators in the adjoint representation then $g(\lambda) = \exp(\lambda^a T_a)$ forms the *adjoint representation of the Lie group.*

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