Effective Action Approach to Quantum Field Theory

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Chapter 1

Classical Field Theory

1.1 Introduction

In these lectures we will use mostly the covariant spacetime approach to the field theory developed mainly by De Witt [7, 12].

The basic object of any physical theory is the *spacetime*. We will denote it by M and assume that it is a d-dimensional manifold with the topological structure

$$M = \mathbb{I} \times \Sigma, \tag{1.1}$$

where $\mathbb{I}$ is an open interval of the real line and Σ is some $(d-1)$-dimensional manifold. Σ can be compact or noncompact. More precisely, we assume the spacetime to be a Riemannian manifold with a hyperbolic metric g of the signature $(- + \cdots +)$ which admits a foliation of spacetime into spacelike sections identical to Σ.

The points of the spacetime are denoted by x and local coordinates by x^μ $(\mu = 0, 1, \ldots, d-1)$, x^0 will be often denoted by t as well.

1.1.1 Superclassical fields

Let us consider a set of some say *real* smooth differentiable functions over the spacetime

$$\varphi^A(x), \qquad A = 1, 2, \ldots, p. \tag{1.2}$$

If these functions transform according to some special rules under the transformation of the coordinates, i.e., if they form a representation of the diffeomorphism group they are said to be a *classical* field.

This can be formulated in a more mathematical language. Let us consider a vector bundle $V_c(M)$ over the spacetime M each fiber of which is a vector space V_c, on which the Lorentz group $O_1(1, d-1)$, subscript 1 denoting the component of $O(1, d-1)$ containing the identity, acts. The sections of this vector bundle are called *classical tensor fields*. They do not need to be irreducible representations of the Lorentz group. In general, the bundle $V_c(M)$ is the direct sum of all bundles with sections being irreducible tensor representations of Lorentz group.

These tensor fields are represented by their components, which form a set of smooth differentiable functions on the spacetime manifold

$$\varphi : M \to \mathbb{R}^p, \tag{1.3}$$

$p = \dim V_c$ being the dimension of the corresponding vector space.

The label A denotes the collection of all possible discrete indices that label the tensor product of irreducible representations.

We will always suppose that there exists also *spin structure* on the spacetime manifold M, i.e., that the second Stiefel-Whitney class of M vanishes, and there is an associated *vector bundle* $V_a(M)$, each fibre of which is a complex vector space V_a, on which the spin group $Spin_1(1, d-1)$, i.e. the covering group of Lorentz group, acts. The sections of this bundle are called *spinor fields*. The bundle $V_a(M)$ we consider is, in general, the direct sum of all spin-tensor bundles, having the sections as spin-tensor fields.

One of the most important theorems in quantum field theory is the theorem about the connection of the spin and statistics. It states that there is a crucial difference between the tensor fields and spin-tensor fields. All tensor fields have bosonic statistics and are called *boson fields* and the spin-tensor fields have fermionic statistics and are called *fermion fields*.

In QFT the classical fields become Hermitian operators on a Hilbert space. The boson fields satisfy some commutation relations and the fermion ones – the anticommutation relations

$$[\hat{B}_1, \hat{B}_2] = \hbar \cdots, \qquad [\hat{F}_1, \hat{F}_2]_+ = \hbar \cdots, \qquad [\hat{B}, \hat{F}] = 0. \tag{1.4}$$

where B and F denote some boson and fermion fields, $[,]$ and $[,]_+$ are the commutator and the anticommutator.

That is why in the classical limit $\hbar \to 0$ of QFT the boson fields are assumed to commute with each other and with the fermion fields

$$B_1 B_2 = B_2 B_1, \tag{1.5}$$

$$BF = FB, \tag{1.6}$$

However, the fermion fields in the classical limit should be taken to *anticommute* with each other

$$F_1 F_2 = -F_2 F_1. \tag{1.7}$$

It is clear that the product of two (and, hence, of any even number) of fermion fields is a boson field.

We do not restrict ourselves only to boson or fermion fields. The set φ^A contains both boson and fermion fields. Such sets of the boson and fermion fields are called super fields.

To deal with the collection of the boson and fermion fields we define the parity $\varepsilon(\varphi^A)$ of the field component φ^A by

$$\varepsilon(A) \equiv \varepsilon(\varphi^A) = \begin{cases} 0, & \text{if } \varphi^A \text{ is bosonic} \\ 1, & \text{if } \varphi^A \text{ is fermionic.} \end{cases} \tag{1.8}$$

Then commutation relations (1.5) — (1.7) can be written in a closed form

$$\varphi^A\varphi^B = (-1)^{\varepsilon(A)\varepsilon(B)}\varphi^B\varphi^A \tag{1.9}$$

or

$$[\varphi^A, \varphi^B]_s = \varphi^A\varphi^B - (-1)^{\varepsilon(A)\varepsilon(B)}\varphi^B\varphi^A = 0. \tag{1.10}$$

This is called *supercommutator*. To simplity the notation one can adopt the convention that an index or symbol appearing in an exponent of (-1) is to be understood as assuming the value 0 or 1 according as the associated quantity is fermionic or bosonic and replace $\varepsilon(A) \to A$.

The variables φ^A satisfying the conditions (1.9) are called the Grassmanian variables or supernumbers. They are said to form a Grassmanian algebra Λ_D of dimension D. Thus the fields $\varphi^A(x)$ at a fixed point $x \in M$ generate a finite dimensional Grassmanian algebra, Λ_D, the fermion fields being the odd elements of it and the boson fields the even ones. If we include the values of the fields at all the points $x \in M$, then we have infinitely dimensional Grassmanian algebra Λ_∞. Therefore

$$\varphi : M \to \Lambda_\infty. \tag{1.11}$$

The classical fields satisfying the commutation (1.10) relations are called *superclassical* fields. That is why the starting point of QFT is not just the classical field theory but rather the superclassical field theory.

1.1.2 Field configurations

A field configuration is defined to be the set of all $\varphi^A(x)$ for all x

$$\varphi = \{\varphi^A(x) : \, x \in M, \, A = 1, \dots, D\}. \tag{1.12}$$

To present this idea in a more visual way we will use the condensed notation of De Witt. In this notation the discrete index A and the spacetime point x are combined in one lable $i \equiv (A, x)$

$$\varphi^i \equiv \varphi^A(x). \tag{1.13}$$

The field φ^i becomes then an infinite-dimensional (continuous) column , i.e., a contravariant vector, the product of two fields, $\varphi^i\varphi^k$, and, in general, any quantity with two upper indices like G^{ik} becomes infinite-dimensional matrix (tensor)

$$G^{ik} = G^{AB}(x, y), \qquad i \equiv (A, x); \, k \equiv (B, y) \tag{1.14}$$

and so on. Intuitively one can use a finite-dimensional analogy. Let M_N be a *lattice* (a finite subset of points) in M

$$M_N = \{x_a, \quad a = 1, \dots, N; \quad x_a \in M\} \subset M. \tag{1.15}$$

Then $i = 1, \dots, D \times N$ and φ^i becomes a $D \times N$ finite-dimensional column (vector)

$$\varphi^i = \begin{pmatrix} \varphi^1(x_1) \\ \vdots \\ \varphi^1(x_N) \\ \vdots \\ \varphi^D(x_1) \\ \vdots \\ \varphi^D(x_N) \end{pmatrix}. \tag{1.16}$$

Thus the field configuration is just the *set of the values* of the field in all points of the manifold. The matrix G^{ik} should be viewed on as a $(D \times N) \times (D \times N)$ - dimensional matrix

$$\left(\begin{array}{ccc|c|ccc} G^{11}(x_1,x_1) & \dots & G^{11}(x_1,x_N) & \dots & G^{1D}(x_1,x_1) & \dots & G^{1D}(x_1,x_N) \\ \vdots & & \vdots & & \vdots & & \vdots \\ G^{11}(x_N,x_1) & \dots & G^{11}(x_N,x_N) & \dots & G^{1D}(x_N,x_1) & \dots & G^{1D}(x_N,x_N) \\ \hline \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \hline G^{D1}(x_1,x_1) & \dots & G^{D1}(x_1,x_N) & \dots & G^{DD}(x_1,x_1) & \dots & G^{DD}(x_1,x_N) \\ \vdots & & \vdots & & \vdots & & \vdots \\ G^{D1}(x_N,x_1) & \dots & G^{D1}(x_N,x_N) & \dots & G^{DD}(x_N,x_1) & \dots & G^{DD}(x_N,x_N) \end{array}\right) \tag{1.17}$$

Further, as usual it will be always assumed that a summation over repeated indices is performed. That is in condensed notation — a combined summation-integration, i.e.

$$J\varphi \equiv J_i\varphi^i \equiv \int\limits_M dx\, J_A(x)\varphi^A(x) \tag{1.18}$$

Thus one can formally consider such objects, as the traces and the determinants of the infinite-dimensional matrices.

The next object that is used extensively in QFT is the *configuration space* $\mathcal{M}$. Configuration space is the set of all possible field configurations

$$\mathcal{M} = \left\{\varphi^i\right\}. \tag{1.19}$$

One can show that the configuration space in an infinite-dimensional supermanifold.

1.1.3 Field functionals

A supernumber-valued function $S(\varphi)$ on the configuration space with

$$S(\varphi) : \mathcal{M} \to \Lambda_\infty \tag{1.20}$$

is called a field functional. Functions on supermanifolds are defined by the formal power series in fermion fields. Denoting the boson fields by χ and the fermion fields by ψ, i.e.

$$\varphi = \begin{pmatrix} \chi \\ \psi \end{pmatrix}, \tag{1.21}$$

one can write

$$\begin{aligned} S(\varphi) &= \sum_{n\geq 0} f_{a_1\ldots a_n}(\chi)\psi^{a_n}\cdots\psi^{a_1} \\ &\equiv \sum_{n\geq 0}\int dx_1\cdots dx_n f_{A_1\cdots A_n}(\chi; x_1\ldots x_n)\psi^{A_n}(x_n)\cdots\psi^{A_1}(x_1), \end{aligned} \tag{1.22}$$

where $a_i \equiv (A_i, x_i)$, with the spinor index A_i running over $A_i = 1, \ldots, q$ for some $q < D$. From the anticommutativity of the fermion fields it is clear that $f_{a_1\ldots a_n}(\chi)$ are antisymmetric in all their indices. These are infinite-dimensional p-forms on supermanifold $\mathcal{M}$.

The functional derivatives of the field functionals are defined as follows. Let us consider an infinitesimal variation

$$\delta\varphi^i \equiv \delta\varphi^A(x) \in C^\infty(M). \tag{1.23}$$

The set of all points of spacetime where $\delta\varphi^i$ is not equal to zero is called the support of $\delta\varphi^i$

$$\Omega \equiv \text{supp } \delta\varphi^i = \left\{x \in M, \quad \delta\varphi^A(x) \neq 0\right\}, \tag{1.24}$$

$$\delta\varphi^A = 0 \text{ for } x \notin \Omega. \tag{1.25}$$

We assume that $\delta\varphi^i$ has a compact support

$$\Omega \subset M. \tag{1.26}$$

Let $\delta S(\varphi)$ denote the corresponding change in $S(\varphi)$. If for all $\varphi \in \mathcal{M}$ and all $\delta\varphi \in C^\infty(M)$ with compact support, $\delta S(\varphi)$ can be written in the form

$$\delta S(\varphi) = \delta\varphi^i{}_{i,}S(\varphi) = S_{,i}(\varphi)\delta\varphi$$

$$= \int\limits_M dx\ \delta\varphi^A(x) \left(\frac{\overrightarrow{\delta}}{\delta\varphi^A(x)} S(\varphi) \right) = \int\limits_M dx \left(S(\varphi) \frac{\overleftarrow{\delta}}{\delta\varphi^A(x)} \right) \delta\varphi^A(x), \tag{1.27}$$

where the coefficients

$$_{i,}S \equiv \frac{\overrightarrow{\delta}}{\delta\varphi^i} S \equiv \frac{\overrightarrow{\delta}}{\delta\varphi^A(x)} S, \tag{1.28}$$

$$S_{,i} \equiv S \frac{\overleftarrow{\delta}}{\delta\varphi^i} \equiv S \frac{\overleftarrow{\delta}}{\delta\varphi^A(x)} \tag{1.29}$$

are independent on the $\delta\varphi^i$, then the $S(\varphi)$ is called *differentiable* functional on M and $_{i,}S$ and $S_{,i}$ are called the *left* and the *right functional derivatives*.

Now consider some finite variation h^i and the value of the functional $S(\varphi)$ at the point $\varphi + h$. At a regular point φ it can be expanded in the *functional Taylor series*

$$\begin{aligned} S(\varphi + h) \ &\stackrel{\text{def}}{=}\ S(\varphi) + S_{,i}(\varphi) h^i + \frac{1}{2} S_{,ik}(\varphi) h^k h^i + \cdots \\ &= \sum_{n \geq 0} \frac{1}{n!} S_{,i_1 \dots i_n}(\varphi) h^{i_n} \cdots h^{i_1}, \end{aligned} \tag{1.30}$$

where all variations are moved to the *right*.

The coefficients of this series are called the *higher right functional derivatives*

$$S_{,i_1 \dots i_n} = S \frac{\overleftarrow{\delta}^n}{\delta\varphi^{i_1} \cdots \delta\varphi^{i_n}}. \tag{1.31}$$

Since the superfields φ^i do not commute, the order of variation in Taylor series is important. By rewriting it in the form

$$S(\varphi + h) = \sum_{n \geq 0} \frac{1}{n!} h^{i_1} \cdots h^{i_n}{}_{i_n \dots i_1,} S(\varphi), \tag{1.32}$$

we define the *higher left functional derivatives*

$$_{i_n \dots i_1,}S \equiv \frac{\overrightarrow{\delta}^n}{\delta\varphi^{i_n} \cdots \delta\varphi^{i_1}} S. \tag{1.33}$$

In the usual notation the term of second order in this series looks more complicated

$$S_{,ik} h^k h^i \equiv \int dx\, dy \left(S(\varphi) \frac{\overleftarrow{\delta}^2}{\delta\varphi^A(x) \delta\varphi^B(y)} \right) h^B(y) h^A(x). \tag{1.34}$$

Changing the order of variations it is easy to find the relation between the left and right derivatives. If the functional S itself is even (bosonic), i.e., $\varepsilon(S) = 0$, then

$$S_{,i} = (-1)^i \ {}_{i,}S \tag{1.35}$$

In general

$$S\frac{\overleftarrow{\delta}}{\delta\varphi^i} = (-1)^{i(1+\varepsilon(S))}\frac{\overrightarrow{\delta}}{\delta\varphi^i}S, \tag{1.36}$$

where $\varepsilon(S)$ is the parity of functional S. Besides

$$_{\cdots ik\cdots,}S = (-1)^{ik} \ {}_{\cdots ki\cdots,}S \tag{1.37}$$

$$S_{,\cdots ik\cdots} = (-1)^{ik} S_{,\cdots ki\cdots} \tag{1.38}$$

In other words one has

$$\frac{\overrightarrow{\delta}}{\delta\varphi^i}\frac{\overrightarrow{\delta}}{\delta\varphi^k} = (-1)^{ik}\frac{\overrightarrow{\delta}}{\delta\varphi^k}\frac{\overrightarrow{\delta}}{\delta\varphi^i}, \tag{1.39}$$

$$\frac{\overleftarrow{\delta}}{\delta\varphi^i}\frac{\overleftarrow{\delta}}{\delta\varphi^k} = (-1)^{ik}\frac{\overleftarrow{\delta}}{\delta\varphi^k}\frac{\overleftarrow{\delta}}{\delta\varphi^i}. \tag{1.40}$$

From these equations it follows, that the mixed left-right second derivative of an even functional possesses the following symmetry relation

$$_{i,}S_{,k} = (-1)^{i+k+ik} \ {}_{k,}S_{,i}. \tag{1.41}$$

A matrix with down indices satisfying such a relation will be called *supersymmetric*. This name is because the bilinear form

$$\eta E h \equiv \eta^i E_{ik} h^k, \tag{1.42}$$

where E_{ik} is a supersymmetric matrix with parity determined only by its indices, $\varepsilon(E_{ik}) = \varepsilon(i) + \varepsilon(k)$, is symmetric.

If we write a supersymmetric matrix E in the block form

$$(E_{ik}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.43}$$

where A and D are bose-bose and fermi-fermi sectors (and, therefore, even) and B and C are the mixed bose-fermi and fermi-bose ones (and, hence, odd), then the supersymmetry means that the matrices A and D are symmetric and B and C satisfy the relations

$$A^T = A, \qquad D^T = D, \tag{1.44}$$

$$B^T = -C. \tag{1.45}$$

Example 1. The simplest functional is the field itself. The derivative of it is defined by

$$\delta\varphi^i = \varphi^i{}_{,k}\delta\varphi^k = \delta\varphi^k{}_{k,}\varphi^i, \qquad (1.46)$$

or

$$\delta\varphi^A(x) = \int dy \left(\varphi^A(x)\frac{\overleftarrow{\delta}}{\delta\varphi^B(y)}\right)\delta\varphi^B(y) = \int dy\delta\varphi^B(y)\left(\frac{\overrightarrow{\delta}}{\delta\varphi^B(y)}\varphi^A(x)\right) \qquad (1.47)$$

Therefore

$$\varphi^i{}_{,k} = \delta^i{}_k, \qquad {}_{k,}\varphi^i = \delta^i{}_k, \qquad (1.48)$$

where

$$\delta^i{}_k = \delta^A{}_B\delta(x,y) \qquad (1.49)$$

is infinite-dimensional Kronecker symbol (continuous identity matrix). We also have obviously the super commutation rule

$$\frac{\overrightarrow{\delta}}{\delta\varphi^i}\varphi^k = (-1)^{ik}\varphi^k\frac{\overrightarrow{\delta}}{\delta\varphi^i} + \delta^k_i. \qquad (1.50)$$

Similary, for any linear functional

$$S = J_i\varphi^i \qquad (1.51)$$

we get

$$S_{,i} = J_i. \qquad (1.52)$$

Example 2. Consider now a quadratic functional.

$$S = \frac{1}{2}\varphi^i E_{ik}\varphi^k \qquad (1.53)$$

where E is a supersymmetric matrix

$$E_{ik} = (-1)^{k+i+ik}E_{ki}. \qquad (1.54)$$

We calculate

$$\delta S = \frac{1}{2}\varphi^i E_{ik}\delta\varphi^k + \frac{1}{2}\delta\varphi^i E_{ik}\varphi^k = \varphi^i E_{ik}\delta\varphi^k. \qquad (1.55)$$

Therefore

$$S_{,k} = \varphi^i E_{ik}. \qquad (1.56)$$

Further

$$\delta S_{,k} = \delta\varphi^i E_{ik} = (-1)^k E_{ki}\delta\varphi^i. \qquad (1.57)$$

Hence

$$S_{,ki} = (-1)^k E_{ki}, \qquad (1.58)$$

$${}_{i,}S_{,k} = E_{ik}. \qquad (1.59)$$

Thus using the functional differentiation one can define the concept of tangent spaces and generalize, at least formally, almost the whole structure of differential geometry to the infinite-dimensional supermanifold. In particular, introducing a supersymmetric nondegenerate matrix $E_{ik}(\varphi)$ that depends only on the values of the fields but not on their derivatives and is diagonal in the continuous part,

$$E_{ik}(\varphi) = E_{AB}(\varphi(x))\delta(x, y), \tag{1.60}$$

one can define the ultra-local Riemannian metric on the supermanifold $\mathcal{M}$ by

$$\begin{aligned} E &= d\varphi^i E_{ik}(\varphi) d\varphi^k \\ &= \int\limits_M dx d\varphi^A(x) E_{AB}(\varphi(x)) d\varphi^B(x). \end{aligned} \tag{1.61}$$

This gives the interval between two field configurations φ and $\varphi + d\varphi$. Then one can define formally the connections, geodesics, curvature etc.

Example 3. Now, let us consider a special class of functionals, namely, local functionals. These are functionals which depend on the values of the fields and finite number of their derivatives.

The local functionals have the following form

$$S(\varphi) = \int\limits_M dx \mathcal{L}(\varphi, \varphi_{,\mu}, \dots, \varphi_{,\mu_1 \dots \mu_N}) \tag{1.62}$$

where

$$\varphi_{,\mu} \equiv \partial_\mu \varphi, \tag{1.63}$$

$$\varphi_{,\mu_1 \dots \mu_N} \equiv \partial_{\mu_1} \cdots \partial_{\mu_N} \varphi, \tag{1.64}$$

and $\mathcal{L}$ is some function of the fields derivatives on one spacetime point. It is not difficult to calculate the functional derivative of local functionals. We calculate

$$\begin{aligned} S(\varphi + \delta\varphi) &= S(\varphi) + \int dx \left\{ \delta\varphi^A \frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A} + \delta\varphi^A{}_{,\mu} \frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A{}_{,\mu}} + \cdots \right\} = \\ &= S(\varphi) + \int dx \delta\varphi^A \left\{ \frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A} - \partial_\mu \left(\frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A} \right) + \cdots \right\} \end{aligned} \tag{1.65}$$

where the dots contain the similar terms with higher derivatives of φ. Thus we obtain the Euler-Lagrange formula

$$\begin{aligned} {}_{i,}S &\equiv \frac{\vec{\delta} S}{\delta\varphi^A(x)} = \frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A(x)} - \partial_\mu \left(\frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A{}_{,\mu}} \right) + \cdots \\ &= \frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A(x)} + \sum_{n=1}^N (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_n} \left(\frac{\vec{\partial}\mathcal{L}}{\partial\varphi^A{}_{,\mu_1 \dots \mu_n}} \right) \end{aligned} \tag{1.66}$$

Thus, the functional derivative of any local functional is given by

$$ {}_{i,}S \equiv \frac{\overrightarrow{\delta}}{\delta\varphi^A(x)} S \equiv \frac{\overrightarrow{\mathcal{D}}}{\mathcal{D}\varphi^A(x)}\mathcal{L}(x), \tag{1.67} $$

where

$$ \frac{\overrightarrow{\mathcal{D}}}{\mathcal{D}\varphi^A} = \frac{\overrightarrow{\partial}}{\partial\varphi^A} + \sum_{n\geq 1}(-1)^n\partial_{\mu_1}\cdots\partial_{\mu_n}\frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\mu_1\dots\mu_n}} \tag{1.68} $$

Similarly,

$$ S_{,i} \equiv S\frac{\overleftarrow{\delta}}{\delta\varphi^A(x)} \equiv \mathcal{L}(x)\frac{\overleftarrow{\mathcal{D}}}{\mathcal{D}\varphi^A(x)}. \tag{1.69} $$

The functional derivative of a local functional is obviously again a local functional

$$ {}_{i,}S \equiv \frac{\overrightarrow{\delta}\, S}{\delta\varphi^A(x)} = \frac{\overrightarrow{\mathcal{D}}}{\mathcal{D}\varphi^A(x)}\mathcal{L}(x) = \int dy\,\delta(x,y)\frac{\overrightarrow{\mathcal{D}\mathcal{L}(y)}}{\mathcal{D}\varphi^A(y)}. \tag{1.70} $$

Thus the second derivative is simply given by

$$ {}_{i,}S_{,k} \equiv \frac{\overrightarrow{\delta}}{\delta\varphi^A(x)}S\frac{\overleftarrow{\delta}}{\delta\varphi^B(y)} = \frac{\overrightarrow{\mathcal{D}}}{\mathcal{D}\varphi^A(y)}(\mathcal{L}(y)\delta(x,y))\frac{\overleftarrow{\mathcal{D}}}{\mathcal{D}\varphi^B(y)}. \tag{1.71} $$

Therefore, the first derivative is a usual function on M but the second derivative is a distribution. It is easy to see that the second derivative is actually the kernel of a differential operator of order $2N$. For the functionals that include only the first derivatives of the fields the second functional derivative looks like

$$ \Delta_{ik} \equiv {}_{i,}S_{,k} = \left(-\partial_\mu A^{\mu\nu}_{AB}\partial_\nu + \frac{1}{2}\left(B^\mu_{AB}\partial_\mu + \partial_\mu B^\mu_{AB}\right) - C_{AB}\right)\delta(x,y), \tag{1.72} $$

where

$$ A^{\mu\nu}_{AB} \equiv \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\mu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\nu}} + \frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\nu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\mu}}\right), \tag{1.73} $$

$$ \begin{aligned} B^\mu_{AB} \quad &\equiv \quad \frac{\overrightarrow{\partial}}{\partial\varphi^A}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\mu}} - \frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\mu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B} \\ &+ \quad \frac{1}{2}\partial_\nu\left(\frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\mu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\nu}} - \frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\nu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\mu}}\right), \end{aligned} \tag{1.74} $$

$$ C_{AB} \equiv -\frac{\overrightarrow{\partial}}{\partial\varphi^A}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B} + \frac{1}{2}\partial_\mu\left(\frac{\overrightarrow{\partial}}{\partial\varphi^A}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B_{,\mu}} + \frac{\overrightarrow{\partial}}{\partial\varphi^A_{,\mu}}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\varphi^B}\right). \tag{1.75} $$

For real functional $S(\varphi)$ and real φ^i the matrices A and C are supersymmetric and the matrix B is antisupersymmetric, and possess the following reality (super-Hermitian) relations

$$ A^{\mu\nu}_{AB} = A^{\nu\mu}_{AB} = (-1)^{A+B+AB}A^{\mu\nu}_{BA} = (-1)^{A+B+AB}A^{\mu\nu\,*}_{AB}, \tag{1.76} $$

$$B^{\mu}_{AB} = -(-1)^{A+B+AB} B^{\mu}_{BA} = (-1)^{A+B+AB} B^{\mu\,*}_{AB}, \tag{1.77}$$

$$C_{AB} = (-1)^{A+B+AB} C_{AB} = (-1)^{A+B+AB} C^{\,*}_{AB}. \tag{1.78}$$

Recalling that $\partial^{+}_{\mu} = -\partial_{\mu}$ it follows from these properties that the operator Δ is self-adjont $\Delta^{+} = \Delta$. This is the consequence of the symmetry and reality properties of the functional differentiation.

1.1.4 Dynamics

The fundamental assumption of the field theory is that any dynamical system can be described by an *action functional*. This means that the nature and dynamical properties of the system are completely determined by the action functional. The action functional is a differentiable real-valued even supernumber-valued scalar field on the configuration space

$$S : \mathcal{M} \to \mathbb{R}_c, \tag{1.79}$$

where $\mathbb{R}_c$ is the set of all real even supernumbers. The choice of dynamical variables, i.e., the fields φ^i, used to describe the system is not unique. Consequently, the configuration space $\mathcal{M}$, i.e., the set of all possible field configurations, is also not unique. It depends on the choice of the dynamical variables φ^i (i.e., on the parametrization of the dynamical system) and on the boundary conditions imposed at the time limits (and at spatial infinity if $\sum$ is noncompact). Analogously, the choice of the action functional is not unique.

However, for a given dynamical system all action functionals describe the same physics, i.e., they must give physically equivalent sets of the *dynamical field configurations.* The dynamical field configurations are defined as the field configurations satisfying the *stationary action principle*: physically admissible values for dynamical variables are those for which the action is stationary under small disturbances with given boundary conditions

$$\delta S = 0. \tag{1.80}$$

In other words, the dynamical field configurations must satisfy the *dynamical equations of motion*

$$\frac{\delta S}{\delta \varphi^i} = 0 \tag{1.81}$$

with given boundary conditions. The set of all dynamical field configurations $\mathcal{M}_0$ is a subspace of the configuration space $\mathcal{M}_0 \subset \mathcal{M}$ which is called the *dynamical subspace*. In QFT it is often called the *mass shell*.

In the local field theory the dynamical equations are local partial differential equations. This means that the action is a local functional

$$S(\varphi) = \int\limits_{\Omega} dx\, \mathcal{L}(\varphi, \partial\varphi, \ldots), \tag{1.82}$$

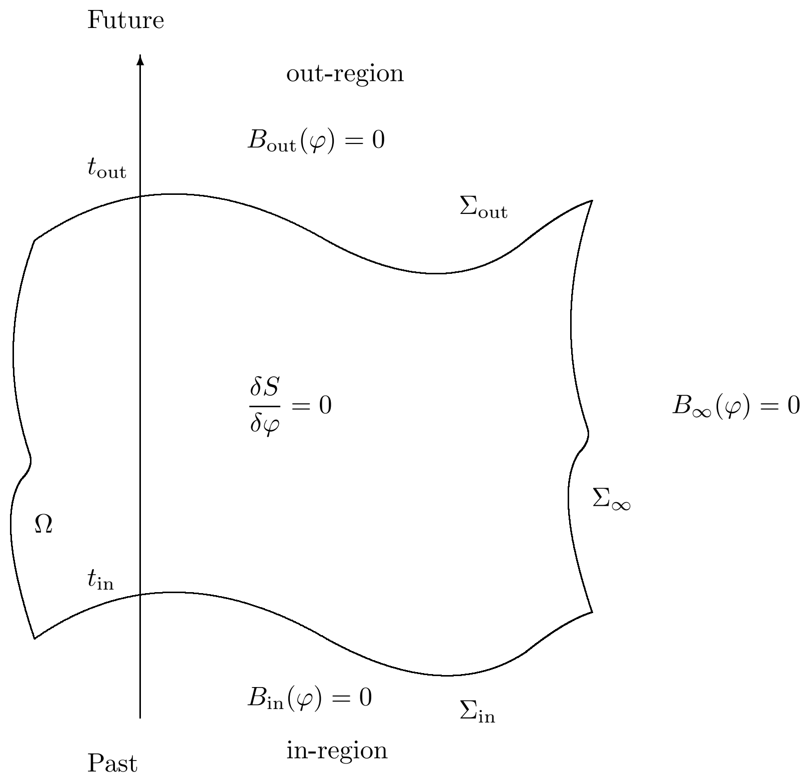

Figure 1.1: Dynamics

where $\Omega \subset M$ is the region of spacetime which we are interested in from the dynamical point of view and $\mathcal{L}$ called the *Langrangian* is a scalar density of unit weight. The whole setting of the problem is illustrated on the Fig. 1.1.

In simple cases the region Ω is just

$$\Omega = (t_{\rm in}, t_{\rm out}) \times \Sigma \tag{1.83}$$

and

$$\partial\Omega = \Sigma_{\rm in} \cup \Sigma_{\rm out} \cup \Sigma_\infty , \tag{1.84}$$

where $\Sigma_\infty = (t_{\rm in}, t_{\rm out}) \times \partial\Sigma$. Besides, in the usual scattering problems of QFT one takes $t_{\rm in}$ and $t_{\rm out}$ first finite but at the very end of calculations let them go to infinity

$$t_{\substack{\rm in \\ \rm out}} \to \mp\infty . \tag{1.85}$$

1.2 Models in field theory

Let us list some simple field theoretical models.

Scalar fields. First of all, a system of scalar fields φ^A, $(A = 1, \dots, D)$, interacting with gravitational and vector gauge fields is described by

$$S_\varphi = \int\limits_M dx\, g^{1/2} \left\{ -\frac{1}{2} g^{\mu\nu} \delta_{AB} \nabla_\mu \varphi^A \nabla_\nu \varphi^B - \frac{1}{2}(m^2 + \xi R)\delta_{AB}\varphi^A\varphi^B - V(\varphi) \right\}, \tag{1.86}$$

where $g_{\mu\nu}$ is the metric of the spacetime, $g = \det g_{\mu\nu}$,

$$\nabla_\mu \varphi^A = \left(\partial_\mu \delta^A_B + A^a_\mu T_a{}^A{}_B \right) \varphi^B \tag{1.87}$$

is the covariant derivative, A^a_μ, $(a = 1, \dots, p)$ are the vector gauge fields, $T_a = (T_a^A{}_B)$ are the generators of the Lie algebra of the gauge group

$$[T_a, T_b] = C^c{}_{ab} T_c, \tag{1.88}$$

$C^c{}_{ab}$ are the structure constants, m^2 is the mass parameter, ξ is the coupling constant to gravity, R is the scalar curvature, and $V(\varphi)$ is a potential for the scalar fields, that does not depend on the derivatives of the fields φ.

A more complicated system of scalar fields is the so called nonlinear σ-model

$$S_\sigma = -\frac{1}{2} \int\limits_M dx g^{1/2} g^{\mu\nu} E_{AB}(\varphi) \nabla_\mu \varphi^A \nabla_\nu \varphi^B, \tag{1.89}$$

where $E_{AB}(\varphi)$ is a local function of the scalar fields.

Yang-Mills fields. The system of vector gauge fields A^a_μ in curved spacetime is described by the Yang-Mills Lagrangian

$$S_{YM} = -\frac{1}{4e^2} \int\limits_M dx\, g^{1/2} g^{\mu\alpha} g^{\nu\beta} \delta_{ab} F^a_{\mu\nu} F^b_{\alpha\beta} \tag{1.90}$$

where e is the coupling constant

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a{}_{bc} A^b_\mu A^c_\nu \tag{1.91}$$

is the field strength of the gauge fields and $C^a{}_{bc}$ are the structure constants of a simple compact Lie group.

Gravity. The gravitational field is described by the metric tensor of the spacetime $g_{\mu\nu}$. The simplest Lagrangian is the Einstein-Hilbert one

$$S_{EH} = \frac{1}{16\pi G} \int\limits_M dx\, g^{1/2} (R - 2\Lambda), \tag{1.92}$$

where G is the Newtonian gravitational constant and Λ is the cosmological constant. This is the only covariant action that leads to the equation of motion of second order. One can, however, consider more complicated gravitational Lagrangians

$$S_{R+R^2} = \int\limits_M dx\, g^{1/2} \left\{ -\frac{1}{2f^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{1}{6\nu^2} R^2 + \frac{1}{16\pi G}(R - 2\Lambda) \right\}, \quad (1.93)$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, f is the tensor coupling constant and ν — the conformal one. This Lagrangian leads to equations of motion of fourth order. That is why this model is also called the higher-derivative gravity. One of the crucial difference between the sigma-model and gravity on the one side and other models on the other side is that the coefficient in front of the derivatives of the fields does depend on the fields, whereas for S_φ, S_{YM} it does not. As we will see in further lectures, this coefficient determines the Riemannian metric of the configuration manifold $\mathcal{M}$. That is for the scalar fields and Yang-Mills fields this metric is constant, i.e., does not depend on the fields. Therefore, the corresponding Riemannian curvature vanishes, i.e., the configuration space is, in fact, flat. For the σ-model and gravity this is not the case. The configuration space metric is not constant, and, hence, the configuration space is curved. This causes serious difficulties in quantizing these theories.

Spinor fields. All the previous models were bosonic. Let us also write down a Lagrangian describing a system of spinor fields ψ^A (which are fermionic) interacting with gravitational and Yang-Mills fields

$$S_\psi = \int\limits_M dx\, g^{1/2} \bar\psi^A \delta_{AB} \left(i\gamma^\mu \nabla_\mu - m \right) \psi^B. \quad (1.94)$$

Here

$$\gamma_\mu = e^a_\mu \gamma_a, \quad (1.95)$$

γ_a are the Dirac $2^{[d/2]} \times 2^{[d/2]}$ matrices, satisfying the anticommutation relations

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2 g_{ab}, \quad (1.96)$$

with $g_{ab} = \text{diag}\,(-1, 1, \dots, 1)$, and e^a_μ are the 1-forms of the local Lorentz frame satisfying the relations

$$g_{\mu\nu} = g_{ab} e^a_\mu e^b_\nu, \quad (1.97)$$

$\bar\psi$ is the Dirac conjugate spinor

$$\bar\psi = \psi^+ \eta, \quad (1.98)$$

where η is the matrix of charge conjugation defined by

$$\gamma^+_\mu = -\eta \gamma_\mu \eta^{-1}. \quad (1.99)$$

The covariant derivative of spinor fields is defined by

$$\nabla_\mu \psi^A = \left(\partial_\mu \delta^A_B + \frac{1}{2} \omega^{ab}{}_\mu \gamma_{ab} \delta^A_B + A^a_\mu T_a{}^A{}_B \right) \psi^B, \tag{1.100}$$

where $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$, $\omega^{ab}{}_\mu$ is the so called spinor connection

$$\begin{aligned} \omega^{ab}{}_\mu &= \frac{1}{2} g^{ac} e^\nu_c \left(e^b{}_{\nu,\mu} - e^b{}_{\mu,\nu} \right) - \frac{1}{2} g^{bc} e^\nu_c \left(e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu} \right) \\ &+ \frac{1}{2} g^{ae} g^{bf} g_{cd} e^\nu_e e^\sigma_f e^d_\mu \left(e^c{}_{\nu,\sigma} - e^c{}_{\sigma,\nu} \right), \end{aligned} \tag{1.101}$$

and e^μ_a is the dual basis of contravariant vectors

$$e^a_\mu e^\mu_b = \delta^a_b, \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{1.102}$$

1.3 Small disturbances and Green functions

Let us consider the equations of motion

$${}_{i,}S = \frac{\delta S}{\delta \varphi^i} = 0. \tag{1.103}$$

They are, in general, complicated nonlinear partial differential equations. Let φ^i be a solution of equations of motion and let us look for another solution in the neighborhood of φ, of the form $\varphi + \delta\varphi$, where $\delta\varphi^i$ is an infinitesimal field. Substifing $\varphi + \delta\varphi$ in the equations of motion

$${}_{i,}S(\varphi + \delta\varphi) = {}_{i,}S(\varphi) + {}_{i,}S_{,j}(\varphi)\delta\varphi^j + \cdots = 0 \tag{1.104}$$

and limiting ourselves to the quantities of the first order we get

$$\Delta_{ij} \delta\varphi^j = 0, \tag{1.105}$$

where

$$\Delta_{ij} = {}_{i,}S_{,j} \tag{1.106}$$

This is the *homogeneous equation of small disturbances*. Its solutions are known as Jacobi fields. In practice it is convenient to introduce infinitesimal external sources δJ_i which cause the small disturbances. Let the action suffer the following change

$$S(\varphi) \to S(\varphi) + \delta J_i \varphi^i. \tag{1.107}$$

Then the equations of motion for the disturbed system becomes

$${}_{i,}S(\varphi) = -\delta J_i. \tag{1.108}$$

In the first order, the solution of these equation of motion is

$$\varphi^i + \delta\varphi^i, \tag{1.109}$$

where φ is the solution of

$$_{i,}S(\varphi) = 0 \tag{1.110}$$

and $\delta\varphi^i$ is the solution of the equation

$$\Delta_{ij}(\varphi)\delta\varphi^j = -\delta J_i. \tag{1.111}$$

This is called the *inhomogeneous equation of small disturbances*. Its general solution is the sum of a particular solution and an arbitrary Jacobi field.

1.4 Wronskian

As we have seen for the local theory without higher derivatives the operator of small disturbances is a differential operator of second order and has the form

$$\Delta_{ik} = \Delta_{AB}(x, \partial)\delta(x, y) \tag{1.112}$$

$$\Delta_{AB}(x, \partial) = \left\{ -\partial_\mu A^{\mu\nu}_{AB}\partial_\nu + \frac{1}{2}\left(B^\mu_{AB}\partial_\mu + \partial_\mu B^\mu_{AB}\right) - C_{AB} \right\}. \tag{1.113}$$

and is formally self-adjoint, i.e., the matrices $A^{\mu\nu}$ and C are supersymmetric

$$A^{\mu\nu}_{AB} = (-1)^{A+B+AB} A^{\mu\nu}_{AB}, \qquad A^{\mu\nu}_{AB} = A^{\nu\mu}_{AB}, \tag{1.114}$$

$$C^\mu_{AB} = (-1)^{A+B+AB} C^\mu_{BA} \tag{1.115}$$

and B^μ is antisupersymmetric

$$B^\mu_{AB} = -(-1)^{A+B+AB} B^\mu_{BA}. \tag{1.116}$$

Besides, for real fields the matrices are super-Hermitian

$$A^{\mu\nu}_{AB} = (-1)^{A+B+BA} A^{\mu\nu\,*}_{AB} \qquad \text{etc.} \tag{1.117}$$

The operator Δ acts on the fields according to

$$\Delta_{ik}h^k = \int dy \Delta_{AB}(x, \partial)\delta(x, y)h^B(y) = \Delta_{AB}(x, \partial)h^B(x). \tag{1.118}$$

$$h^i\Delta_{ik} = \int dy h^A(y)\Delta_{AB}(y, \partial)\delta(y, x) = h^A(x)\Delta_{AB}\left(x, -\overleftarrow{\partial}\right). \tag{1.119}$$

On the other hand

$$h^i\Delta_{ik} = (-1)^k\Delta_{ki}h^i = (-1)^B\Delta_{BA}(y, \partial)h^A(y), \tag{1.120}$$

$$\Delta_{BA}(y, \partial)h^A = (-1)^B h^A \Delta_{AB}\left(y, -\overleftarrow{\partial}\right). \tag{1.121}$$

The formally adjoint operator is

$$\Delta^+_{AB}(x, \partial) = \Delta^*_{BA}(x, -\partial) = -\partial_\mu A^{\mu\nu\,*}_{BA}\partial_\nu - \frac{1}{2}\left(B^{\mu *}_{BA}\partial_\mu + \partial_\mu B^{\mu *}_{BA}\right) - C^*_{BA} \tag{1.122}$$

Let us consider a bilinear form

$$
\begin{aligned}
I(g,h) \quad &\equiv \quad g^i\left(\Delta_{ik}h^k\right) - \left(g^i\Delta_{ik}\right)h^k \\
&= \int\limits_{\Omega} dx\, g^A\left(\Delta_{AB}\left(x,\overrightarrow{\partial}\right) - \Delta_{AB}\left(x,-\overleftarrow{\partial}\right)\right)h^B, \quad (1.123)
\end{aligned}
$$

where Ω is a compact region of spacetime M with smooth boundary $\partial\Omega$.

For the second order operators this can be shown to be

$$
I(g,h) = \int\limits_{\Omega} dx\, \partial_\mu\left(g^A \overleftrightarrow{W}^\mu{}_{AB}\, h^B\right) = \int\limits_{\partial\Omega} d\Sigma_\mu g^A \overleftrightarrow{W}^\mu{}_{AB}\, h^B \qquad (1.124)
$$

where

$$
\overleftrightarrow{W}^\mu{}_{AB} = -A^{\mu\nu}_{AB}\overrightarrow{\partial}_\nu + \overleftarrow{\partial}_\nu A^{\mu\nu}_{AB} + B^\mu_{AB} \qquad (1.125)
$$

is called Wronskian operator associated with Δ.

For the operator Δ to be self-adjoint this antisymmetric bilinear form must vanish. This means that formally self-adjoint operator is self-adjoint indeed on the fields satisfying such boundary conditions that this surface integral vanishes. (For example Dirichlet).

1.5 Retarded and advanced Green functions

Let us consider now the inhomogeneous equation of small disturbances

$$
\Delta_{ik}\delta\varphi^k = -\delta J_i. \qquad (1.126)
$$

Suppose that Δ is a nonsingular differential operator, i.e., with some boundary conditions the solution of this equation exists and is unique.

This is not the case in the field theories with local gauge symmetries, such as Yang-Mills theory and gravity. We will deal with such theories in the further lectures. Anyway after imposing the corresponding supplementary gauge conditions the operator Δ becomes non-singular in these theories too.

The solution of the equation (1.126) can be expressed then in terms of Green functions

$$
\delta\varphi^i = G^{ij}\delta J_j = \int\limits_{\Omega} dy\, G^{AB}(x,y)\delta J_B(y), \qquad (1.127)
$$

where G^{ij} is the Green function, i.e., the solution of the equation

$$
\Delta_{ik}G^{kj} = -\delta^j{}_i \qquad (1.128)
$$

with some boundary conditions.

In classical field theory one considers the retarded and advanced boundary conditions, i.e.,

$$
\delta\varphi^+|_{\Sigma_{\rm out}} = 0, \qquad (1.129)
$$

$$\delta\varphi^-|_{\Sigma_{\rm in}} = 0. \tag{1.130}$$

That is the retarded G^{-ij} and advanced G^{+ij} Green functions satisfy the following boundary conditions

$$\begin{aligned} G^{-ij} &= 0 \quad \text{if} \quad i < j, \\ G^{+ij} &= 0 \quad \text{if} \quad i > j. \end{aligned} \tag{1.131}$$

Here $i < j$ $(i > j)$ means that the time t_i associated with the index i lies to the past (future) of the time t_j associated with the index j.

Consequently, $G^{-ij}(G^{+ij})$ is nonvanishing only when the spacetime point x_i associated with i lies on or inside the future (past) light cone emanating from the spacetime point x_j associated with j.

Future light cone

Past light cone

The self-adjointness of Δ gives rise to simple relations between the retarded and the advanced Green functions. One can show that

$$G^{\pm ij} = (-1)^{ij} G^{\mp ji}. \tag{1.132}$$

This is called reciprocity relations. The derivation is

$$\begin{aligned} 0 &= (-1)^{ik} G^{-ik} \left[\Delta_{ke} - (-1)^{k+e+ke}\Delta_{ek}\right] G^{+ej} \\ &= -(-1)^{ij} G^{-ji} - (-1)^{e(i+1)}\Delta_{ek} G^{-ki} G^{+ej} \\ &= -(-1)^{ij} G^{-ji} + G^{+ij}. \end{aligned} \tag{1.133}$$

Using the advanced and retarded Green functions one can define other Green functions. First one can define a specific solution of the homogeneous equation of small disturbances

$$\tilde{G}^{ij} \stackrel{\rm def}{=} G^{+ij} - G^{-ij}. \tag{1.134}$$

By definition it is antisupersymmetric

$$\tilde{G}^{ij} = -(-1)^{ij}\tilde{G}^{ji}. \tag{1.135}$$

This function satisfies obviously

$$\Delta_{ik}\tilde{G}^{kj} = 0 \tag{1.136}$$

and is called Pauli-Jordan (sometimes also Schwinger) *supercommutator function*. It will give the supercommutator of linear field operators in quantum theory

$$\left[\hat{\varphi}^i, \hat{\varphi}^j\right]_s = i\hbar\tilde{G}^{ij} \tag{1.137}$$

It is clear that $\tilde{G}(x, y)$ is nonvanishing only inside the light cone emanating from the point y.

This means that for two spacetime points x and y which are separated by a spacelike interval the field operators (super) commute. That is there are no physical correlations between the fields in such points. This must be so in any reasonable field theory because of the causality principle — the information cannot be transferred faster than light.

1.6 Cauchy problem for Jacobi fields

The supercommutator function gives the solution of the Cauchy problem for the Jacobi fields:

$$\Delta_{ik}\delta\varphi^k = 0 \tag{1.138}$$

$$\delta\varphi_J^A(x) = \int\limits_{\Sigma_{\text{in}}} d\Sigma_\mu \tilde{G}^{AB}(x, y)\overleftrightarrow{W}_{BC}{}^{\mu}(y, \partial)\delta\varphi_J^C(y), \tag{1.139}$$

where Σ_{in} is an arbitrary spacelike surface. Thus the Jacobi fields are completely determined by the values of $\delta\varphi$ on Σ_{in} and its first derivatives induced by the Wronskian operator.

1.7 Feynman propagator

The most important boundary condition used in QFT are the causal (Feynman) ones, which lead to the Feynman propagator. They can be described as follows. The Feynman propagator $G(x, y)$ is defined by the requirement that it should be expanded in negative frequency modes in the in-region and in positive frequency modes in the out-region, i.e., roughly speaking

$$G(x, y) = \begin{cases} \sum\limits_n e^{-i\omega_n t} u_n & , t \to -\infty \\ \sum\limits_n e^{+i\omega_n t} v_n & , t \to +\infty \end{cases} \quad t = x^0 \tag{1.140}$$

In other words, the Feynman propagator is defined by the requirement that it should be finite when

$$t \to \pm i\infty \tag{1.141}$$

This becomes formally correct by the following procedure. Let us consider the complexified spacetime when the time coordinate can take complex values. Let us go in this complexified spacetime to the so called Euclidean section, when the time is purely imaginary

$$t = i\tau. \tag{1.142}$$

This is called the Wick rotation.

The spacetime metric of the Euclidean section becomes Riemannian with the positive signature

$$g \to g_E \tag{1.143}$$

$$\operatorname{sign} g_{E\mu\nu} = (+\cdots+). \tag{1.144}$$

Further, we also define the Euclidean Lagrangian and the action functional

$$\mathcal{L} \to -\mathcal{L}_E \tag{1.145}$$

$$S \to iS_E. \tag{1.146}$$

The operator of small disturbances becomes *elliptic differential* operator

$$\Delta \to \Delta_E. \tag{1.147}$$

If , additionally, the Euclidean action S_E is a bounded functional, that is the case in most 'normal' field theories, then the operator Δ_E is positive elliptic operator. Such an operator has a unique *Euclidean Green function* defined by the equation

$$\Delta_E G_E = 1. \tag{1.148}$$

The corresponding boundary condition is the regularity of G_E at Euclidean infinity $x_E \to \pm\infty$, i.e., for $\tau \to \pm\infty$ too. But these are exactly the Feynman boundary conditions. Therefore, the Feynman propagator is obtained by the analytical continuation back to the Lorentzian spacetime

$$G_E \stackrel{\tau \longrightarrow -it}{\longrightarrow} G. \tag{1.149}$$

If the Euclidean action is not bounded from below then the operator Δ_E is not positive any longer — it can have zero modes as well as negative modes. The Euclidean Green function as well as the Feynman propagator are not well defined then. This causes difficulties in quantizing such models and could break the stability and the unitarity of the theory.

There are many other Green functions obtained by linear combinations from the advanced, retarded and Feynman ones.

For example, there is a symmetric Green function

$$\bar{G} = \frac{1}{2}\left(G^+ + G^-\right), \tag{1.150}$$

$$\Delta G = -1, \tag{1.151}$$

$$\bar{G}^{ij} = (-1)^{ij}\bar{G}^{ji}. \tag{1.152}$$

Further one defines the Hadamard Green function $G^{(1)}$ by

$$G = \bar{G} + \frac{i}{2}G^{(1)}, \tag{1.153}$$

which is a symmetric solution of the homogeneous equation

$$\Delta G^{(1)} = 0, \tag{1.154}$$

$$G^{(1)ij} = (-1)^{ij}G^{(1)ji} \tag{1.155}$$

The Wightman functions $G^{(\pm)}$ are defined by

$$G^{(\pm)} = G - iG^{\pm}. \tag{1.156}$$

All these Green function define in QFT the vacuum averages of the form $<$ out$|\,\hat{\varphi}^i\hat{\varphi}^j\,|$in $>$ for different boundary conditions.

1.8 Classical perturbation theory

Let J_i be some finite external functions and the action functional suffer the change

$$S(\varphi) \to S(\varphi) + J_i\varphi^i. \tag{1.157}$$

The equations of motion for this system are

$$S_{,i}(\varphi) = -J_i. \tag{1.158}$$

Let ϕ be the solution of this equation:

$$S_{,i}(\phi) = -J_i. \tag{1.159}$$

This means that ϕ is a functional of the sources J. The field ϕ is called the background field.

Let us look for another solution

$$\varphi = \phi + h \tag{1.160}$$

where h is a finite disturbance.

Expanding the action in h

$$\begin{aligned} S(\phi + h) &= \sum_{n\ge 0}\frac{1}{n!}S_{,i_1\dots i_n}(\phi)h^{i_n}\cdots h^{i_1} \\ &= S(\phi) + S_{,i}(\phi)h^i + \frac{1}{2}h^i{}_{i,}S_{,k}(\phi)h^k + \\ &+ \sum_{n\ge 3}\frac{1}{n!}S_{,i_1\dots i_n}(\phi)h^{i_n}\cdots h^{i_1} \end{aligned} \tag{1.161}$$

and differentiating with respect to h we obtain

$$\begin{aligned}\frac{\delta}{\delta h^i} S(\phi+h) &= \sum_{n\geq 0} \frac{1}{n!}\, {}_{i,}S_{,i_1\dots i_n} h^{i_n}\cdots h^{i_1} \\ &= {}_{i,}S + {}_{i,}S_{,k}h^k + \sum_{n\geq 2}\frac{1}{n!}\, {}_{i,}S_{,i_1\dots i_n} h^{i_n}\cdots h^{i_i}. \qquad (1.162)\end{aligned}$$

Therefore, defining $\Delta_{ik} = {}_{i,}S_{,k}$ and recalling that ${}_{i,}S = (-1)^i S_{,i} = -(-1)^i J_i$ we obtain

$$\Delta_{ik}h^k = (-1)^i J_i - \sum_{n\geq 2}\frac{1}{n!}\, {}_{i,}S_{,i_1\dots i_n} h^{i_n}\cdots h^{i_1}. \qquad (1.163)$$

If Δ is a nonsingular operator this nonlinear differential equation may be rewritten as an integro-differential one

$$h^k = h^k_J + G^{ki}\left(\sum_{n\geq 2}\frac{1}{n!}\, {}_{i,}S_{,i_1\dots i_n} h^{i_n}\cdots h^{i_1}\right), \qquad (1.164)$$

where h_J is the solution of the linear inhomogeneous equation,

$$h^k_J = h^k_0 - (-1)^i G^{ki} J_i, \qquad (1.165)$$

h_0 being a Jacobi field and G^{ki} some Green function of the operator Δ with appropriate boundary conditions.

This integro-differential equation may be solved formally by iteration. The result is a power series in h_J

$$h^k = h^k_J + G^{ki}\sum_{n\geq 2}\frac{1}{n!}\, {}_iT_{i_1\dots i_n} h^{i_n}_J\cdots h^{i_1}_J. \qquad (1.166)$$

The coefficients ${}_iT_{i_1\dots i_n}$ are called the *tree functions*. It is not difficult to calculate some first tree functions substituting the expansion (1.166) into the equation (1.164).

$${}_iT_{km} = {}_{i,}S_{,km}, \qquad (1.167)$$

$${}_iT_{kmn} = {}_{i,}S_{kmn} + {}_{i,}S_{,kp}G^{pq}S_{,qmn}. \qquad (1.168)$$

Each tree function ${}_iT_{i_1\dots i_n}$ can be presented as the sum of all tree graphs having one trunk and $n\geq 2$ terminal branches.

Each *internal* line represents a Green function (propagator)

$$G^{ik} \Longleftrightarrow \;...... \qquad (1.169)$$

and each verfex represents a vertex function

$$S_{,i_1\dots i_n} \Longleftrightarrow \;.... \qquad n\geq 3 \qquad (1.170)$$

Indices of the Green functions and vertex functions are paired together as the combinatorics of the graph indicate, and summation-integrations are performed over all pairs. If G^{ik} is not supersymmetric each internal line may have an orientation (e.g. for complex, i.e., charged, fields). Along each path from the trunk to a terminal branch the orientation are all required to be the same.

Finally, a summation is carried out over all distinct permutations of the free indices borne by the terminal branches with a factor (-1) included for each interchange of a pair of fermionic indices.

$$ {}_1T_2 = \tag{1.171}$$

$$ {}_1T_3 = \tag{1.172}$$

$$ {}_1T_4 = \tag{1.173}$$

$$ {}_1T_5 = \tag{1.174}$$

The graphs for some low-order tree functions are given on the Fig (1.174)

If we multiply the tree functions by Green function for each index we obtain the *tree multi-point Green functions*

$$G^{k_1 \dots k_n} = (-1)^P T_{i_1 \dots i_n} G^{i_n k_n} G^{i_{n-1} k_{n-1}} \cdots G^{i_1 k_1} \tag{1.175}$$

The diagram for the multi-point Green functions are the same except for now not only the internal lines but also the *external* ones represent Green function.

The multi-point Green functions appear for example, if $h_0 = 0$ and, hence, $h_J^{\ k} = -(-1)^k G^{ki} J_i$ and the solution is expanded in the external sources

$$h^k = -(-1)^i G^{ki} J_i + \sum_{n \geq 2} (-1)^{n+i_1+\cdots+i_n} \frac{1}{n!} G^{k i_1 \dots i_n} J_{i_n} \cdots J_{i_1}. \tag{1.176}$$

This solution is non-vanishing only when the sources are present.

In quantum scattering theory one encounters structures having the same general form as $h^i_J\, {}_i T_{i_1 \dots i_n} h^{i_n}_J \cdots h^{i_1}_J$. These terms are called *tree amplitudes.* In the scattering theory they become physical quantities that yield transition probabilities and transition rates.

From the structure of tree amplitudes it is clear that the whole scattering process is divided in some elementary processes, namely the propagation of small disturbances h in a given background ϕ from one spacetime point x to another y. This process is described by the propagator $G(x, y)$. Another elementary process is the local interaction of the disturbances h (in the background ϕ) between themselves at a fixed spacetime point. These processes are described by the tree vertex functions

$$S_{,i_1 \dots i_n}. \tag{1.177}$$

In local theories the vertex functions $S_{,i_1 \dots i_n}$ are ultra-local, i.e., they contain $(n-1)$ δ-functions, i.e., the terms like

$$ {}_{i,}S_{,i_1 \dots i_n} h^{i_n} \cdots h^{i_1} = f_A\left(h, \partial h, \dots, \partial^m h\right) \tag{1.178}$$

are local functionals.

In polynomial field theories there are only finite number of different types of interaction, since

$$S_{,i_1\dots i_n} = 0 \qquad \text{for } n \geq N+1, \tag{1.179}$$

N being the highest degree of the nonlinear terms in the action. However, in non-polynomial theories like gravity there are infinite-many types of interactions. This also causes difficulties in QFT by renormalizing such theories.

Chapter 2

Quantization of non-gauge field theories

In this lecture we are going to describe the *formal* structure of the usual non-gauge field theories.

Any dynamical system, both classical and quantum, is described by the set of *states* and the *dynamical evolution*. The state of a classical dynamical system at some time is characterized by the values of the fields and momentums (or velocities), i.e., the first time derivatives at this time, more precisely, on a spacelike surface. In other words, the state is a point in the *phase space*:

$$\mathbb{P} = \{(\varphi(x), \dot{\varphi}(x)) | x \in \Sigma_t,\, t \in (t_{\rm in}, t_{\rm out})\}\,. \tag{2.1}$$

Given a state at an initial time one is able to determine from the dynamical equation of motion the states at all other times, which defines the dynamical evolution of the classical dynamical system, so called dynamical trajectory. Symbolically this is show on the Fig. 2

Thus each *dynamical trajectory* is a solution of the classical equation of motion and as such defines a *point* in the configuration space $\mathcal{M}$. The set of all dynamical trajectories defines the dynamical configuration subspace or the mass shell, $\mathcal{M}_0 \subset \mathcal{M}$.

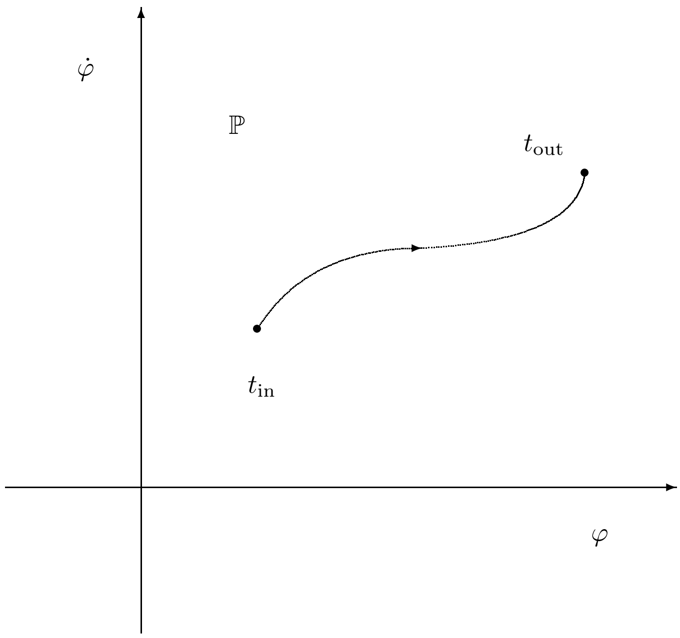

Figure 2.1: Phase space

Figure 2.2: Configuration space

In other words, the dynamical subspace is the set of all solutions of equations of motion with all possible initial conditions. Each solution, i.e., each point of $\mathcal{M}_0$, is parametrized by the initial state. Therefore, one can also call each point in the dynamical configuration subspace a 'state' of the dynamical system.

Thus the values of the fields and their first time derivatives are independent dynamical variables that completely describe the system. Any physical observable A is some functional of the dynamical variables $A(\varphi)$. The value of the physical observable in a given state is just the value of this functional on the dynamical trajectory

$$A(\varphi)|_P = A(\varphi_P), \tag{2.2}$$

where φ_P is the solution of the dynamical equations of motion with given initial conditions P.

2.1 Quantum Field Theory.

In QFT this classical picture is modernized. In short, one has three postulates:

1. The phase space $\mathbb{P}$ is substituted by a *Hilbert* space $\mathcal{H}$. The state of the system is described by a *vector* $|\psi>$ in this Hilbert space.

2. The physical observables A are represented by *Hermitian operators* $\hat{A}$ acting on the vectors of this Hilbert space

$$\hat{A} : \mathcal{H} \to \mathcal{H}, \tag{2.3}$$

$$\hat{A}^{\dagger} = \hat{A}. \tag{2.4}$$

3. The mean value of an observable A in the state $|\psi>$ is defined in terms of the inner product of Hilbert space

$$< A >=< \psi|\hat{A}|\psi > . \tag{2.5}$$

The fields also become quantum operators

$$\hat{\varphi}^i \equiv \hat{\varphi}^A(x), \tag{2.6}$$

which do not supercommute any longer. *Formally* they satisfy the same dynamical equation of motion.

$$S_{,i}(\hat{\varphi}) = 0. \tag{2.7}$$

Here there appears, of course, the known difficulty of ordering the non-commuting factors in classical expressions. However, we will not take much attention to this. The interested reader is referred to [].

If we split the field into a classical background part ϕ and a quantum one $\hat{h}$,

$$\hat{\varphi}^i = \phi^i + \sqrt{\hbar}\hat{h}^i, \tag{2.8}$$

where $\hbar$ is the Planck constant, then the supercommutation relations can be written in form

$$[\hat{\varphi}^i, \hat{\varphi}^k]_s = \hbar[\hat{h}^i, \hat{h}^k]_s = i\hbar\hat{\tilde{G}}^{ik}. \tag{2.9}$$

The supercommutator $\hat{\tilde{G}}^{ik}$ is, in general, not a function, but also an operator. In the lowest order approximation, however, $\hat{\tilde{G}}$ is just the supercommutator function of Pauli-Joirdan (or Schwinger) described in the first lecture.

2.2 S-matrix.

Most of the problems of standard QFT deal with the scattering processes. This means that in the remote past one has well defined measurable physical states. These can be, for example, two beams of free noninteracting particles that are far away from each other in the space. These beams approach each other at some finite time and do interact in some finite region Ω. After the interaction the beams go away again to infinity. (See Fig. 2.2). The particles at remote future infinity are again free, i.e., they do not interact with each other.

Figure 2.3: Scattering process

Free particles are described by the linearized equations of motion. Therefore, it is not difficult to construct the states of free particles. The essential nontrivial physical phenomena occur inside the dynamical region Ω. This processes are described by the nonlinear equations that are impossible to solve exactly, in general.

To describe formally this kind of physics one introduces the so called *scattering matrix*, or shortly S-matrix. Let $A \subset \mathcal{H}$ be the subspace of all initial states and let $|\alpha; \mathrm{in} >$ be an orthonormal complete set of initial state vectors with α being some labels. That means

$$< \alpha; \mathrm{in}|\alpha'; \mathrm{in} >= \delta_{\alpha\alpha'} \tag{2.10}$$

and any *initial* state $|\mathrm{in} >$ can be presented in form

$$|\mathrm{in} >= \sum_{\alpha} |\alpha; \mathrm{in} >< \mathrm{in}; \alpha|\mathrm{in} > . \tag{2.11}$$

Further, let $B \subset \mathcal{H}$ be the subspace of all final states and let $|\beta; \mathrm{out} >$ be an orthonormal complete set of final state vectors with other labels β, i.e.,

$$< \beta; \mathrm{out}|\beta'; \mathrm{out} >= \delta_{\beta\beta'} \tag{2.12}$$

and

$$|\mathrm{out} >= \sum_{\beta} |\beta; \mathrm{out} >< \mathrm{out}; \beta|\mathrm{out} > \tag{2.13}$$

for any *final* vector $|\mathrm{out} >$. Here the summation over the labels α and β is understood to include as usual the integration over continuous variables.

The scattering processes are described by the transitions amplitudes

$$< \mathrm{out}|\mathrm{in} > . \tag{2.14}$$

It is clear that such transition amplitudes would be known if one knows all the transition amplitudes

$$S(\beta, \alpha) \stackrel{\mathrm{def}}{=} < \mathrm{out}; \beta|\alpha; \mathrm{in} > . \tag{2.15}$$

The matrix with such elements is called the *scattering matrix*, or S-matrix. Note that if $A \neq B$ then the S-matrix is not a square matrix. This could happen, for example, if in the out-region there are some exotic states, such as bound states, that cannot be presented as a linear combination of the initial vectors $|\alpha; \mathrm{in} >$. Moreover, if the labels α and (or) β contain continuous labels then the S-matrix is infinite-dimensional matrix.

If $A = B$ and both sets are complete then one can define an operator, called the *scattering operator*,

$$\mathbf{S} = \sum_{\alpha} |\alpha; \mathrm{in} >< \mathrm{out}; \alpha|. \tag{2.16}$$

The S-matrix is then a square matrix with the entries determined by the matrix elements of this operator

$$S(\beta, \alpha) =< \text{out}; \beta|\mathbf{S}|\alpha; \text{out} >=< \text{in}; \beta|\mathbf{S}|\alpha; \text{in} > . \tag{2.17}$$

The scattering operator must be unitary

$$\mathbf{S}^\dagger \mathbf{S} = 1 \tag{2.18}$$

and the sets $|\alpha; \text{in} >$ and $|\beta; \text{in} >$ are said to be unitary equivalent. The scattering operator transforms the initial vectors in the final ones and vice versa

$$|\alpha; \text{in} >= \mathbf{S}|\alpha; \text{out} > \tag{2.19}$$

$$< \text{out}; \alpha| =< \text{in}; \alpha|\mathbf{S}. \tag{2.20}$$

2.3 Schwinger variational principle.

As we have seen the objects of main interest in QFT are the $< \text{out}|\text{in} >$ transition amplitudes. We are now going to describe a very elegant and general approach for calculating such amplitudes. Let $|\text{in} >$ and $|\text{out} >$ be some initial and final states of a quantum dynamical system. Let us consider the transition amplitude

$$< \text{out}|\text{in} > \tag{2.21}$$

and ask the question: how does $< \text{out}|\text{in} >$ change under a variation of the action δS of the form

$$\delta S = \int\limits_\Omega dx \delta \mathcal{L}(x), \tag{2.22}$$

where $\delta\mathcal{L}(x)$ has a compact support in Ω, i.e., $t_{\text{out}} > \operatorname{supp} \mathcal{L} > t_{\text{in}}$. We will often call below the support of a local functional (like the action) simply the support of the integrand, i.e.,

$$\operatorname{supp} \delta S \stackrel{\text{def}}{=} \operatorname{supp} \delta \mathcal{L} = \{x \in M : \delta \mathcal{L}(x) \neq 0\} . \tag{2.23}$$

The answer to this question gives the Schwinger's variational principle which states that

$$\delta < \text{out}|\text{in} >= i < \text{out}|\delta \hat{S}|\text{in} > . \tag{2.24}$$

This principle gives a very powerful tool to study the transition amplitudes. One can say that it is the *quantization postulate*, because the whole information about the quantum fields will be derived from the only equation (2.24).

Let us change the external conditions by adding a linear interaction with external classical sources in the dynamical region Ω to the action

$$S(\varphi) \to S(\varphi) + J_i \varphi^i \tag{2.25}$$

with $t_{\rm out} > {\rm supp} J_i > t_{\rm in}$. The amplitude $< {\rm out}|{\rm in} >$ becomes a functional of the sources $Z(J)$:

$$Z(J) =< {\rm out}|\,{\rm in} > |_{S\to S+J\varphi}. \tag{2.26}$$

By using the Schwinger variational principle one can obtain the derivatives of the functional $Z(J)$.

Consider a specific variation of the action of the form

$$\delta S = \delta J_k \varphi^k \tag{2.27}$$

with $t_{\rm out} > {\rm supp}\,\delta J_k > t_{\rm in}$. From the Schwinger variational principle we have in this case

$$\delta < {\rm out}\,|\,{\rm in} >= i(-1)^{k\varepsilon({\rm out})}\delta J_k < {\rm out}\,|\,\hat\varphi^k\,|{\rm in} > \tag{2.28}$$

where $\varepsilon({\rm out})$ is the parity of the vector $|{\rm out} >$. Hence

$$\frac{1}{i}\frac{\delta}{\delta J_k} Z = (-1)^{k\varepsilon({\rm out})} < {\rm out}|\,\hat\varphi^k\,|{\rm in} > . \tag{2.29}$$

Now let us consider this amplitude and another variation of the form (2.27) with δJ_j with support in the future with respect to the time t_k

$$t_{\rm out} > {\rm supp}\,\delta J_j > t_k > t_{\rm in}. \tag{2.30}$$

Then by defining a new initial state

$$\hat\varphi^k|\,{\rm in} >= |\,\varphi^k\,;{\rm in} > \tag{2.31}$$

one can again apply the Schwinger principle to get

$$\begin{aligned} \delta < {\rm out}\,|\,\varphi^k|{\rm in} > &= \delta < {\rm out}|\varphi^k;{\rm in} > \\ &= i(-1)^{j\varepsilon({\rm out})}\delta J_j < {\rm out}|\hat\varphi^j|\varphi^k;{\rm in} > \\ &= i(-1)^{j\varepsilon({\rm out})}\delta J_j < {\rm out}|\hat\varphi^j\hat\varphi^k|{\rm in} > . \end{aligned} \tag{2.32}$$

Therefore,

$$\left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J_j}\frac{\delta}{\delta J_k} Z = (-1)^{(j+k)\varepsilon({\rm out})} < {\rm out}|\hat\varphi^j\hat\varphi^k|{\rm in} > \tag{2.33}$$

for $t_{\rm out} > t_j > t_k > t_{\rm in}$. In the opposite case

$$t_{\rm out} > t_k > t_j > t_{\rm in}, \tag{2.34}$$

i.e., if the support of the second variation is in the past with respect to the time t_k we calculate

$$\begin{aligned} \delta < {\rm out}|\hat\varphi^k|{\rm in} > &= \delta < {\rm out};\varphi^k|{\rm in} > \\ &= i < {\rm out};\varphi^k|\delta J_j\hat\varphi^j|{\rm in} > \\ &= i < {\rm out}|\hat\varphi^k\delta J_j\hat\varphi^j|{\rm in} > \\ &= i(-1)^{jk+j\varepsilon({\rm out})}\delta J_j < {\rm out}|\hat\varphi^k\hat\varphi^j|{\rm in} > . \end{aligned} \tag{2.35}$$

That is

$$\left(\frac{1}{i}\right)^2 \frac{\delta^2}{\delta J_j \delta J_k} Z = (-1)^{jk+(j+k)\varepsilon(\mathrm{out})} < \mathrm{out}|\hat{\varphi}^k \hat{\varphi}^j|\mathrm{in} > \tag{2.36}$$

for $t_{\mathrm{out}} > t_k > t_j > t_{\mathrm{in}}$.

One can combine both cases in one formula by writing

$$\left(\frac{1}{i}\right)^2 \frac{\delta^2}{\delta J_j \delta J_k} Z = (-1)^{(j+k)\varepsilon(\mathrm{out})} < \mathrm{out}|T(\hat{\varphi}^j \hat{\varphi}^k)|\mathrm{in} > \tag{2.37}$$

where

$$T\left(\hat{\varphi}^j \hat{\varphi}^k\right) = \begin{cases} \hat{\varphi}^j \hat{\varphi}^k, & t_j > t_k \\ (-1)^{jk} \hat{\varphi}^k \hat{\varphi}^j, & t_k > t_j \end{cases} \tag{2.38}$$

is the *chronological product*.

One can show that in general

$$\left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J_{i_n} \cdots \delta J_{i_1}} Z = < \mathrm{out}|T\left(\hat{\varphi}^{i_n} \cdots \hat{\varphi}^{i_1}\right)|\mathrm{in} > \tag{2.39}$$

where we assumed for simplicity that

$$\varepsilon(\mathrm{out}) = \varepsilon(\mathrm{in}) = 0. \tag{2.40}$$

In other words the functional $Z(J)$ is the *generating functional* for *chronological amplitudes*

$$Z(J+\eta) = \sum_{n\geq 0} \frac{i^n}{n!} \eta_{i_1} \cdots \eta_{i_n} < \mathrm{out}|T\left(\hat{\varphi}^{i_n} \cdots \hat{\varphi}^{i_1}\right)|\mathrm{in} > . \tag{2.41}$$

The chronological amplitude of any (analytical) functional $A(\varphi)$

$$A(\varphi) = \sum_{n\geq 0} \frac{1}{n!} A_{i_1 \ldots i_n} \varphi^{i_n} \cdots \varphi^{i_1}, \tag{2.42}$$

is given by

$$\begin{aligned} < \mathrm{out}|T(A(\varphi)|\mathrm{in} > &= \sum_{n\geq 0} \frac{1}{n!} A_{i_1 \ldots i_n} < \mathrm{out}|T\left(\hat{\varphi}^{i_n} \cdots \hat{\varphi}^{i_1}\right)|\mathrm{in} > \\ &= \sum_{n\geq 0} \frac{1}{n!} A_{i_1 \ldots i_n} \frac{1}{i^n} \frac{\delta^n}{\delta J_{i_n} \cdots \delta J_{i_1}} Z \\ &= A\left(\frac{1}{i}\frac{\delta}{\delta J}\right) Z(J). \end{aligned} \tag{2.43}$$

Let us now define another functional $W(J)$ by

$$Z(J) = e^{iW(J)} \tag{2.44}$$

and consider its Taylor expansion

$$W(J+\eta)=\sum_{n\geq 0}\frac{1}{n!}\eta_{i_1}\cdots\eta_{i_n}\mathcal{G}^{i_n\dots i_1}(J) \tag{2.45}$$

with

$$\mathcal{G}^{i_n\dots i_1}(J)=\frac{\delta^n}{\delta J_{i_n}\cdots\delta J_i}W(J). \tag{2.46}$$

Further we denote

$$\phi^i(J)\stackrel{\rm def}{=}\frac{\delta W(J)}{\delta J_i} \tag{2.47}$$

and introduce the *chronological mean value* by

$$<A(\hat\varphi)>=\frac{<{\rm out}|T(A(\hat\varphi))|{\rm in}>}{<{\rm out}|{\rm in}>}. \tag{2.48}$$

It is easy to show that

$$\begin{aligned} <A(\varphi)> &= e^{-iW}A\left(\frac{1}{i}\frac{\delta}{\delta J}\right)e^{iW} \\ &= A\left(\phi+\frac{1}{i}\frac{\delta}{\delta J}\right)\cdot 1. \end{aligned} \tag{2.49}$$

Therefore,

$$\begin{aligned} <\hat\varphi^{i_n}\cdots\hat\varphi^{i_1}> &= \left(\frac{1}{i}\right)^n e^{-iW}\frac{\delta^n}{\delta J_{i_n}\cdots\delta J_{i_1}}e^{iW} \\ &= \left(\phi^{i_n}+\frac{1}{i}\frac{\delta}{\delta J_{i_n}}\right)\cdots\left(\phi^{i_2}+\frac{1}{i}\frac{\delta}{\delta J_{i_2}}\right)\phi^{i_1}. \end{aligned} \tag{2.50}$$

In particular,

$$<\hat\varphi^i>=\phi^i, \tag{2.51}$$

$$<\hat\varphi^i\hat\varphi^k>=\phi^i\phi^k+\frac{1}{i}\mathcal{G}^{ik}, \tag{2.52}$$

$$<\hat\varphi^i\hat\varphi^k\hat\varphi^j>=\phi^i\phi^k\phi^j+\frac{3}{i}\phi^{(i}\mathcal{G}^{kj)}+\left(\frac{1}{i}\right)^2\mathcal{G}^{ikj}, \tag{2.53}$$

etc. Here the indices in the brackets are *supersymmetrized*, i.e., one has to sum over all permutation of the indices adding factor (-1) for each term having odd number of fermionic permutations, e.g.

$$\phi^{(i}\mathcal{G}^{kj)}=\frac{1}{3}\left\{\phi^i\mathcal{G}^{kj}+(-1)^{i(j+k)}\phi^k\mathcal{G}^{ji}+(-1)^{j(i+k)}\phi^j\mathcal{G}^{ik}\right\}. \tag{2.54}$$

Thus we see that ϕ is actually the *mean field*, $\mathcal{G}^{ik}$ is called the *one-point Green function*, or *propagator*, and $\mathcal{G}^{i_i\cdots i_n}$ — *the multi-point Green functions*. They describe the extent to which the mean values of products of field operators differ form products of the mean values. That is why they are also called *correlation functions.*

Thus, whilst $Z(J)$ is the generating functional for chronological amplitudes the functional $W(J)$ is the generating functional for the Green functions. The Green functions satisfy the boundary conditions which are determined by the states $|\text{in}>$ and $|\text{out}>$.

2.4 The effective action.

The mean field itself is a functional of the sources, $\phi = \phi(J)$, the derivative of the mean field being the propagator

$$\frac{\delta\phi^i}{\delta J_j} = \frac{\delta^2 W}{\delta J_j \delta J_i} = \mathcal{G}^{ji}. \tag{2.55}$$

Therefore, if $\mathcal{G}^{ij}$ is a non-degenerate matrix one can change the variables and consider ϕ as independent variable and $J(\phi)$ (as well as all ofter functionals) as the functional of ϕ . The derivative with respect to J is then

$$\frac{\delta}{\delta J_i} = \frac{\delta\phi^k}{\delta J_i}\frac{\delta}{\delta\phi^k} = \mathcal{G}^{ik}\frac{\delta}{\delta\phi^k}. \tag{2.56}$$

In particular,

$$< A(\varphi) >= A\left(\phi^j + \frac{1}{i}\mathcal{G}^{jk}\frac{\delta}{\delta\phi^k}\right)\cdot 1. \tag{2.57}$$

Also

$$\mathcal{G}^{i_n \dots i_1} = G^{i_n k_n}\frac{\delta}{\delta\phi^{k_n}}\cdots\mathcal{G}^{i_3 k_3}\frac{\delta}{\delta\phi^{k_3}}\mathcal{G}^{i_2 i_1}. \tag{2.58}$$

Let us consider now the operator equations of motion

$$S_{,i}(\hat\varphi) = -J_i. \tag{2.59}$$

The mean value of these equations reads

$$< S_{,i}(\hat\varphi) >= -J_i. \tag{2.60}$$

Differentiating this equation with respect to J_j and using eq. (2.56) we have

$$\mathcal{G}^{jk}{}_{k}, < S_{,i}(\hat\varphi) >= \delta^j_i, \tag{2.61}$$

where the comma outside the brackets means differentiation with respect to the background field ϕ^k, whereas the comma inside the brackets — the differentiation with respect to the quantum field $\hat\varphi^i$. If we assume Feynman boundary conditions the propagator is supersymmetric

$$\mathcal{G}^{ik} = (-1)^{i+k+ik}\mathcal{G}^{ki}. \tag{2.62}$$

This means that the matrix

$$ {}_{k},< S_{,i}(\hat{\varphi}) > \qquad (2.63)$$

is supersymmetric too

$$ {}_{k},< S_{,i}(\hat{\varphi}) >= (-1)^{i+k+ik}\ {}_{i},< S_{,k}(\hat{\varphi}) >, \qquad (2.64)$$

which can be also rewritten in form

$$\left\langle S(\hat{\varphi})\frac{\overleftarrow{\delta}}{\delta\hat{\varphi}^i}\right\rangle\frac{\overleftarrow{\delta}}{\delta\phi^k} = (-1)^{ik}\left\langle S(\hat{\varphi})\frac{\overleftarrow{\delta}}{\delta\hat{\varphi}^k}\right\rangle\frac{\overleftarrow{\delta}}{\delta\phi^i}. \qquad (2.65)$$

This means that there exists a functional $\Gamma(\phi)$ such that

$$< S(\hat{\varphi})\frac{\overleftarrow{\delta}}{\delta\hat{\varphi}^i} >= \Gamma\frac{\overleftarrow{\delta}}{\delta\phi^i}. \qquad (2.66)$$

Therefore, the equations (2.60) and (2.61) take the form

$$\Gamma_{,i} = -J_{i,}, \qquad (2.67)$$

$$ {}_{k},\Gamma_{,i}\mathcal{G}^{ij} = -\delta^j_k. \qquad (2.68)$$

One can also express the generating functional W directly in terms of the functional Γ. We have

$$\phi^i = \frac{\delta}{\delta J_i}W = \mathcal{G}^{ij}\frac{\delta}{\delta\phi_j}W. \qquad (2.69)$$

Using eq. (2.68) we obtain therefrom

$$\frac{\delta}{\delta\phi_i}W = -{}_{j},\Gamma_{,i}\phi^i = \frac{\delta}{\delta\phi_j}(\Gamma - \Gamma_{,i}\phi^i). \qquad (2.70)$$

Therefore,

$$W(J) = \Gamma(\phi(J)) - \Gamma(\phi(J))\frac{\overleftarrow{\delta}}{\delta\phi_i}\phi^i(J) \qquad (2.71)$$

up to some additive nonessential normalization constant. Using eqs. (2.67), (2.69) this can also be rewritten as

$$\Gamma(\phi) = W(J(\phi)) - J_i(\phi)\frac{\delta}{\delta J_i}W(J(\phi)). \qquad (2.72)$$

The equations (2.71) and (2.72) are nothing but the functional Legendre transform.

The eqs. (2.67) are the effective equations of motion determining the dynamics of the background field $\phi = \phi_0$. That is why the functional $\Gamma(\phi)$ is called the effective action. The equation (2.68) determines the propagator on the background ϕ_0 and higher order Green functions may be expressed in terms of the propagator and the derivatives of the effective action (called vertex functions) by using the equation (2.58) and the identity

$$\frac{\delta}{\delta\phi^m}\mathcal{G}^{ij} = (-1)^{mi}\mathcal{G}^{ik}{}_{km,}\Gamma_{,n}\mathcal{G}^{nj}. \tag{2.73}$$

For example,

$$\mathcal{G}^{kij} = \mathcal{G}^{km}\frac{\delta}{\delta\phi^m}\mathcal{G}^{ij} = (-1)^{mi}\mathcal{G}^{km}\mathcal{G}^{ip}{}_{pm,}\Gamma_{,n}\mathcal{G}^{nj}. \tag{2.74}$$

2.5 Graphical representation.

There is a very convenient graphical representation of the Green functions. Let us represent the propagator $\mathcal{G}$ by a thick line

$$\mathcal{G}^{ij} \Longleftrightarrow, \tag{2.75}$$

which can have orientation in case $\mathcal{G}^{ij}$ is not supersymmetric, and the derivatives of the effective action of order 3 and higher by vertexes having prongs equal in number to the number of functional differentiations

$$\Gamma_{,i_1\ldots i_n} \Longleftrightarrow n \geq 3 \tag{2.76}$$

Then the Green functions are represented by diagrams in which lines are joined together at vertices in the same ways as the propagators in the explicit expressions are coupled to derivatives of the effective action by dummy indices.

These diagrams are obtained by application of two rules:

1. The differentiation with respect to the source corresponds to the insertion of an *external line* in all possible ways into a given diagram.
2. The differentiation with respect to the mean field corresponds to the insertion of a *vertex prong* in all possible ways into a given diagram.

Each Green function of a given order is expressible as the sum of all *simply connected* (or tree) diagrams having a fixed number of external lines. Representing the Green functions $\mathcal{G}_n, (n \geq 3)$ by a polygon with n external lines

$$\mathcal{G}^{i_1\ldots i_n} \Longleftrightarrow n \geq 3 \tag{2.77}$$

one can draw some low-order Green functions.

Here P_N indicates that the indices associated with the external lines are to be permuted just sufficiently to yield complete supersymmetry, N being the number of permutations required.

Structure of the diagrams:

These are exactly the *same* diagrams of the classical perturbation theory for the bare Green functions. They are *tree* diagrams. The only difference is in substituting the bare (classical) propagator by the full (or dressed, exact) one

$$G\} \Longrightarrow \{\mathcal{G} \tag{2.78}$$

and the bare vertex functions by the full (exact) ones

$$S_{,i_1\dots i_n}\} \Longrightarrow \{\Gamma_{,i_1\dots i_n}\ . \tag{2.79}$$

The difference with the classical theory is that the full vertexes $\Gamma_(n)$ are *nonlocal* and, in general, *do not vanish* for any n, also for polynomial theories. The similarity with classical tree diagrams occurs because of the nature of the problem: one is trying to solve the effective equations

$$\Gamma_{,i} = -J_i \tag{2.80}$$

instead of the classical ones

$$S_{,i} = -J_i. \tag{2.81}$$

So, the only difference is in substituting

$$S \Longrightarrow \Gamma. \tag{2.82}$$

Summarizing one can say that the knowledge of the effective action enables one to compute all the scattering amplitudes, i.e., the S-matrix.

i) First of all, it determines the mean fields $\phi =< \varphi >$ by means of the eq. (2.67).

ii) Second, it determines the propagator, i.e., the one-point Green function $\mathcal{G}^{ij}$.

iii) Further, it gives the vertex functions $\Gamma_{,i_1 \dots i_n}$ $(n \geq 3)$ that together with the propagator determine the multi-point Green functions $\mathcal{G}^{i_1 \cdots i_n}$ $(n \geq 3)$ by means of the tree diagrams.

iv) Finally, the effective action determines the functional W, or the amplitude $<$ out|in $>$, that together with the multi-point Green functions determine all the chronological amplitudes $<$ out$|\mathrm{T}(A(\hat{\varphi}))|$in $>$ and, hence, the S-matrix.

2.6 Computation of the chronological mean values.

Thus we have seen how all the Green functions can be calculated in terms of the propagator and the vertex functions.

Let us now show how the chronological values of any functional can be calculated in terms of the Green functions. Consider some analytic functional

$$A(\varphi) = \sum_{n \geq 0} \frac{1}{n!} A_{i_n \dots i_1} \varphi^{i_1 \dots i_n}. \tag{2.83}$$

From the equation (2.49) we know that the chronological mean value of this functional can be presented in form

$$< A(\varphi) >= e^{-iW(J)} A\left(\frac{1}{i}\frac{\delta}{\delta J}\right) e^{iW(J)}. \tag{2.84}$$

We calculate

$$
\begin{aligned}
< A(\hat\varphi) >&= A\left(\frac{1}{i}\frac{\delta}{\delta\eta}\right) e^{i[W(J+\eta)-W(J)]}\Big|_{\eta=0}\\
&= A\left(\frac{1}{i}\frac{\delta}{\delta\eta}\right)\exp\left\{i\left[\eta_k\phi^k+\sum_{n\ge 2}\frac{1}{n!}\eta_{i_n}\cdots\eta_{i_1}\mathcal{G}^{i_1\dots i_n}\right]\right\}\Bigg|_{\eta=0}\\
&= A\left(\frac{1}{i}\frac{\delta}{\delta\eta}\right) e^{i\eta_k(\phi^k+h^k)}\exp\left\{i\sum_{n\ge 2}\frac{1}{n!}\left(\frac{1}{i}\right)^n\frac{\overleftarrow{\delta}}{\delta h^{i_n}}\cdots\frac{\overleftarrow{\delta}}{\delta h^{i_1}}\mathcal{G}^{i_1\dots i_n}\right\}\Bigg|_{\substack{\eta=0\\h=0}}\\
&= A(\phi+h)\exp\left\{i\sum_{n\ge 2}\frac{1}{n!}\left(\frac{1}{i}\right)^n\frac{\overleftarrow{\delta}}{\delta h^{i_n}}\cdots\frac{\overleftarrow{\delta}}{\delta h^{i_1}}\mathcal{G}^{i_1\dots i_n}\right\}\Bigg|_{h=0}. \qquad (2.85)
\end{aligned}
$$

This result can be also rewritten in a slightly different form

$$
< A(\hat\varphi) >= A(\phi)\colon \exp\left\{i\sum_{n\ge 2}\frac{1}{n!}\left(\frac{1}{i}\right)^n\frac{\overleftarrow{\delta}}{\delta \phi^{i_n}}\cdots\frac{\overleftarrow{\delta}}{\delta \phi^{i_1}}\mathcal{G}^{i_1\dots i_n}\right\}\colon \qquad (2.86)
$$

where the colon denotes the normal ordering, i.e., in the expansion of the exponent all the functional derivatives should be moved to the left and act to the left. In other words, although the Green functions $G^{i_1\cdots i_n}$ are also functionals of ϕ, in the expansion of the normal ordered exponent the functional derivatives are treated not to act on the Green functions.

2.7 Functional integration.

We are going now to introduce the notion of the functional integration, i.e., the integration over the configuration space.

To do this let us consider first the *finite* dimensional approximation. That is we substitute the spacetime manifold M with a *finite* subset of points $M_N \subset M$. Consider first the *boson* fields. Then any field configuration φ^i becomes a finite-dimensional column-vector, i.e., $i = 1,\dots, D\times N$. Thus the configuration space $\mathcal{M}$ becomes a finite dimensional manifold $\mathcal{M}_N \subset \mathcal{M}$ with local coordinates φ^i . We assume that the values of fields vary from $-\infty$ to $+\infty$. So, in this approximation, the configuration space $\mathcal{M}_N$ is just $\mathbb{R}^{D\times N}$

$$
\mathcal{M}_N = \mathbb{R}^{D\times N}. \qquad (2.87)
$$

In some cases the values of the fields can be restricted by some constraints. The configuration space $\mathcal{M}_N$ can be then a *region* of $\mathbb{R}^{D\times N}$, or, more generally, can be some compact Riemannian space with some metric and so on. But we will not consider such complications.

All such complications are connected with the *global* structure of the configuration space, the problem that is far away from its solution. In other words, our consideration is *purely local in the configuration space.* We consider actually the points of the configuration space that lie in the neighborhood of the dynamical subspace $\mathcal{M}_0$. This is the typical approach of the perturbation theory — one has a classical background and some small quantum fluctuations around this background. In the case when the weight of large fluctuations is suppressed one can extent this small neighbourhood of the mass shell by the whole tangent space. The error of such approximation is asymptotically small in the semiclassical limit.

Any functional of the fields $A(\varphi)$ is just a *function* of finite number of variables φ^i . Let us suppose that this function falls off sufficiently rapidly at the infinity, so that

$$\lim_{\varphi\to\pm\infty}\left|\varphi^{i_1}\cdots\varphi^{i_n}\frac{\partial}{\partial\varphi^{k_1}}\cdots\frac{\partial}{\partial\varphi^{k_m}}A(\varphi)\right|\to 0 \tag{2.88}$$

for any n and m. Let us consider the *finite* dimensional integral

$$\int\limits_{\mathbb{R}^{D\times N}}\mathcal{D}\varphi A(\varphi) \tag{2.89}$$

with some measure

$$\mathcal{D}\varphi\equiv\frac{d\varphi^1}{\sqrt{2\pi}}\cdots\frac{d\varphi^{D\times N}}{\sqrt{2\pi}}. \tag{2.90}$$

Such integrals have a number of crucial properties that do not depend much on the dimension of the space $\mathbb{R}^{D\times N}$.

i) First of all, transformation rule of the measure under the change of variables

$$\mathcal{D}\varphi=\mathcal{D}\varphi'\det\left|\frac{\partial\varphi}{\partial\varphi'}\right|. \tag{2.91}$$

ii) Second, there is the integration by parts *without* the off-integral terms

$$\int\mathcal{D}\varphi A(\varphi)\frac{\partial}{\partial\varphi^i}B(\varphi)=-\int\mathcal{D}\varphi A(\varphi)\frac{\overleftarrow{\partial}}{\partial\varphi^i}B(\varphi) \tag{2.92}$$

iii) Third, there is the well defined Fourier transform

$$B(J)=\int\mathcal{D}\varphi e^{iJ\varphi}A(\varphi), \tag{2.93}$$

$$A(\varphi)=\int\mathcal{D}Je^{-iJ\varphi}B(J) \tag{2.94}$$

where $J\varphi=J_k\varphi^k$.

iv) Fourth, the Fourier transform of the unity defines the delta-function

$$\tilde{\delta}(J) = \int \mathcal{D}\varphi e^{iJ\varphi}, \tag{2.95}$$

so that

$$\int \mathcal{D}J\tilde{\delta}(J - J')A(J) = A(J'). \tag{2.96}$$

v) Finally, there is a particular but very important class of such integrals, so called Gaussian integrals. With our normalization of the measure we have

$$\int \mathcal{D}\varphi e^{-\frac{1}{2}|\varphi|^2} = 1, \tag{2.97}$$

where $|\varphi|^2 = \varphi^i \delta_{ik} \varphi^k$. More generally,

$$\int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi A\varphi} = (\det A)^{-1/2}, \tag{2.98}$$

where $\varphi A\varphi \equiv \varphi^i A_{ik} \varphi^k$. The determinant, $\det A$, appears actually as the Jacobian of the change of variables $\varphi \to A^{-1/2}\varphi$.

This formula is valid for any nondegenerate matrix A having eigenvalues with *positive* real part:

$$\mathrm{Re}\, A > 0. \tag{2.99}$$

If

$$\lambda_a(A), \quad |\arg \lambda_a(A)| < \frac{\pi}{2} \tag{2.100}$$

denote the eigenvalues of the matrix A then the formula (2.98) can be also rewritten in the form

$$\int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi A\varphi} = |\det A|^{-1/2} \exp(-i\, \mathrm{ind}(A)) \tag{2.101}$$

where

$$\mathrm{ind}(A) = \frac{1}{2}\sum_a \arg \lambda_a(A) \tag{2.102}$$

is the index of the matrix A. By presenting the matrix A in the polar coordinates

$$A = \sqrt{AA^*}\, e^{i \arg(A)}, \tag{2.103}$$

where

$$\arg(A) = \frac{1}{i}\log \frac{A}{\sqrt{AA^*}}\,, \tag{2.104}$$

we find that the index is determined by the trace of the phase

$$\mathrm{ind}(A) = \frac{1}{2}\mathrm{tr}\, \arg(A). \tag{2.105}$$

For a nondegenerate *Hermitian* matrix Δ having non-zero real eigenvalues one has also

$$\int \mathcal{D}\varphi e^{\frac{i}{2}\varphi\Delta\varphi} = (\det(-i\Delta))^{-1/2} = |\det\Delta|^{-1/2} \exp\left[\frac{i\pi}{4}\,\mathrm{sign}(\Delta)\right] \qquad (2.106)$$

where

$$\mathrm{sign}(\Delta) = N_+(\Delta) - N_-(\Delta) \qquad (2.107)$$

is the signature of the matrix Δ and $N_+(\Delta)$ and $N_-(\Delta)$ are the numbers of the positive and negative eigenvalues. Note that the formula (2.106) follows from (2.107) with account of

$$\mathrm{ind}(-i\Delta) = -\frac{\pi}{4}\,\mathrm{sign}(\Delta). \qquad (2.108)$$

By shifting the integration variable $\varphi \to \varphi + \mathrm{const}$ in Gaussian integrals we obtain more general formulas

$$\int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi A\varphi + iJ\varphi} = (\det A)^{-1/2} \exp\left(-\frac{1}{2} J A^{-1} J\right). \qquad (2.109)$$

$$\int \mathcal{D}\varphi e^{\frac{1}{2}\varphi\Delta\varphi + iJ\varphi} = (\det(-i\Delta))^{-1/2} \exp\left(\frac{i}{2} JGJ\right) \qquad (2.110)$$

where $G = -\Delta^{-1}$.

From these equation by expanding in power series in J we obtain a series of integrals

$$\int \mathcal{D}\varphi e^{\frac{i}{2}\varphi\Delta\varphi} \varphi^{i_1} \cdots \varphi^{i_{2n+1}} = 0. \qquad (2.111)$$

$$\int \mathcal{D}\varphi e^{\frac{i}{2}\varphi\Delta\varphi} \varphi^{i_1} \cdots \varphi^{i_{2n}} = (\det(-i\Delta))^{-1/2} \frac{(2n)!}{n!} \left(\frac{i}{2}\right)^n G^{(i_1 i_2} \cdots G^{i_{2n-1} i_{2n})}. \qquad (2.112)$$

Using these integrals one can calculate, at least formally, integrals of arbitrary analytical functions with Gaussian measure

$$\int \mathcal{D}\varphi e^{\frac{i}{2}\varphi\Delta\varphi + iJ\varphi} B(\varphi) = \int \mathcal{D}\varphi e^{\frac{i}{2}\varphi\Delta\varphi} B\left(\frac{1}{i}\frac{\partial}{\partial J}\right) e^{iJ\varphi} = B\left(\frac{1}{i}\frac{\partial}{\partial J}\right) \det(-i\Delta)^{-1/2} \exp\left(\frac{i}{2} JGJ\right). \qquad (2.113)$$

2.8 Stationary phase method.

Let us consider now integrals depending on a small parameter $\hbar$

$$Z(J) = \int \mathcal{D}\varphi \, \exp\left\{ \frac{i}{\hbar}[S(\varphi) + J\varphi] \right\} \qquad (2.114)$$

where $S(\varphi)$ is a *real* valued function. Our aim is to calculate this integral in the limit $\hbar \to 0$.

It is clear that as $\hbar \to 0$ the integral oscillates very fast and gives an asymptotically small contribution. The main contribution comes from the critical point φ_0^i where the phase $S(\varphi) + J\varphi$ is stationary. The critical points φ_0 are the solutions of the equations

$$\frac{\partial S}{\partial \varphi^i} = -J_i. \qquad (2.115)$$

and are, of course, some functions of J, $\varphi_0 = \varphi_0(J)$. We assume that there is only finite number of critical points $\varphi_{0,\alpha}(J)$, $(\alpha = 1, \dots, p)$, all of them being *isolated* points. Then one can divide the whole integration region in the non-overlapping neighborhoods of the critical points $\mathcal{M}_\alpha$,

$$\bigcup_{\alpha=1}^{p} \mathcal{M}_\alpha \subset \mathbb{R}^{N\times D} \qquad (2.116)$$

$$\mathcal{M}_\alpha \bigcap \mathcal{M}_\beta = \emptyset, \qquad \alpha \neq \beta. \qquad (2.117)$$

The whole integral becomes the sum of the integrals over the neighborhoods of the critical points

$$\begin{aligned} Z(J) &= \sum_\alpha \int\limits_{\mathcal{M}_\alpha} \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar}[S(\varphi) + J\varphi] \right\} \\ &+ \text{ asymptotically small terms.} \end{aligned} \qquad (2.118)$$

In each $\mathcal{M}_\alpha$ we change the integration variables $\varphi^i = \varphi^i_{0,\alpha} + h^i$ and expand the exponent in power series in h

$$\begin{aligned} S(\varphi) + J\varphi &= S(\varphi_{0,\alpha}) + J_i\varphi^i_{0,\alpha} + \frac{1}{2}S_{,ik}(\varphi_{0,\alpha})h^k h^i + \\ &+ \sum_{n\geq 3} \frac{1}{n!} S_{,i_1\dots i_n}(\varphi_{0,\alpha}) h^{i_n} \cdots h^{i_1}. \end{aligned} \qquad (2.119)$$

Then with the same accuracy we extend each integration region $\mathcal{M}_\alpha$ to the whole space $\mathbb{R}^{D\times N}$ obtaining

$$Z(J) = \sum_\alpha \exp\left[\frac{i}{\hbar}\left(S(\varphi_{0,\alpha}) + J\varphi_{0,\alpha}\right) \right]$$

$$\times \int \mathcal{D}h \exp\left\{\frac{i}{\hbar}\left(\frac{1}{2}S_{,ik}(\varphi_{0,\alpha})h^k h^i + \sum_{n\geq 3}\frac{1}{n!}S_{,i_1\dots i_n}(\varphi_{0,\alpha})h^{i_n}\cdots h^{i_1}\right)\right\}. \tag{2.120}$$

This integral can be calculated by using the formula (2.113):

$$Z(J) = \sum_\alpha e^{\frac{i}{\hbar}(S(\varphi_{0,\alpha})+J\varphi_{0,\alpha})}\det\left(-\frac{i}{\hbar}S_{,ik}(\varphi_{0,\alpha})\right)^{-1/2}$$
$$\times \quad \exp\left\{i\sum_{n\geq 3}\frac{\hbar^{(n-1)/2}}{n!}S_{,i_1\dots i_n}(\varphi_{0,\alpha})\frac{\partial}{i\partial p_{i_n}}\cdots\frac{\partial}{i\partial p_{i_1}}\right\}\exp\left(\frac{i}{2}p_i G^{ik}_{0,\alpha}p_k\right)\Bigg|_{p=0}, \tag{2.121}$$

where $G^{ik}_{0,\alpha}$ is the inverse of $S_{,ik}(\varphi_0)$

$$S_{,ik}(\varphi_{0,\alpha})G^{kj}_{0,\alpha} = -\delta^j_i. \tag{2.122}$$

Note that the critical points φ_α are determined from the equation (2.115) and do, therefore, depend on J. Let us rebuild the asymptotic expansion by replacing $J = \hbar\bar{J}$, i.e.,

$$Z(\bar{J}) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}S(\varphi)+i\bar{J}\varphi}. \tag{2.123}$$

The critical points, denoted now by $\bar{\varphi}_{0,\alpha}$, are defined as the solutions of the equation

$$\frac{\partial S}{\partial \varphi} = 0, \tag{2.124}$$

and do not depend on $\bar{J}$.

In this case we have another asymptotic expansion

$$\begin{aligned} Z(J) &= \sum_\alpha \exp\left(\frac{i}{\hbar}S(\bar{\varphi}_{0,\alpha}) + i\bar{J}\bar{\varphi}_{0,\alpha}\right) \\ &\times \int \mathcal{D}h \exp\Bigg\{\frac{i}{\hbar}\frac{1}{2}S_{,ik}(\bar{\varphi}_{0,\alpha})h^k h^i + i\bar{J}_k h^k \\ &+ \frac{i}{\hbar}\sum_{n\geq 3}\frac{1}{n!}S_{,i_1\dots i_n}(\bar{\varphi}_{0,\alpha})h^{i_n}\cdots h^{i_1}\Bigg\}. \end{aligned} \tag{2.125}$$

Using the formula (2.113) we obtain

$$\begin{aligned} Z(J) &= \sum_\alpha \exp\left[\frac{i}{\hbar}S(\bar{\varphi}_{0,\alpha}) + i\bar{J}\bar{\varphi}_{0,\alpha}\right]\det\left(-\frac{i}{\hbar}S_{,ik}(\bar{\varphi}_{0,\alpha})\right)^{-1/2} \\ &\times \exp\left\{\frac{i}{\hbar}\sum_{n\geq 3}\frac{1}{n!}S_{,i_1\dots i_n}(\bar{\varphi}_{0,\alpha})\frac{\partial}{i\partial p_{i_n}}\cdots\frac{\partial}{i\partial p_{i_1}}\right\}\exp\left\{\frac{i}{2}\hbar p_i\bar{G}^{ik}_0 p_k\right\}\Bigg|_{p=\bar{J}} \end{aligned} \tag{2.126}$$

where $\bar{G}_0^{ik}$ is the inverse of $S_{,ik}(\bar{\varphi}_{0,\alpha})$:

$$S_{,ik}(\bar{\varphi}_{0,\alpha})\bar{G}_0^{kj} = -\delta_i^j. \tag{2.127}$$

Obviously, for $J = 0$ we have $\varphi_\alpha(0) = \bar{\varphi}_{0,\alpha}, G_0 = \bar{G}_0$ and the expansions (2.121) and (2.125) do coincide.

Let us rebuild the asymptotic expansion a bit more. We assume that there is the only critical point that depends on J and it is itself an asymptotic series in $\hbar$. We do not know first how to determine it. Later we will get an equation for it. So, let us denote this *true* critical point by $\phi(J)$ and expand the exponent around it. By using the equation (2.113) we obtain analogously

$$\begin{aligned} Z(J) &= \exp\left[\frac{i}{\hbar}\Big(S(\phi) + J\phi\Big)\right] \det\left(-\frac{i}{\hbar}S_{,ik}(\phi)\right)^{-1/2} \\ &\times \exp\left\{i\sum_{n\geq 3}\frac{\hbar^{n-1}}{n!}S_{,i_1\dots i_n}(\phi)\frac{\partial}{i\partial p_{i_n}}\cdots\frac{\partial}{i\partial p_{i_1}}\right\} \\ &\times \exp\left\{\frac{i}{2}\hbar\Big(p_i + S_{,i}(\phi)\Big)G^{ik}(\phi)\Big(p_k + S_{,k}(\phi)\Big)\right\}\Bigg|_{p=J} \end{aligned} \tag{2.128}$$

where $G^{ik}(\phi)$ is defined by

$$S_{,ik}(\phi)G^{kj} = -\delta_i^j. \tag{2.129}$$

Now let us define another function

$$Z(J) = e^{\frac{i}{\hbar}W(J)} \tag{2.130}$$

From (2.128) we have the asymptotic expansion of $W(J)$:

$$\begin{aligned} W(J) &= S(\phi) + J\phi + \frac{i\hbar}{2}\log\det\left(-\frac{iS_{,ik}(\phi)}{\hbar}\right) \\ &- i\hbar\log\left\{\exp\left[i\sum_{n\geq 3}\frac{\hbar^{n-1}}{n!}S_{,i_1\dots i_n}(\phi)\frac{\partial}{i\partial p_{i_n}}\cdots\frac{\partial}{i\partial p_{i_1}}\right]\right. \\ &\times \left.\exp\left[\frac{i}{2}\hbar(p_i + S_{,i}(\phi))G^{ik}(\phi)(p_k + S_{,k}(\phi))\right]\Bigg|_{p=J}\right\}. \end{aligned} \tag{2.131}$$

Now we demand $\phi(J)$ to be defined from the equation

$$\phi^i(J) = \frac{\partial W}{\partial J_i} = \frac{\int \mathcal{D}\varphi e^{\frac{i}{\hbar}(S(\varphi)+J\varphi)}\varphi^i}{\int \mathcal{D}\varphi e^{\frac{i}{\hbar}(S(\varphi)+J\varphi)}}. \tag{2.132}$$

Then in the lowest approximation the matrix

$$\mathcal{G}^{ik} = \frac{\partial\phi^i}{\partial J_k} = \frac{\partial^2 W}{\partial J_k \partial J_i} \tag{2.133}$$

is just G_0^{ik} and, therefore, is nondegenerate. Then one can invert the function $\phi(J)$ and treat J as a function of the independent variable ϕ: $J = J(\phi)$.

Defining yet another function of ϕ by the Legendre transform

$$\Gamma(\phi) = W(J(\phi)) - J(\phi)\phi \tag{2.134}$$

we have

$$\begin{aligned}\Gamma(\phi) &= S(\phi) + \hbar\frac{i}{2}\log\,\det\left(-\frac{i}{\hbar}S_{,ik}(\phi)\right)\\ &- i\,\hbar\log\left\{\exp\left[i\sum_{n\geq 3}\frac{\hbar^{n-1}}{n!}S_{,i_1\dots i_n}(\phi)\frac{\partial}{i\partial p_{i_n}}\cdots\frac{\partial}{i\partial p_{i_1}}\right]\right.\\ &\times \left.\exp\left[\frac{i}{2}\hbar(p_i + S_{,i}(\phi))G^{ik}(\phi)(p_k + S_{,k}(\phi))\right]\Big|_{p=J(\phi)}\right\}.\end{aligned} \tag{2.135}$$

The critical point ϕ is determined now from the equation

$$\frac{\partial\Gamma(\phi)}{\partial\phi^i} = -J_i. \tag{2.136}$$

In other words, the function $\Gamma(\phi)$ is defined as the solution of the equation

$$\exp\left\{\frac{i}{\hbar}\Gamma(\phi)\right\} = \int\mathcal{D}\varphi\exp\left\{\frac{i}{\hbar}\left[S(\varphi) - (\varphi-\phi)\frac{\delta\Gamma}{\delta\phi}\right]\right\}. \tag{2.137}$$

2.9 Anticommuting variables

In the exposition above the variables φ were assumed to be *boson*. That is why the integral over $\mathcal{M}_N$ was just the usual Riemann (or Lebesgue) integral.

On the first glance it seems to be impossible to generalize the concept of integration on the fermion *anticommuting* variables. However, it turns out to be possible to define the integral over anticommuting variables purely formally, i.e., by demanding some properties of this object to be valid (postulates) and proving that the definition is consistent. The resulting object is still called integral although it has nothing to do with the Riemann (or Lebesgue) measure — these is *no measure* for anticommuting variables. The integral over fermion variables was introduced mainly in the papers of F. Berezin [1, 2].

Let us consider just one anticommuting variable θ. From the anticommutativity with itself

$$\theta\theta = -\theta\theta \tag{2.138}$$

it follows that it is nilpotent

$$\theta^2 = 0. \tag{2.139}$$

Therefore, any function of it is linear

$$f(\theta) = a + \theta b. \tag{2.140}$$

The derivative of this function is defined as usual

$$\frac{\partial f(\theta)}{\partial\theta} = b. \tag{2.141}$$

Consider a change of the variable θ

$$\theta = \theta(\eta) = e + \eta c \tag{2.142}$$

where e and c are some constants. Then, as usual

$$\frac{\vec{\partial}}{\partial\eta} = \frac{\vec{\partial}\,\theta}{\partial\eta}\frac{\vec{\partial}}{\partial\theta}. \tag{2.143}$$

Let us now define a linear functional,

$$I(f) = \int d\theta f(\theta), \tag{2.144}$$

called *formally* integral, that satisfies the following rules:

$$I(f\cdot c) = I(f)\cdot c, \tag{2.145}$$

$$I(f_1 + f_2) = I(f_1) + I(f_2), \tag{2.146}$$

$$I(1) = \int d\theta = 0, \tag{2.147}$$

$$I(\theta) = \int d\theta\,\theta = 1. \tag{2.148}$$

By linearity this sufficies to calculate the integral of any function

$$I(a + \theta b) = aI(1) + I(\theta)b = b. \tag{2.149}$$

In other words, this functional is *nothing but the derivative.*

$$I(f) = \int d\theta f(\theta) = \frac{\partial}{\partial\theta} f(\theta). \tag{2.150}$$

One can prove that with such definition of the integral the following *usual* properties remain valid in anticommuting case too

1. Integration by parts

$$\int d\theta f(\theta)\left(\frac{\vec{\partial}}{\partial\theta} g(\theta)\right) = +\int d\theta \left(f(\theta)\frac{\overleftarrow{\partial}}{\partial\theta}\right) g(\theta). \tag{2.151}$$

   Note the 'wrong' sign + here! Usually, for boson case, one has $-$ in the right hand side.

2. Defining the Fourier transform by

$$\tilde{f}(\psi) = \int \mathcal{D}\theta e^{i\theta\psi} f(\theta), \tag{2.152}$$

ψ being a fermion variable,

$$\psi\theta = -\theta\psi, \quad \psi^2 = 0. \tag{2.153}$$

and

$$\mathcal{D}\theta = \frac{d\theta}{\sqrt{i}} = e^{-i\frac{\pi}{4}} d\theta, \tag{2.154}$$

we also have the usual property

$$f(\theta) = \int \mathcal{D}\psi e^{+i\psi\theta} \tilde{f}(\psi) = \int \mathcal{D}\psi e^{-i\theta\psi} \tilde{f}(\psi). \tag{2.155}$$

i.e.

$$\tilde{\tilde{f}} = f. \tag{2.156}$$

3. The Fourier transform of the unity defines a δ-functional

$$\delta(\psi) = \int \mathcal{D}\theta e^{i\theta\psi} \tag{2.157}$$

which has the expected property

$$\int d\psi\, \delta(\psi) f(\psi) = f(0). \tag{2.158}$$

4. Using the definition we calculate formally

$$\begin{aligned} \int d\theta f(\eta(\theta)) &= \frac{\partial}{\partial\theta} f(\eta(\theta)) = \frac{\partial\eta(\theta)}{\partial\theta} \frac{\partial f(\eta)}{\partial\eta} \\ &= \frac{\partial\eta(\theta)}{\partial\theta} \int d\eta f(\eta). \end{aligned} \tag{2.159}$$

Therefore, formally we have an *unusual* behavior under the change of the variables

$$d\theta(\eta) = \left(\frac{\partial\theta}{\partial\eta}\right)^{-1} d\eta. \tag{2.160}$$

This is to compare with the usual rule

$$df(x) = \left(\frac{\partial f}{\partial x}\right) dx \tag{2.161}$$

for commuting variables and is the *main difference* between integration over commuting and anticommuting variables.

Having defined the one-dimensional integral over anticommuting variable one can define the integral for many anticommuting variables and, in general, by combing with the usual integral, one defines integrals over supervariables that can be both boson and fermion ones.

Let us consider now many anticommuting variables θ^i, $(i = 1, \dots p)$, forming a Grassmanian algebra Λ_p

$$\theta^i\theta^k + \theta^k\theta^i = 0. \tag{2.162}$$

Let us take a function of θ and expand it in the power series in θ

$$f(\theta) = \sum_{n\geq 0} \frac{1}{n!}\theta^{i_1}\cdots\theta^{i_n}a_{i_n\dots i_1}. \tag{2.163}$$

From the anticommutativity of θ it is easy to see that the coefficients of this series $a_{i_1\dots i_n}$ are completely antisymmetric tensors in all their indices (so called p-forms), i.e.,

$$a_{i_1\dots i_n} = a_{[i_1\dots i_n]}, \tag{2.164}$$

where square brackets mean the antisymmetrization.

If the number of the anticommuting variables is finite, say p, then it is clear that the rank of the p-forms is restricted from above $(n \leq p)$ — there are no antisymmetric tensors of rank more than the dimension of the Grassmanian algebra.

Therefore, any function of θ is actually a polynomial

$$f(\theta) = a + \theta^i a_i + \frac{1}{2}\theta^i\theta^k a_{ki} + \cdots + \frac{1}{p!}\theta^{i_1}\cdots\theta^{i_p}a_{i_p\dots i_1}. \tag{2.165}$$

The derivatives of such polynomials are defined as usual. And the integrals are defined again *pure formally* as linear functionals using the rules (2.144)-(2.148) for each variable θ^i:

$$\int d\theta^i f(\theta) = \frac{\partial}{\partial\theta^i}f(\theta) \tag{2.166}$$

Moreover, now one can also define the multiple integrals

$$\int d\theta^i d\theta^k f(\theta) = \int d\theta^i \frac{\partial}{\partial\theta^k}f(\theta) = \frac{\partial^2}{\partial\theta^i\partial\theta^k}f(\theta) \tag{2.167}$$

$$\int d\theta^{i_{p-1}}\cdots d\theta^{i_1} = \frac{\partial^{p-1}}{\partial\theta^{i_{p-1}}\cdots\partial\theta^{i_1}}f(\theta). \tag{2.168}$$

$$\int d\theta f(\theta) = \frac{\partial^p}{\partial\theta^1\cdots\partial\theta^p}f(\theta) \tag{2.169}$$

where $d\theta \equiv d\theta^1\cdots d\theta^p$.

The last integral is called the integral over the whole Grassmanian algebra Λ_p. Since any function $f(\theta)$ is, in fact, a polynomial, this integral does not depend on θ and is just the highest order coefficient

$$\int d\theta f(\theta) = a_{1\dots p}. \tag{2.170}$$

The integral over anticommuting variables in multidimensional case possesses all basic properties:

1. integration by parts

$$\int \mathcal{D}\theta f(\theta) \left(\frac{\overrightarrow{\partial}}{\partial\theta^i} g(\theta) \right) = + \int \mathcal{D}\theta \left(f(\theta) \frac{\overleftarrow{\partial}}{\partial\theta^i} \right) g(\theta), \tag{2.171}$$

2. Fourier transform

$$\tilde{f}(\psi) = \int \mathcal{D}\theta e^{i\theta\psi} f(\theta), \tag{2.172}$$

$$f(\theta) = \int \mathcal{D}\psi e^{i\psi\theta} \tilde{f}(\psi), \tag{2.173}$$

where ψ^i are anticommuting variables

$$\psi^i\psi^k + \psi^k\psi^i = 0, \qquad \psi^i\theta^k + \psi^k\theta^i = 0, \tag{2.174}$$

and

$$\mathcal{D}\theta = \frac{d\theta^1}{\sqrt{i}} \cdots \frac{d\theta^p}{\sqrt{i}} = e^{-i\frac{\pi}{4}p} d\theta. \tag{2.175}$$

3. δ-function

$$\delta(\psi) = \int \mathcal{D}\theta e^{i\theta\psi} \tag{2.176}$$

$$\int \mathcal{D}\psi \delta(\psi) f(\psi) = f(0). \tag{2.177}$$

4. Change of variables

$$\theta^i = \theta^i(\eta), \tag{2.178}$$

$$\mathcal{D}\theta = \det \left| \frac{\partial\theta^i}{\partial\eta^k} \right|^{-1} \mathcal{D}\eta. \tag{2.179}$$

Note the inverse power of the Jacobian!

Let us prove eq. (2.179). This formula is easy to obtain from the definition of the integral in term of the highest order derivative (2.169). Under linear transformations

$$\theta^i = A^i{}_k \eta^k, \tag{2.180}$$

with A being a matrix with boson elements, we have easily

$$\begin{aligned} \theta^1 \cdots \theta^p &= A^1_{[i_1} \cdots A^p_{i_p]} \eta^{i_1} \cdots \eta^{i_p} = \\ &= \det A \, \eta^1 \cdots \eta^p \end{aligned} \tag{2.181}$$

Therefore,

$$\frac{\partial^p}{\partial\theta^1 \cdots \partial\theta^p} = (\det A)^{-1} \frac{\partial^p}{\partial\eta^1 \cdots \partial\eta^p}. \tag{2.182}$$

and

$$\mathcal{D}\theta = (\det A)^{-1}\mathcal{D}\eta. \tag{2.183}$$

For a general nonlinear change of variables it suffices to prove (2.179) for infinitesimal form

$$\theta^i = \eta^i + \xi^i(\eta). \tag{2.184}$$

We have

$$\int d\theta f(\theta) = \int d\eta J(\eta)\bar{f}(\eta) \tag{2.185}$$

where

$$\bar{f}(\eta) \stackrel{\text{def}}{=} f(\theta(\eta)) = f(\eta + \xi(\eta)). \tag{2.186}$$

and J is the fermionic generalization of the Jacobian.

On the right hand side we can just replace the integration variable by θ

$$\int d\theta f(\theta) = \int d\theta J(\theta)\bar{f}(\theta). \tag{2.187}$$

To first order in η we have

$$\bar{f}(\theta) = f(\theta + \xi(\theta)) = f(\theta) + \xi^i(\theta)\frac{\partial f(\theta)}{\partial\theta} \tag{2.188}$$

By writing

$$J(\theta) = 1 + \varepsilon(\theta) \tag{2.189}$$

we have from (2.187)

$$\int d\theta \left\{\varepsilon(\theta)f(\theta) + \xi^i\frac{\partial f(\theta)}{\partial\theta^i}\right\} = 0. \tag{2.190}$$

Integrating by parts we rewrite this as

$$\int d\theta \left\{\varepsilon(\theta)f(\theta) + \left(\xi^i\frac{\overleftarrow{\partial}}{\partial\theta^i}\right)f(\theta)\right\} = 0 \tag{2.191}$$

and, therefore, we have finally

$$\varepsilon(\theta) = -\xi^i\frac{\overleftarrow{\partial}}{\partial\theta^i}. \tag{2.192}$$

Note that the sign here is determined by the sign in the integration by parts formula. For boson variables this sign would be +1. Thus

$$\begin{aligned} J &= 1 - \xi^i_{,i} = \exp\left(1 - \xi^i_{,i}\right) = \exp\left[-\mathrm{tr}\log\left(\delta^i_k + \xi^i_{,k}\right)\right] \\ &= \left(\det\left(1 + \xi^i_{,i}\right)\right)^{-1} = \left(\det\theta^i_{,k}\right)^{-1}. \end{aligned} \tag{2.193}$$

So we convinced ourselves that, indeed, the fermionic Jacobian is just the inverse bosonic one (2.179).

Thus we see that all the formulas of integration look almost the same in boson and fermion case. The only difference is the *sign* in the formula of integration by parts (2.171) and the *inverse power* of the Jacobian in the formula of the change of variables (2.179), which is actually the cosequence of the integration by parts.

Moreover, one can generalize all the formulas to the supervalued variables

$$\varphi^i = \begin{pmatrix} \chi^a \\ \psi^A \end{pmatrix}, \quad a = 1, \ldots, q;\ A = 1, \ldots, p; \quad p + q = D. \tag{2.194}$$

where χ^a are even (boson) and ψ^A are odd (fermion) variables.

The functions of supervariables $f(\varphi)$ are the polynomial in odd variables with coefficients depending on the even variables

$$f(\varphi) = \sum_{o \le n \le p} \frac{1}{n!} a_{A_1 \ldots A_n}(\chi) \psi^{A_n} \cdots \psi^{A_1}. \tag{2.195}$$

Let us assume the functions $a_{A_1 \ldots A_n}(\chi)$ to satisfy the conditions (2.88), i.e., for any k and m

$$\lim_{\chi \to 0} \left| \chi^{b_1} \cdots \chi^{b_k} \frac{\partial^m}{\partial \chi^{a_1} \cdots \partial \chi^{a_m}} a \right| = 0. \tag{2.196}$$

Then by introducing the measure

$$\mathcal{D}\varphi = \mathcal{D}\chi \mathcal{D}\psi \frac{d\chi^1}{\sqrt{2\pi}} \cdots \frac{d\chi^q}{\sqrt{2\pi}} \frac{d\psi^1}{\sqrt{i}} \cdots \frac{d\psi^p}{\sqrt{i}}. \tag{2.197}$$

we define integral over supervariables

$$\begin{aligned} \int \mathcal{D}\varphi f(\varphi) &\stackrel{\text{def}}{=} \int \mathcal{D}\chi \left(\int \mathcal{D}\psi f(\chi, \psi) \right) \\ &= \int \mathcal{D}\chi \frac{\partial^p}{\sqrt{i}\partial\psi^1 \cdots \sqrt{i}\partial\psi^p} f(\chi, \psi) \\ &= \int \mathcal{D}\chi\, e^{-i\frac{\pi}{4}p} a_{1\ldots p}(\chi). \end{aligned} \tag{2.198}$$

Using this definition it is not difficult to prove the following properties. Integration by parts takes the form

$$\int \mathcal{D}\varphi f(\varphi) \frac{\partial}{\partial \varphi^i} g(\varphi) = -(-1)^i \int \mathcal{D}\varphi f(\varphi) \frac{\overleftarrow{\partial}}{\partial \varphi^i} g(\varphi), \tag{2.199}$$

where, according to our convention, the index i in the exponent of (-1) equals 0 for even variables and 1 for odd ones.

There is a well defined super Fourier transform

$$\tilde{f}(J) = \int \mathcal{D}J e^{iJ\varphi} f(\varphi) \tag{2.200}$$

$$f(\varphi) = \int \mathcal{D}\varphi e^{-iJ\varphi} \tilde{f}(J) \tag{2.201}$$

and the super-delta-functional

$$\delta(\varphi) = \int \mathcal{D}J e^{iJ\varphi} \tag{2.202}$$

$$\int \mathcal{D}\varphi \delta(\varphi - \varphi') f(\varphi) = f(\varphi'). \tag{2.203}$$

The behavior of the super measure under a change of supervariables $\varphi = \varphi(\bar{\varphi})$ reads

$$\int \mathcal{D}\varphi f(\varphi) = \int \mathcal{D}\bar{\varphi} J(\bar{\varphi}) f(\varphi(\bar{\varphi})) \tag{2.204}$$

where

$$J(\bar{\varphi}) = \mathrm{sdet}\,\left(\frac{\partial \varphi^i}{\partial \bar{\varphi}^k}\right) \tag{2.205}$$

is the super-Jacobian.

Here sdet is the *superdeterminant* (sometimes also called *Berezinian*) defined as follows. Let

$$E = (E^i{}_k) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.206}$$

be a supermatrix where A and D are even nondegenerate matrices and B and C are odd matrices. Then the superdeterminant is defined by

$$\begin{aligned} \mathrm{sdet}\, E &= \det\,(A - BD^{-1}C) \det D^{-1} \\ &= \det A \det\,(D - CA^{-1}B)^{-1}. \end{aligned} \tag{2.207}$$

It is clear that if $B = C = 0$ then

$$\mathrm{sdet}\, E = \det A\, (\det D)^{-1}. \tag{2.208}$$

The superdeterminant possesses very important properties: multiplicativity

$$\mathrm{sdet}\,(E_1 E_2) = \mathrm{sdet}\, E_1 \mathrm{sdet}\, E_2 \tag{2.209}$$

and the relation to the supertrace

$$\mathrm{sdet}\,(\exp E) = \exp(\mathrm{str} E). \tag{2.210}$$

The supertrace, str, is defined by

$$\mathrm{str} E = (-1)^i E^i{}_i = \mathrm{tr}\, A - \mathrm{tr}\, D \tag{2.211}$$

Using the definition of the integral over supervariables one can show that the formulas for the Gaussian integrals are still valid with the only replacement: the superdeterminant instead of the determinant. This is so because the Gaussian integrals are calculated , in fact, just by the change of coordinates.

If one normalizes the measure by

$$\int \mathcal{D}\varphi \exp\left\{-\frac{1}{2}\varphi^i E_{ik}\varphi^k\right\} = 1 \tag{2.212}$$

for some fixed supersymmetric matrix E

$$E_{ik} = (-1)^{ik+i+k}E_{ki}, \tag{2.213}$$

then any Gaussian integral is

$$\int \mathcal{D}\varphi \exp\left\{\frac{i}{2}\varphi\Delta\varphi + iJ\varphi\right\} = \mathrm{sdet}\,(-i\hat{\Delta})^{-1/2} \exp\left\{\frac{i}{2}JGJ\right\}, \tag{2.214}$$

where $\hat{\Delta} = E^{-1}\Delta$ and $G = -\Delta^{-1}$. It can be calculated by the change of variables

$$\varphi^i \to \left[(-iE^{-1}\Delta)^{-1/2}\right]^i{}_k h^k - J_k(\Delta^{-1})^{ki} \tag{2.215}$$

and taking into account the super Jacobian (2.204).

All other formulas (also for the stationary phase method) are the consequences of this Gaussian integral and also remain valid.

2.10 Functional integral

The functional integral (called also path integral, or Feynman integral) is the integral over the configuration space $\mathcal{M}$. In field theory $\mathcal{M}$ is *infinitely* dimensional. Besides, it contains also fermion field configurations, i.e., it is a superspace.

Formally it can be defined by the continuous limit of the finite-dimensional case $\mathcal{M}_N \to \mathcal{M}$ when the number of the points N in the spacetime goes to infinity.

A very important property of the Gaussian integrals consists in the fact that their form *does not depend much on the dimension* of $\mathcal{M}_N$. In the continuum limit $N \to \infty$ the finite-dimensional matrix Δ_{ik} becomes a differential operator and the inverse $G = -\Delta^{-1}$ — its Green function. This Green function can be well defined if one imposes some *boundary conditions*. This means that the boundary conditions do actually enter the definition of the functional measure $\mathcal{D}\varphi$ — one integrates over some field configurations with some boundary conditions. Without the boundary conditions the functional measure is not well defined. Further, there is a superdeterminant that enters the formula for the Gaussian integral. If we manage to generalize the notion of the superdeterminant to the infinite-dimensional (functional) case, then we will have a well defined Gaussian functional integral.

Thus, formally *all the formulas* are the same as in the finite-dimensional case and we are allowed to do the change of variables and the integration by parts.

One has to note that many (almost all) expressions are formally divergent — if one tries to evaluate the integrals, one encounters the meaningless divergent expressions. These divergences are purely local and are due to the local nature of the quantum field theory. This difficulty can be overcome in the framework of the renormalization theory, that will be discussed a bit in next lectures.

2.11 Functional representation of the generating functional

Let us now write the mean value of the operator equations of motion

$$< S_{,i}(\hat\varphi) >= -J_i \tag{2.216}$$

Using the formula for the mean values (2.49)

$$< A(\hat\varphi) >= A\left(\frac{1}{i}\frac{\delta}{\delta J}\right) Z(J) \tag{2.217}$$

one can rewrite this equation in the form

$$\left\{ S_{,i}\left(\frac{1}{i}\frac{\delta}{\delta J}\right) + J_i \right\} Z(J) = 0 \tag{2.218}$$

This is a functional differential equation for $Z(J)$. Let us try a functional Fourier transform

$$Z(J) = \int \mathcal{D}\varphi f(\varphi) e^{iJ\varphi}. \tag{2.219}$$

Substituting this integral in the equation (2.218) we calculate

$$\begin{aligned} 0 &= \int \mathcal{D}\varphi f(\varphi)\left(S_{,i}\left(\frac{1}{i}\frac{\delta}{\delta J}\right) + J_i \right) e^{iJ\varphi} \\ &= \int \mathcal{D}\varphi f(\varphi)\left(S_{,i}(\varphi) + J_i \right) e^{iJ\varphi} \\ &= \int \mathcal{D}\varphi f(\varphi)\left(S_{,i}(\varphi) + \frac{1}{i}(-1)^i \frac{\delta}{\delta\varphi^i} \right) e^{iJ\varphi} \\ &= \int \mathcal{D}\varphi f(\varphi)\left(S_{,i}(\varphi) - \frac{1}{i}\frac{\overleftarrow{\delta}}{\delta\varphi^i} \right) e^{iJ\varphi} \end{aligned} \tag{2.220}$$

Thus we get a functional equation for the functional $f(\varphi)$

$$f(\varphi)\frac{\overleftarrow{\delta}}{\delta\varphi^i} = i f(\varphi) S_{,i}(\varphi). \tag{2.221}$$

The solution of this equation is obviously

$$f(\varphi) = \mathcal{N} e^{iS(\varphi)} \tag{2.222}$$

with some normalization 'constant' $\mathcal{N}$. Thus we obtained the generating functional in form of a functional integral

$$Z(J) = e^{iW(J)} =< \mathrm{out}|\mathrm{in} >= \mathcal{N} \int \mathcal{D}\varphi \exp\left\{i[S(\varphi) + J\varphi]\right\}. \tag{2.223}$$

The chronological amplitudes and the mean values of any functional $A(\varphi)$ are then defined by

$$< \mathrm{out}|\, T(A(\hat{\varphi}))\, |\mathrm{in} >= \mathcal{N} \int \mathcal{D}\varphi e^{i(S(\varphi)+J\varphi)} A(\varphi) \tag{2.224}$$

$$< A(\hat{\varphi}) >= \frac{\int \mathcal{D}\varphi e^{i(S(\varphi)+J\varphi)} A(\varphi)}{\int \mathcal{D}\varphi e^{i(S+J\varphi)}}. \tag{2.225}$$

Using the relation of the functional W to the effective action (2.71) one can obtain from (2.223) a functional equation for the effective action

$$\exp\left\{\frac{i}{\hbar}\Gamma(\phi)\right\} = \int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar}[S(\varphi) - \Gamma_{,i}(\varphi^i - \phi^i)]\right\}, \tag{2.226}$$

where the Planck constant $\hbar$ is introduced for convenience. This will be helpful to make the semiclassical expansion.

2.12 Relation between the effective action and the classical action

Let us consider again the left hand side of the effective equations of motion

$$< S_{,i}(\hat{\varphi}) >= \Gamma_{,i}(\phi) \tag{2.227}$$

Using the formula for the mean values (2.86) one can rewrite this in form

$$\Gamma_{,i}(\phi) = S_{,i}(\phi) : \exp\left\{ i \sum_{n\geq 2} \frac{1}{n!} \frac{1}{i^n} \frac{\overset{\leftarrow n}{\delta}}{\delta\phi^{i_n} \cdots \delta\phi^{i_1}} \mathcal{G}^{i_1\cdots i_n} \right\} :, \tag{2.228}$$

where $\mathcal{G}^{i_1\cdots i_n}$ are the full (exact) Green functions. This can be called *mixed* perturbation theory since it includes both the *full* propagator $\mathcal{G}^{ik}$ and Green functions $\mathcal{G}^{i_1\cdots i_n}$ $(n \geq 3)$ and the bare vertex functions $S_{,i_1\dots i_n}$ $(n \geq 3)$. It is not so convenient because one has to express additionally the full Green functions in terms of the bare propagator and vertex functions.

There is a more suitable expansion of the effective action *directly* in terms of the classical action, i.e., in terms of the bare propagator and the classical vertex functions. This expansion is obtained by solving the functional equation (2.121) in form of a power series in the small parameter $\hbar$:

$$\Gamma(\phi) = S(\phi) + \Sigma(\phi), \tag{2.229}$$

where

$$\Sigma(\phi) = \sum_{k\geq 1} \hbar^k \Gamma_{(k)}(\phi) \tag{2.230}$$

is called the *self-energy functional*.

One should note from the beginning that this expansion is purely *formal* (or asymptotic). There is no guaranty at all that it converges. Moreover, there are indications that it diverges, in general.

Substituting the expansion (2.230) into the equation (2.121) , making the change of variables

$$\varphi = \phi + \sqrt{\hbar}\, h \tag{2.231}$$

and expanding the action in Taylor series in h, we obtain

$$e^{i\Gamma_{(1)}(\phi)} \exp\left(i \sum_{k\geq 2} \hbar^{k-1}\Gamma_{(k)}(\phi)\right) = \int \mathcal{D}h \exp\left\{ i\frac{1}{2} h^i \Delta_{ik}(\phi) h^k \right\}$$
$$\times \exp\left\{ i \sum_{k\geq 3} \hbar^{k/2-1} S_{,i_1\dots i_n}(\phi) h^{i_n}\cdots h^{i_1} - i\sum_{k\geq 1} \hbar^{k-1/2}\Gamma_{(k),i}(\phi) h^i \right\}, \tag{2.232}$$

where $\Delta_{ik} =_{i,} S_{,k}$. By expanding both sides of this equation in $\hbar$ and equating the coefficients we obtain an infinite set of equations that determine recursively all the contributions $\Gamma_{(n)}$. All the functional integrals appearing in this expansion have the form

$$\int \mathcal{D}h e^{\frac{i}{2}h\Delta h} h^{i_1}\cdots h^{i_n}. \tag{2.233}$$

These integrals are Gaussian and can be calculated in terms of the bare propagator $G = -\Delta^{-1}$ using the formula (2.112).

It is this point that enables one to define the functional integration well — one does not need other integrals in perturbation theory. As a result we express the effective action in terms of the bare propagator and the vertex functions only.

In particular,

$$\Gamma_{(1)} = -\frac{i}{2}\log \operatorname{sdet}\, \Delta, \tag{2.234}$$

$$\Gamma_{(2)} = -\frac{1}{12} S_{,ijk} G^{(kj} G^{ie} G^{nm)}\,{}_{mne,}S - \frac{1}{8} S_{,ijkm} G^{(ij} G^{km)} \tag{2.235}$$

etc.

Using the graphical representation one can present the result in the form

We see immediately that each order in $\hbar$ is represented by loop diagrams the number of loops being equal to the order of perturbation theory. That is why the semiclassical expansion in the Planck constant is called also the *loop expansion.* We could, in principle, put $\hbar = 1$ from the beginning. The appearance of $\hbar$ in the eq. (2.229) is only to order the quantum corrections.

Thus, the self-energy functional Σ describes all radiative corrections to the classical theory and gives rise to the *nonlocality* of the effective dynamical equations

$$S_{,i} = -J_i - \Sigma_{,i}. \tag{2.236}$$

All the loop diagrams of the perturbation theory are actually divergent. Therefore, it must ultimately be dealt with the methods of *renormalization theory.*

Let us make finally some general remarks.

1. Thus the quantum mechanics is basically a theory of small disturbances. The S-matrix may be regarded as a mathematical tool which goes beyond the simple linear approximation and describes the interaction of the disturbances.

2. The entire quantum theory is summed up in the *functional structure* of the effective action. The Green functions built from the effective action contain, in fact, more information than just the S-matrix amplitudes. Therefore, instead of asking separate questions about each distinct physical process we may ask equivalent questions about the effective action $\Gamma(\phi)$. For this it is necessary, however, that the background fields ϕ vary over all permissible values.

3. Neither the classical action S nor the self-energy functional Σ have physical sense separately. Only the effective action

$$\Gamma = S + \Sigma \tag{2.237}$$

has physical meaning and describes real physics. If the self-energy functional contains terms similar to those in the classical action then only the sums of their coefficients can be determined experimentally as observable coupling constants.

If some terms in the self-energy functional have divergent coefficients then they can be compensated by the *counterterms* in the classical action. In other words, if one decomposes

$$\Sigma = \Sigma^{\rm div} + \Sigma^{\rm fin}, \tag{2.238}$$

where $\Sigma^{\rm div}$ and $\Sigma^{\rm fin}$ are the divergent and finite parts of the self-energy functional, then the effective action can be equivalently rewritten as

$$\Gamma = S + \Sigma = (S + \Sigma^{\rm div}) + \Sigma^{\rm fin} = S^{\rm ren} + \Sigma^{\rm fin} \tag{2.239}$$

where

$$S^{\rm ren} = S + \Sigma^{\rm div} \qquad (2.240)$$

is the renormalized classical action. It is supposed that S is also divergent initially, so that $S^{\rm ren}$ is finite. This is the basis of the renormalization theory.

To be more precise, let us write the classical action in the form

$$S(\phi) = \sum_{1\le i\le r} c_i I_i(\phi), \qquad (2.241)$$

where $I_i(\phi)$, $(i = 1, 2, \dots, r)$ is a finite set of some functionals and c_i are the coupling constants. If the divergent part of the self-energy functional has the same functional structure, i.e.,

$$\Sigma^{\rm div} = \sum_{1\le i\le r} \beta_i I_i(\phi), \qquad (2.242)$$

then all the divergencies can be compensated, so that the renormalized coupling constants

$$c_i^{\rm ren} = c_i + \beta_i \qquad (2.243)$$

are finite. Such field theories are called renormalizable QFT.

In non-renormalizable field theories there appear infinitely many divergent terms of different functional type. Therefore, the effective action cannot be made finite within the renormalization procedure.

4. We stress once again that to construct the S-matrix with the help of the effective action one needs only tree diagrams. No closed loops appear. The vertices generated by the effective action are *self-vertices* (or full, exact or dressed). All the quantum corrections are already included in them. Since analogous tree diagrams appear in the classical perturbations theory (with the substitution $\Gamma \to S$) one can say that the effective action describes the dynamics of the coherent fields of large amplitude with due regard to quantum corrections. This also remains true in the case when there is no well defined S-matrix at all.

Chapter 3

Quantization of gauge field theories

In the previous lecture it has been shown how to quantize a non-gauge field theory. We defined the generating functionals $Z(J)$ and $W(J)$ and the effective action $\Gamma(\phi)$ and constructed the perturbation theory for these objects, i.e., the diagramatic technique. All the diagrams are constructed of two kinds of the constituent blocks — the propagator and the vertexes. If these objects turn out to be well defined then all the diagrams are well defined (at least formally). Thus the effective action (and consequently the S-matrix) is well defined, at least perturbatively.

Of course, to do practical calculations, this is not enough and one has to employ the apparatus of the renormalization theory. But on the formal level the construction in the previous lecture is consistent. It gives simply a raw framework that should be filled with further details and methods.

The bare vertex functions $S_{,i_1\dots i_n}$ $(n \geq 3)$ are simple ultralocal objects — there are no difficulties at all in defining them correctly. As far as the propagator $G^{ik} = -(S_{,ik})^{-1}$ is concerned we simply *assumed* that there exists some propagator that can be well defined by fixing some appropriate boundary conditions.

It is this condition that determines the non-gauge field theory. However, it is not always the case. Moreover, the most interesting field models (which are also most important from the physical point of view) belong to another class of field theories, so called *gauge field theories*, where this condition is not fulfilled.

To formulate this more precisely let us consider a dynamical system that is described by a set of (for simplicity boson) fields φ^i and an action functional $S(\varphi)$. The classical dynamics of the system is described by the equations of motion

$$S_{,i} \equiv \frac{\delta S}{\delta \varphi^i} = 0. \tag{3.1}$$

All possible field configurations build the configuration space $\mathcal{M} = \{\varphi^i\}$. The solutions of the classical equation of motion determine the dynamical subspace (mass-shell) $\mathcal{M}_0 \subset \mathcal{M}$,

$$\mathcal{M}_0 = \left\{ \varphi_0^i : \quad S_{,i}(\varphi_0) = 0 \right\}. \tag{3.2}$$

Let φ_0 be a point in $\mathcal{M}_0$, i.e., a solution of the equations of motion (3.1) with some boundary conditions. It should be noted that the boundary conditions are, in general, not arbitrary. The equations of motion can impose some *constraints* on possible boundary conditions. Let us consider the neighbourhood of φ_0 in $\mathcal{M}_0$, i.e., let us consider another solution of the form $\bar{\varphi}_0 = \varphi_0 + \delta\varphi$ with *the same boundary conditions.* The infinitesimal disturbance $\delta\varphi$ satisfies obviously the homogeneous equation of small disturbances

$$S_{,ik}(\varphi_0)\delta\varphi^k = 0 \tag{3.3}$$

and *zero* boundary conditions.

If all the equations (3.1) are functionally independent, i.e., if there is a *unique* solution for given boundary conditions, then the matrix $S_{,ik}(\varphi_0)$ (so called Hessian), is nondegenerate. In discrete language this means

$$\operatorname{rank} S_{,ik}(\varphi_0) = D \times N, \tag{3.4}$$

where D is the number of field components and N is the number of points in the spacetime. Hence the homogeneous equation of small disturbances with zero boundary conditions has only trivial solution $\delta\varphi = 0$, i.e., it does not have any solutions of compact support.

In other words this means that all the solutions of the equations of motion are *isolated critical points* of the action, i.e., in a sufficiently small neighbourhood of any solution there is no other solution with the same boundary conditions. (In the Euclidean formulation of QFT this is exactly what happens, also in the continuum version). It might be useful to note that in these cases the matrix (1.73)

$$A^{00}_{AB} = \frac{\partial}{\partial\dot{\varphi}^A}\frac{\partial}{\partial\dot{\varphi}^B}\mathcal{L} \tag{3.5}$$

determining the second time derivatives in the equation of small disturbances is nondegenerate and the equations of motion can be, in principle, rewritten in form

$$\ddot{\varphi}^A = f^A(\varphi, \dot{\varphi}). \tag{3.6}$$

In general, the equations (3.1) are not independent — there some linear identities, called Noether identities, between them

$$S_{,i}R^i_\alpha \equiv 0, \tag{3.7}$$

where $\alpha = (a, x)$ is a condensed index that also includes the spacetime point and $a = 1, \dots, p$. This means that in the dynamical subspace $\mathcal{M}_0$ the Hessian $S_{,ik}(\varphi_0)$ is degenerate

$$S_{,ik}(\varphi_0)R^i{}_\alpha(\varphi_0) \equiv 0, \tag{3.8}$$

i.e., there are nontrivial solutions of homogeneous equations of small disturbances (3.3). In the discrete approximation $\alpha = 1, \dots, p \times N$, and R^i_α are some rectangular matrices of the rank

$$\operatorname{rank} R^i_\alpha \le p \times N. \tag{3.9}$$

The rank of the Hessian is then

$$\operatorname{rank} S_{,ik}(\varphi_0) = D \times N - \operatorname{rank} R^i{}_\alpha \le (D-p) \times N. \tag{3.10}$$

This determines the 'number' of the identities (3.7). The number of dynamical degrees of freedom is equal to the number of independent equations and is less or equal to $(D-p)$. It is clear that it should be $p < D$, otherwise the system would not have any dynamical degrees of freedom at all.

More generally, R^i_α is a set of vector fields on the configuration space $\mathcal{M}$. Defining

$$\mathbb{R}_\alpha = R^i_\alpha \frac{\delta}{\delta \varphi^i} \tag{3.11}$$

one can rewrite eq. (3.7) in the form

$$\mathbb{R}_\alpha S \equiv 0. \tag{3.12}$$

The action $S(\varphi)$ is a scalar on $\mathcal{M}$. Therefore, the equation (3.12), rewritten in form

$$\mathcal{L}_{\mathbb{R}_\alpha} S = 0, \tag{3.13}$$

where $\mathcal{L}_{\mathbb{R}_\alpha}$ is the Lie derivative, means that $\mathbb{R}_\alpha$ are the *invariant flows* on $\mathcal{M}$. This means nothing but there are some specific transformations of the fields

$$\delta_\xi \varphi^i = R^i_\alpha \xi^\alpha \tag{3.14}$$

that leave the action functional invariant:

$$\delta_\xi S = S_{,i} \delta_\xi \varphi^i = S_{,i} R^i_\alpha \xi^\alpha \equiv 0. \tag{3.15}$$

Here ξ^α are some infinitesimal parameters

$$\xi^\alpha = \xi^a(x), \tag{3.16}$$

that are functions over spacetime with compact support. Such transformations are called *invariance transformations* and $\mathbb{R}_\alpha$ are called the *generators of invariance transformations*. It is clear that the generators are defined not uniquely. If we transform them according to

$$R^i_\alpha \to R^i_\alpha + F^{ij}_{\ \ \alpha} S_{,j}, \tag{3.17}$$

with $F^{ij}_{\ \ \alpha}$ being antisymmetric tensor fields on $\mathcal{M}$

$$F^{ij}_{\ \ \alpha} = -F^{ji}_{\ \ \alpha}, \tag{3.18}$$

then the invariance condition

$$S_{,i}R^i_\alpha \equiv 0 \tag{3.19}$$

still holds.

Moreover, even in the case when there are no invariance flows the vector fields

$$A^i{}_\alpha(\varphi) = F^{ij}{}_\alpha S_{,j} \tag{3.20}$$

in the tangent space T_φ at some point φ that does not lie in the dynamical subspace, $\varphi \notin \mathcal{M}_0$, i.e., $S_{,j} \neq 0$, are orthogonal to $S_{,i}$,

$$A^i{}_\alpha S_{,i} \equiv 0. \tag{3.21}$$

However such vector fields are nonessential physically because they vanish on the dynamical subspace, $A^i{}_\alpha(\varphi_0) = 0$, and do not lead to the degeneracy of the Hessian.

Besides, the generators $\mathbb{R}_\alpha$ are not independent. Up to transformations (3.17) the commutator of two invariant flows is an invariance flow again. To see this let us take the commutator of two invariance transformations. We have obviously

$$(\delta_{\xi_1}\delta_{\xi_2} - \delta_{\xi_2}\delta_{\xi_1})S \equiv 0. \tag{3.22}$$

This means that

$$[\mathbb{R}_\alpha, \mathbb{R}_\beta]S = (\mathbb{R}_\alpha\mathbb{R}_\beta - \mathbb{R}_\beta\mathbb{R}_\alpha)S \equiv 0, \tag{3.23}$$

or in components

$$\left(R^i_{\alpha,k}R^k_\beta - R^i_{\beta,k}R^k_\alpha\right) S_{,i} \equiv 0. \tag{3.24}$$

More elegant the same equation follows from the property of the Lie derivative

$$\left[\mathcal{L}_{\mathbb{R}_\alpha}, \mathcal{L}_{\mathbb{R}_\beta}\right] S = \mathcal{L}_{[\mathbb{R}_\alpha,\mathbb{R}_\beta]}S. \tag{3.25}$$

This identity says that the commutator of invariance transformations is again a vector field on $\mathcal{M}$ that is orthogonal to $S_{,i}$. Let us ask the question: What can one say about the set of the flows $\mathbb{R}_\alpha$ from this identity? In such setting the problem is too general. Therefore, we will make some restrictive assumptions.

In the tangent space T_φ at some point φ there is a 1-form $S_{,i}$ and the vector fields R^i_α. The equation (3.19) means that all the vectors R^i_α are orthogonal to $S_{,i}$.

1. We assume that the vector fields $\mathbb{R}_\alpha(\varphi)$ are *linearly independent* in the tangent space T_φ at φ. This means that from

$$R^i_\alpha\xi^\alpha = 0 \tag{3.26}$$

it follows $\xi = 0$. Put it in another way: there are no $\xi^\alpha \neq 0$ of compact support such that equation (3.26) holds. This also means that the dimension of the subspace of the linear combinations of the vectors R^i_α is simply the number of these vectors and is equal to

$$\operatorname{rank} R^i_\alpha = p \times N. \tag{3.27}$$

2. We also assume that the invariance flows $\mathbb{R}_\alpha$ are *complete*, i.e., they generate *all* invariant flows. This means that the subspace of all vectors orthogonal to $S_{,i}$ is covered by linear combination of the vectors R^i_α.

Now it is clear that any vector field orthogonal to $S_{,i}$ must be a linear combination of the generators R^i_α and the transformations like (3.17). Therefore

$$[\mathbb{R}_\alpha, \mathbb{R}_\beta] = C^\gamma{}_{\alpha\beta}\mathbb{R}_\gamma + S_{,j}\mathbb{T}^j{}_{\alpha\beta} \tag{3.28}$$

or in components

$$R^i{}_{\beta,k}R^k{}_\alpha - R^i{}_{\alpha k}R^k{}_\beta = C^\gamma{}_{\alpha\beta}R^i{}_\gamma + S_{,j}T^{ji}{}_{\alpha\beta} \tag{3.29}$$

where $C^\gamma{}_{\alpha\beta} = C^\gamma{}_{\alpha\beta}(\varphi)$ are some scalar fields functionals on the configuration space, satisfying the condition

$$C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha} \tag{3.30}$$

and

$$\mathbb{T}^j{}_{\alpha\beta} = T^{ji}{}_{\alpha\beta}(\varphi)\frac{\delta}{\delta\varphi^i} \tag{3.31}$$

are some vector fields (differential operators) on $\mathcal{M}$, satisfying the conditions

$$T^{ij}{}_{\alpha\beta} = -T^{ji}{}_{\alpha\beta} = -T^{ij}{}_{\beta\alpha}. \tag{3.32}$$

Thus we see that the vector fields $\mathbb{R}_\alpha$ form an algebra (3.28) which is called the *gauge algebra*. Moreover, one can check by explicit calculations that the vector fields $\mathbb{R}_\alpha$ satisfy the Jacobi identity

$$[\mathbb{R}_\alpha[\mathbb{R}_\beta, \mathbb{R}_\gamma]] + [\mathbb{R}_\beta, [\mathbb{R}_\gamma, \mathbb{R}_\gamma]] + [\mathbb{R}_{\gamma,}[\mathbb{R}_\alpha, \mathbb{R}_\beta]] \equiv 0. \tag{3.33}$$

There are three essentially different kinds of gauge algebras.

I. The simplest case is when there exists such a field redefinition and a transformation of the generators that $T^{ij}{}_{\alpha\beta}$ vanish,

$$T^{ij}{}_{\alpha\beta} = 0, \tag{3.34}$$

and $C^\alpha{}_{\beta\gamma}$ do not depend on the fields

$$\frac{\delta}{\delta\varphi^i}C^\alpha{}_{\beta\gamma} = 0. \tag{3.35}$$

The gauge algebra takes the form of a infinite-dimensional Lie algebra

$$[\mathbb{R}_\alpha, \mathbb{R}_\beta] = C^\gamma{}_{\alpha\beta}\mathbb{R}_\gamma, \tag{3.36}$$

where $C^\gamma{}_{\alpha\beta}$ are some *constant* functionals, satisfying the Jacobi identity

$$C^\gamma{}_{\beta[\lambda}C^\beta{}_{\mu\nu]} \equiv 0. \tag{3.37}$$

In this case the gauge transformations form an infinite-dimensional Lie group, so called gauge group G. The $C^{\alpha}{}_{\beta\gamma}$ are the structure constants of this group. If $C^{\alpha}{}_{\beta\gamma}$ also vanish, $C^{\alpha}{}_{\beta\gamma} = 0$, then gauge group is Abelian Lie group and the field theory is called Abelian gauge theory (electrodynamics). The most interesting gauge field models are non-Abelian (Yang-Mills, gravity), when $C^{\alpha}{}_{\beta\gamma} \neq 0$.

The flow vectors $\mathbb{R}_{\alpha}$ decompose the configuration space into the *orbits*. An orbit is a subspace of $\mathcal{M}$ consisting of the points that are connected by the gauge transformations. The space of orbits is then $\bar{\mathcal{M}} = \mathcal{M}/G$. The linear independence of the vectors $\mathbb{R}_{\alpha}$ at each point implies that each orbit is a copy of the group manifold.

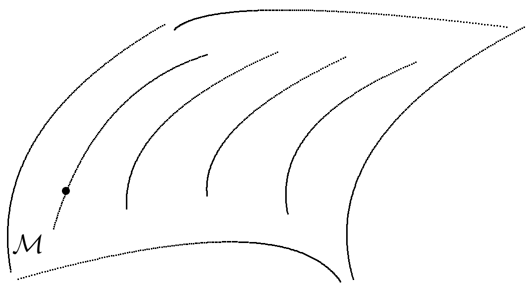

Figure 3.1: Configuration space of gauge fields

This case is the simplest and the most important one. It includes such important systems as the Yang-Mills model and gravity.

II. The second class of gauge theories consists of such algebras when the functionals $T^{ij}{}_{\alpha\beta}$ vanish but the structure coefficients $C^{\alpha}{}_{\beta\gamma}$ do depend on the fields

$$T^{ij}{}_{\alpha\beta} = 0, \tag{3.38}$$

$$\frac{\delta}{\delta\varphi^i} C^{\alpha}{}_{\beta\gamma}(\varphi) \neq 0. \tag{3.39}$$

Therefore, they are called not the structure constants but the structure functions (or functionals). One should note that the structure functions can not be made constant by a redefinition of the flow vectors and the reparametrization of the fields. If this is the case then we have again the case I.

Now we have a closed algebra

$$[\mathbb{R}_\alpha, \mathbb{R}_\beta] = C^\alpha{}_{\alpha\beta}(\varphi)\mathbb{R}_\gamma \tag{3.40}$$

and the flows still decompose the configuration space $\mathcal{M}$ into the orbits. But the orbits are not group manifold. The relation (3.40) does not define any Lie algebra and a Lie group.

III. This is the most general case when $T^{ij}{}_{\alpha\beta} \neq 0$ and $C^\alpha{}_{\beta\gamma}$ depend on φ. In this case the flow vectors do not form a close system in general. Only on the dynamical subspace $\mathcal{M}_0$, where $S_{,i} = 0$, the gauge algebra closes. Otherwise it is said to be *open gauge algebra*. Thus only $\mathcal{M}_0$ is decomposed into the orbits.

Although we restricted ourselves in this lecture to the boson fields, the whole exposition can be generalized to include also fermion fields. Then the supergravity models are typical examples of the gauge theories of the second and the third classes.

The absence of an explicit group structure in these cases causes serious difficulties in quantizing such theories. Only recently there were found effective methods to quantize general gauge theories — so called Batalin-Fradkin-Vilkovisky method. We will not consider in these lectures the such systems and refer the interested reader to the appropriate literature [12].

3.1 Physical observables.

The gauge field theories are characterized by the presence of some transformations of the fields, gauge transformations, that leave the action invariant. Therefore, such transformations do not play any role in solving the equations of motion, i.e., in determining the dynamical subspace $\mathcal{M}_0$. Two field configurations that can be connected by a gauge transformation, i.e., two points in an orbit, are physically equivalent. This means that physical dynamical variables are the classes of gauge equivalent field configurations, i.e., the orbits. The physical configuration space is, hence, the space of orbits $\bar{\mathcal{M}} = \mathcal{M}/G$. In other words the physicall observables must be the invariants of the gauge group.

Let us show that the invariance flows map the dynamical subspace $\mathcal{M}_0$ into itself. Varying the identity

$$S_{,i}R^i{}_\alpha \equiv 0 \tag{3.41}$$

we have

$$S_{,ik}R^i{}_\alpha + S_{,i}R^i{}_{\alpha,k} \equiv 0. \tag{3.42}$$

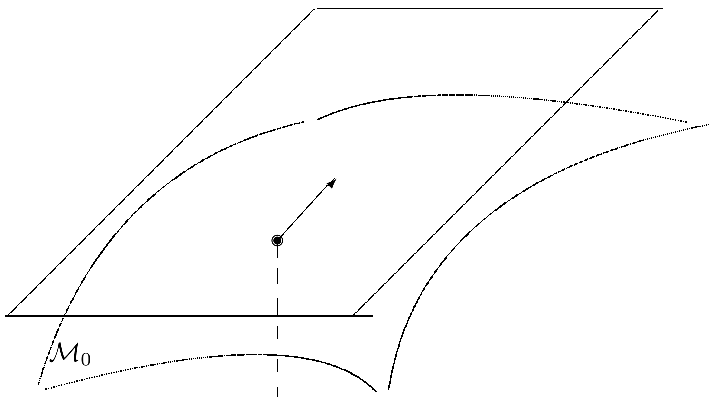

Figure 3.2:

Therefore, the gauge transformation of the left-hand side of the equations of motion is

$$\delta_\xi S_{,k} = S_{,ki} R^i{}_\alpha \xi^\alpha = -S_{,i} R^i{}_{\alpha,k} \xi^\alpha. \quad (3.43)$$

This means that a point φ_0 of the dynamical subspace $\mathcal{M}_0$ is not lead out of $\mathcal{M}_0$ by gauge transfornations. In other words the orbits can not intersect $\mathcal{M}_0$, they can either lie completely in $\mathcal{M}_0$ or not to have any common point with $\mathcal{M}_0$. If only one point of an orbit lies in $\mathcal{M}_0$, then the whole orbit does. The vector fields $R^i{}_\alpha(\varphi_0)$ at the tangent space at a point $\varphi_0 \in \mathcal{M}_0$ do not have any orthogonal component to this tangent space. They only cover a part of the tangent space, and, therefore, are all tangent vectors to $\mathcal{M}_0$.

This becomes clear if we note that $\mathcal{M}_0$ is defined by vanishing of the functional $S_{,i}(\varphi)$. Thus the normal vector to $\mathcal{M}_0$ is determined by the Hessian on

$\mathcal{M}_0$

$$N_k(\varphi_0) = S_{,ik}(\varphi_0)A^i \tag{3.44}$$

where φ_0 is a point in $\mathcal{M}_0$ and A^i is an orbitrary constant functional. The equation (3.42) on $\mathcal{M}_0$, i.e., for $S_{,i} = 0$, means then that $R^i{}_\alpha(\varphi_0)$ are orthogonal to the normal vectors (3.43), $R^i{}_\alpha(\varphi_0)N_i(\varphi_0)$, and are tangent to $\mathcal{M}_0$.

Thus the field transformations of the form

$$\delta\varphi^i = R^i{}_\alpha \xi^\alpha \tag{3.45}$$

are unphysical. Following De Witt [12] we will call functionals $A(\varphi)$ which are identically invariant,

$$A_{,i}R^i{}_\alpha = 0 \tag{3.46}$$

i.e., everywhere on $\mathcal{M}$, (like the action functional $S(\varphi)$), *absolute* invariants and functionals $B(\varphi)$ that are invariant only on $\mathcal{M}_0$, i.e.,

$$B_{,i}(\varphi_0)R^i{}_\alpha(\varphi_0) = 0 \qquad \text{for } \varphi_0 \in \mathcal{M}_0 \tag{3.47}$$

but, in general,

$$B_{,i}R^i{}_\alpha \equiv S_{,j}F^j{}_\alpha, \tag{3.48}$$

conditional invariants. It sufficies for physical observables to be only conditional invariants.

3.2 Invariant measure on the configuration space

As in the previous lecture, to quantize the gauge field theories, we will need to integrate over the configuration space $\mathcal{M}$. One needs, thus, a measure $\mu(\varphi)$ on $\mathcal{M}$. In the gauge field theories this measure should be gauge-invariant, at least formally. Gauge invariance means that $\mu(\varphi)$ should satisfy the condition

$$\mathcal{L}_{\mathbb{R}_\alpha}\mu = 0 \tag{3.49}$$

where $\mathcal{L}_{\mathbb{R}_\alpha}$ is the Lie derivative along $\mathbb{R}_\alpha$. In component language this reads:

$$\left(\mu R^i{}_\alpha\right)_{,i} = \mu_{,i}R^i{}_\alpha + \mu R^i{}_{\alpha,i}. \tag{3.50}$$

It is easy to show that this equation guarantees that the measure

$$\mathcal{D}\varphi\,\mu(\varphi) \tag{3.51}$$

is invariant under the gauge transformations

$$\varphi^i \to \varphi^i + R^i{}_\alpha\xi^\alpha. \tag{3.52}$$

If we introduce a symmetric nondegenerate matrix $E_{ik}(\varphi)$,

$$E_{ik} = E_{ki}, \quad \det E \neq 0 \tag{3.53}$$

that plays the role of the Riemannian metric on the configuration space $\mathcal{M}$, then one can always define the covariant measure in the usual way

$$\mu = (\det E)^{1/2}. \tag{3.54}$$

This measure will be gauge invariant, i.e., will satisfy the condition (3.49), if the metric E is gauge invariant,

$$\mathcal{L}_{\mathbb{R}_\alpha} E = 0 \tag{3.55}$$

or in components

$$E_{ik,j} R^j{}_\alpha + E_{ij} R^j{}_{\alpha,k} + E_{kj} R^j{}_{\alpha,i} = 0. \tag{3.56}$$

These are nothing but the Killing equations. Thus the vector fields $\mathbb{R}_\alpha$ must be the Killing vectors of the metric E_{ik}.

3.3 Ward identities

By differentiating the identity (3.7) one can get an infinite series of higher-order identities

$$\sum_{0\le m\le n} \binom{n}{m} S_{,i(k_1\dots k_m} R^i{}_{|\alpha|,k_{m+1}\dots k_n)} \equiv 0, \qquad n = 0,1,2,\dots \tag{3.57}$$

or, explicitly,

$$S_{,i} R^i{}_\alpha = 0, \tag{3.58}$$

$$S_{,ik_1} R^i{}_\alpha + S_{,i} R^i{}_{\alpha,k_1} \equiv 0, \tag{3.59}$$

$$S_{,ik_1k_2} R^i{}_\alpha + 2S_{,i(k_1} R^i{}_{|\alpha|,k_2)} + S_{,i} R^i{}_{\alpha,k_1k_2} \equiv 0. \tag{3.60}$$

These identities express the variation of the bare vertex functions $S_{,k_1\dots k_n}$ in terms of $S_{,i}$ and $S_{,ij}$ and the vertex functions of lower order. Namely,

$$\begin{aligned} \delta_\xi S_{,k_1\dots k_n} &= S_{,k_1\dots k_n i} R^i{}_\alpha \xi^\alpha \\ &= -\sum_{0\le m\le n-1} \binom{n}{m} S_{,i(k_1\dots k_m} R^i{}_{|\alpha|,k_{m+1}\dots k_n)} \xi^\alpha. \end{aligned} \tag{3.61}$$

In many important cases it turns out to be possible to find a linear representation of the gauge group, i.e., to choose the field variables in such a way that $R^i{}_\alpha(\varphi)$ are linear in φ

$$R^i{}_\alpha(\varphi) = A^i{}_\alpha + R^i{}_{\alpha,k}\varphi^k, \tag{3.62}$$

$A^i{}_\alpha$ and $R^i{}_{\alpha,k}$ being constant functionals. Then all the derivatives of $R^i{}_\alpha$ of higher orders vanish,

$$R^i{}_{\alpha,jk_1\dots k_n} \equiv 0, \qquad n = 1,2,\dots \tag{3.63}$$

and the first derivatives ${R^i}_{\alpha,j}$ are some constant matrices, forming a representation of the gauge Lie group, i.e.,

$$ {R^i}_{\alpha,k} {R^k}_{\beta,j} - {R^i}_{\beta,k} {R^k}_{\alpha,j} = {C^\gamma}_{\alpha\beta} {R^i}_{\gamma,j}. \tag{3.64} $$

This representation is called the defining representation and the contragradient representation (formed by the matrices ${F_{k,\alpha}}^i = -{R^i}_{\alpha,k}$) the co-defining representation.

For linear representations the identities (3.57) take especially simple form

$$ S_{,k_1\ldots k_n i} {R^i}_\alpha = -S_{,k_2\ldots k_n i} {R^i}_{\alpha,k_1} - \cdots - S_{,k_1\ldots k_{n-1} i} {R^i}_{\alpha,k_n}. \tag{3.65} $$

These identities are called *Ward identities*.

The action of the gauge group on the vertices looks then

$$ \begin{aligned} \delta_\xi S_{,k_1\ldots k_n} &= S_{,k_1\ldots k_n i} {R^i}_\alpha \xi^\alpha \\ &= -S_{,ik_2\ldots k_n} {R^i}_{\alpha,k_1} \xi^\alpha - \cdots - S_{,k_1\ldots k_{n-1} i} {R^i}_{\alpha,k_n} \xi^\alpha. \end{aligned} \tag{3.66} $$

These transformation rules hold obviously for any absolute invariant. Thus the functional derivatives of an absolute invariant transform as the direct product of co-defining representations.

The generators ${R^i}_\alpha$ transform according to

$$ \begin{aligned} \delta_\xi {R^i}_\alpha &= {R^i}_{\alpha,k} {R^k}_\beta \xi^\beta \\ &= {R^i}_{\beta,k} \xi^\beta {R^k}_\alpha - {C^\gamma}_{\beta\alpha} \xi^\beta {R^i}_\gamma. \end{aligned} \tag{3.67} $$

This means that ${R^i}_\alpha$ transform as direct product of defining representation and the co-adjont representation.

In general, each field (Latin) index means the defining representation, or co-defining representation if it is up or down, and each group (Greek) index denotes the adjont or co-adjont representation if it is up or down. This means that any functional

$$ A^{i\ldots\alpha\ldots}_{j\ldots\beta\ldots}(\varphi) \tag{3.68} $$

consisting of the product of the functional derivatives of absolute invariants (e.g. the action) $S_{,i_1\ldots i_n}(\varphi)$, the generators ${R^i}_\alpha(\varphi)$, their first derivative ${R^i}_{\alpha,j}$ and the structure constants ${C^\gamma}_{\alpha\beta}$, is transforming according to the direct product of defining, co-defining, adjont and co-adjont representation of the gauge group:

$$ \begin{aligned} \delta_\xi A^{i\ldots\alpha\ldots}_{j\ldots\beta\ldots} &= {R^j}_{\gamma,k} \xi^\gamma A^{k\ldots\alpha\ldots}_{j\ldots\beta\ldots} - {R^k}_{\gamma,j} \xi^\gamma A^{i\ldots\alpha\ldots}_{k\ldots\beta\ldots} \\ &\quad + {C^\alpha}_{\gamma\lambda} \xi^\gamma A^{i\ldots\lambda\ldots}_{j\ldots\beta\ldots} - {C^\lambda}_{\gamma\beta} \xi^\gamma A^{i\ldots\alpha\ldots}_{j\ldots\lambda\ldots} \end{aligned} \tag{3.69} $$

3.4 Special choice of field variables

Thus we have seen that the configuration space $\mathcal{M}$ is decomposed by the invariance flows into the orbits. To describe the local geometry of the configuration space it is convencient to reparametrize it by introducing new local coordinates $I^A(\varphi)$ and $\chi^\alpha(\varphi)$, so that the variables I^A enumerate the orbits and the variables χ^α label the points in the orbits. The variables $I^A(\varphi)$ are obviously absolute invariants which are, in general, very complicated *nonlocal* functionals satisfying the identities

$$\mathbb{R}_\alpha I^A = I^A_{\ ,i} R^i{}_\alpha = 0. \tag{3.70}$$

The change of variables $\bar\varphi^j = (I^A(\varphi), \chi^\alpha(\varphi))$ should be nondegenerate. This means that the matrix

$$\left(\bar\varphi^j_{,i}\right) = \left(I^A_{\ ,i} \quad \chi^\alpha{}_{,i}\right) \tag{3.71}$$

is nondegenerate

$$\det\left(I^A_{\ ,i} \quad \chi^\alpha{}_{,i}\right) \neq 0. \tag{3.72}$$

The vector fields $\frac{\delta}{\delta\varphi^i}$ are expressed as

$$\frac{\delta}{\delta\varphi^i} = \chi^\beta{}_{,i}\frac{\delta}{\delta\chi^\beta} + I^A{}_{,i}\frac{\delta}{\delta I^A}. \tag{3.73}$$

Since I^A are absolute invariants it is clear that the vector fields $\mathbb{R}_\alpha$ are parallel to $\frac{\delta}{\delta\chi^\alpha}$

$$\mathbb{R}_\alpha = R^i_\alpha\frac{\delta}{\delta\varphi^i} = R^i_\alpha\chi^\beta{}_{,i}\frac{\delta}{\delta\chi^\beta} + R^i_\alpha I^A{}_{,i}\frac{\delta}{\delta I^A} = F^\beta{}_\alpha\frac{\delta}{\delta\chi^\beta} \tag{3.74}$$

where

$$F^\beta{}_\alpha = \chi^\beta{}_{,i}R^i_\alpha. \tag{3.75}$$

Introducing the notation

$$\mathbb{X}_\beta = \frac{\delta}{\delta\chi^\beta} \tag{3.76}$$

we have

$$\mathbb{R}_\alpha = F^\beta{}_\alpha\mathbb{X}_\beta. \tag{3.77}$$

Since the vector fields $\mathbb{X}_\beta$ are linearly independent and complete the matrix $F^\alpha{}_\beta$ is nondegenerate. Therefore,

$$\mathbb{X}_\beta = X^i{}_\beta\frac{\delta}{\delta\varphi^i} \tag{3.78}$$

where

$$X^i{}_\beta = R^i{}_\alpha F^{-1\alpha}{}_\beta. \tag{3.79}$$

The vector fields $\mathbb{X}_\beta$ do obviously form an Abelian algebra

$$[\mathbb{X}_\alpha, \mathbb{X}_\beta] = \left[\frac{\delta}{\delta\chi^\alpha}, \frac{\delta}{\delta\chi^\beta}\right] = 0. \tag{3.80}$$

In terms of the fields φ this equation takes the form

$$X^i{}_{\beta,k} X^k{}_\alpha - X^i{}_{\alpha,k} X^k{}_\beta = 0. \tag{3.81}$$

This can be proved explicitly.

We calculate first

$$\begin{aligned}
[\mathbb{X}_\alpha, \mathbb{X}_\beta] &= F^{-1\mu}{}_\alpha \mathbb{R}_\mu F^{-1\nu}{}_\beta \mathbb{R}_\nu - F^{-1\nu}{}_\beta \mathbb{R}_\nu F^{-1\mu}{}_\alpha \mathbb{R}_\mu \\
&= F^{-1\mu}{}_\alpha [\mathbb{R}_\mu, F^{-1\nu}{}_\beta] \mathbb{R}_\nu - F^{-1\nu}{}_\beta [\mathbb{R}_\nu, F^{-1\mu}{}_\alpha] \mathbb{R}_\mu \\
&\quad + F^{-1\mu}{}_\alpha F^{-1\nu}{}_\beta [\mathbb{R}_\mu, \mathbb{R}_\nu] \\
&= \Big\{ F^{-1\mu}{}_\alpha [\mathbb{R}_\mu, F^{-1\gamma}{}_\beta] - F^{-1\mu}{}_\beta [\mathbb{R}_\mu, F^{-1\gamma}{}_\alpha] \\
&\quad + F^{-1\mu}{}_\alpha F^{-1\nu}{}_\beta C^\gamma{}_{\mu\nu} \Big\} \mathbb{R}_\gamma.
\end{aligned} \tag{3.82}$$

Further

$$\begin{aligned}
[\mathbb{R}_\mu, F^{-1\gamma}{}_\beta] &= -F^{-1\gamma}{}_\lambda [\mathbb{R}_\mu, F^\lambda{}_\sigma] F^{-1\sigma}{}_\beta \\
&= -F^{-1\gamma}{}_\lambda [\mathbb{R}_\sigma, F^\lambda{}_\mu] F^{-1\sigma}{}_\beta - F^{-1\gamma}{}_\lambda C^\delta{}_{\sigma\mu} F^\lambda{}_\delta F^{-1\sigma}{}_\beta.
\end{aligned} \tag{3.83}$$

Thus

$$\begin{aligned}
F^{-1\mu}{}_\alpha \mathbb{R}_\mu F^{-1\gamma}{}_\beta = &- F^{-1\gamma}{}_\lambda F^{-1\mu}{}_\alpha [\mathbb{R}_\sigma, F^\lambda{}_\mu] F^{-1\sigma}{}_\beta \\
&- F^{-1\mu}{}_\alpha C^\gamma{}_{\sigma\mu} F^{-1\sigma}{}_\beta.
\end{aligned} \tag{3.84}$$

Then

$$\left[\mathbb{R}_\sigma, F^\lambda{}_\mu\right] = F^\lambda{}_{\mu,k} R^k{}_\sigma = \chi^\lambda{}_{,ik} R^i{}_\sigma R^k{}_\mu + \chi^\lambda{}_{,i} R^i{}_{\mu,k} R^k{}_\sigma. \tag{3.85}$$

But this is symmetric in σ, μ. Therefore,

$$F^{-1\mu}{}_\alpha \left[\mathbb{R}\ {}_\mu, F^{-1\gamma}{}_\beta\right] - F^{-1\mu}{}_\beta \left[\mathbb{R}\ {}_\mu, F^{-1\gamma}{}_\alpha\right] = -F^{-1\mu}{}_\alpha C^\gamma{}_{\sigma\mu} F^{-1\sigma}{}_\beta. \tag{3.86}$$

Substituting this into equation (3.82) we prove that equation (3.80) really holds.

Let us now define a matrix

$$N_{\alpha\beta} = E_{ik} R^i{}_\alpha R^k{}_\beta. \tag{3.87}$$

In local field theories this is actually a differential operator. It is always assumed that the metric E_{ik} is *ultralocal*, i.e., it depends locally only on the fields but not on their derivatives,

$$E_{ik} = E_{AB}(\varphi(x))\delta(x, x'). \tag{3.88}$$

In practical cases of interest the generators $R^i{}_\alpha$ are the first order differential operators. Then $N_{\alpha\beta}$ is the differential operator of second order.

Moreover, since the vector fields $R^i{}_\alpha$ are linearly independent and the metric E_{ik} is nondegenerate, the matrix $N_{\alpha\beta}$ is nondegenerate too:

$$\det N_{\alpha\beta} \neq 0. \tag{3.89}$$

This means that there is an inverse operator $N^{-1\alpha\beta}$. In field theory this is a Green function of N. To define it properly one has to specify some boundary conditions.

Now consider an infinitesimal displacement $\delta\varphi^i$ and define

$$\delta_\perp \varphi^i = \delta\varphi^i + R^i{}_\alpha \xi^\alpha \tag{3.90}$$

with

$$\xi^\alpha = -N^{-1\alpha\beta} R^i{}_\beta E_{ik} \delta\varphi^k. \tag{3.91}$$

Then it is easy to see that $\delta_\perp\varphi^i$ is orthogonal to the vector fields $R^i{}_\alpha$ in the metric E_{ik}

$$\delta_\perp \varphi^i E_{ik} R^k{}_\alpha = 0. \tag{3.92}$$

In other words

$$\delta_\perp \varphi^i = \Pi^i{}_k \delta\varphi^k \tag{3.93}$$

where

$$\Pi^i{}_k = \delta^i_k - R^i{}_\alpha N^{-1\alpha\beta} R^j{}_\beta E_{jk} \tag{3.94}$$

is the component of $\delta\varphi^i$ that is perpendicular to the orbit and

$$\delta_\xi \varphi^i = -R^i{}_\alpha N^{-1\alpha\beta} R^i{}_\beta E_{ik} \delta\varphi^k \tag{3.95}$$

is the tangent component.

The operator $\Pi^i{}_j$ is obviously an orthogonal projector satisfying the conditions

$$\Pi^i{}_j \Pi^j{}_k = \Pi^i{}_k, \tag{3.96}$$

$$\Pi^i{}_j R^j{}_\alpha = 0, \tag{3.97}$$

$$R^i{}_\alpha E_{ik} \Pi^k{}_j = 0. \tag{3.98}$$

Using this projector one can define a metric $E^\perp_{ij}$

$$E^\perp_{ij} = \Pi^n{}_i E_{nk} \Pi^k{}_j = E_{ik} \Pi^k{}_j \tag{3.99}$$

that measures the perpendicular distance between two orbits labelled by I and $I + \delta I$

$$\delta_\perp s^2 = E^\perp{}_{ij} \delta\varphi^i \delta\varphi^j = E_{ij} \delta_\perp \varphi^i \delta_\perp \varphi^j. \tag{3.100}$$

This can also be rewritten in terms of a metric in the space of orbits $\bar{\mathcal{M}}$

$$\delta_\perp s^2 = g_{AB}(I) \delta I^A \delta I^B. \tag{3.101}$$

We have obviously

$$E^\perp{}_{ij}(\varphi) = g_{AB}(I(\varphi)) I^A{}_{,i}(\varphi) I^B{}_{,j}(\varphi) \tag{3.102}$$

and

$$\operatorname{rank} E^\perp{}_{ij} = (D - p) \times N. \tag{3.103}$$

Since $I^A(\varphi)$ are gauge-invariant

$$\mathcal{L}_{\mathbb{R}_\alpha} I^A = I^A{}_{,i} R^i{}_\alpha \equiv 0 \tag{3.104}$$

we also have

$$\mathcal{L}_{\mathbb{R}_\alpha} g_{AB} = 0 \qquad (\text{or} \quad g_{AB,i} R^i{}_\alpha = 0) \tag{3.105}$$

$$\mathcal{L}_{\mathbb{R}_\alpha} I^A{}_{,i} = \mathcal{L}_{\mathbb{R}_\alpha} \frac{\delta}{\delta\varphi^i} I^A = \frac{\delta}{\delta\varphi^i} \left(\mathcal{L}_{\mathbb{R}_\alpha} I^A \right) = 0. \tag{3.106}$$

In components

$$I^A{}_{,ik} R^k{}_\alpha + I^A{}_{,k} R^k{}_{\alpha,i} = \left(I^A{}_{,k} R^k{}_\alpha \right)_{,i} = 0. \tag{3.107}$$

Therefore, the orthogonal metric is also invariant

$$\mathcal{L}_{\mathbb{R}_\alpha} E^\perp{}_{ij} = 0. \tag{3.108}$$

Together with the invariance of the metric E_{ij} this also leads to the invariance of the projector

$$\mathcal{L}_{\mathbb{R}_\alpha} \Pi^k{}_j = \mathcal{L}_{\mathbb{R}_\alpha} \left(E^{ki} E^\perp{}_{ij} \right) = \left(\mathcal{L}_{\mathbb{R}_\alpha} E^{ki} \right) E^\perp{}_{ij} + E^{ki} \mathcal{L}_{\mathbb{R}_\alpha} E^\perp{}_{ij} = 0. \tag{3.109}$$

In components this equation means

$$\Pi^k{}_{j,i} R^i{}_\alpha + \Pi^k{}_i R^i{}_{\alpha,j} - \Pi^i{}_j R^k{}_{\alpha,i} = 0. \tag{3.110}$$

3.5 Small disturbances

Let us study now in short the theory of small disturbances in gauge theories. In the same way as for non-gauge theories we consider two close solutions, φ_0 and $\varphi = \varphi_0 + \delta\varphi$, of the equations of motion. That is we have

$$S_{,i}(\varphi_0) = 0 \tag{3.111}$$

and to first order in $\delta\varphi$ a homogeneous equation of small disturbances

$$S_{,ik}(\varphi_0) \delta\varphi^k = 0. \tag{3.112}$$

Modifycing slightly the problem, we consider two close field configurations ϕ and $\varphi = \phi + \delta\varphi$ where ϕ is a general point in $\mathcal{M}$, i.e., it is not a solution of the equation of motion (3.111) but satisfies the equation with some small external disturbances

$$S_{,i}(\phi) = -\delta J_i. \tag{3.113}$$

One can also treat δJ_i just as the extent to which ϕ differs from a solution φ_0. We get then the inhomogeneous equation of small disturbances

$$S_{,ik}(\phi)\delta\varphi^k = -\delta J_i. \tag{3.114}$$

In the case of non-gauge theories the operator $S_{,ik}$ is non-degenerate, i.e., fixing some boundary conditions there exists a well defined unique solution of this equation.

The solution can be expressed in terms of the Green functions of the operator $S_{,ik}$

$$\delta\varphi^i = G^{ik}\delta J_k \tag{3.115}$$

where G^{ik} satisfies the equation for the Green functions

$$S_{,ik}G^{kn} = -\delta^n_{\ i} \tag{3.116}$$

with some boundary conditions.

The main difference (and the problem) of the gauge theories is that the *operator $S_{,ik}$ is degenerate on mass shell.* That is even by fixing the boundary conditions the solution of the equation (3.114) is not unique. Indeed from the identities

$$S_{,ik}R^k_{\ \alpha} = -R^k_{\ \alpha,i}S_{,k} \tag{3.117}$$

we have that any fields of the form

$$\delta_\xi\varphi^k = R^k_{\ \alpha}\xi^\alpha, \tag{3.118}$$

with ξ^α being small functions of compact support, satisfy the equation

$$S_{,ik}\delta_\xi\varphi^k = R^k_{\ \alpha,i}\xi^\alpha\delta J_k. \tag{3.119}$$

The right hand side of this equation is of second order in disturbances. Therefore, in the first order

$$S_{,ik}\delta\varphi^k = 0 \tag{3.120}$$

and $\delta\varphi^k$ are the zero-modes of the operator $S_{,ik}$. This means, if $\delta\varphi$ is a solution then

$$\delta\varphi + \delta_\xi\varphi \tag{3.121}$$

is also a solution for any ξ. Thus, the operator $S_{,ik}$ does not have well defined Green functions. Note that, since ξ^α has a compact support, adding of $\delta_\xi\varphi$ does not change the boundary conditions for $\delta\varphi$.

Strictly speaking the operator $S_{,ik}(\phi)$ is non degenerate for $\phi \notin \mathcal{M}_0$, i.e., if $S_{,i}(\phi) \neq 0$, because the right hand side of equation (3.119) is not strictly zero. But the limit to the physical field configurations, i.e., to the dynamical subspace $\mathcal{M}_0$, is *singular* — there appear infinitely many zero modes and $S_{,ik}$ becomes degenerate. Thus instead of having a well defined unique solution for fixed boundary conditions we have a class of physically equivalent solutions, an orbit.

In discrete version the number of the zero modes is equal to the rank of the generators. Therefore, the rank of the operators $S_{,ik}$ is

$$\text{rank} S_{,ik} = (D - p) \times N. \tag{3.122}$$

To deal with such situations one has to choose a representative solution in each orbit. This can be done by imposing some $p \times N$ supplementary conditions, so called *gauge conditions*. We choose the supplementary conditions in the form

$$\chi^\alpha{}_i \delta\varphi^i = 0. \tag{3.123}$$

The matrix $\chi^\alpha{}_i$ here is a rectangular matrix of rank

$$\text{rank} \chi^\alpha{}_i = p \times N. \tag{3.124}$$

This guaranties that the matrix

$$F^\alpha{}_\beta = \chi^\alpha{}_i R^i{}_\beta \tag{3.125}$$

is non-degenerate.

Further, let us define

$$\Delta_{ik} = S_{,ik} + \chi^\alpha{}_i \beta_{\alpha\beta} \chi^\beta{}_k. \tag{3.126}$$

where $\beta_{\alpha\beta}$ is a local symmetric nondegenerate matrix.

The operator Δ is a symmetric non-degenerate operator even on mass-shell. Indeed, from the Ward identities eq. (3.42) rewritten in the form

$$\Delta_{ik} R^k{}_\alpha = -S_{,j} R^j{}_{\alpha,i} + \chi^\gamma{}_i \beta_{\gamma\beta} F^\beta{}_\alpha \tag{3.127}$$

or

$$R^i{}_\alpha \Delta_{ik} = -S_{,j} R^j{}_{\alpha,k} + F^\gamma{}_\alpha \beta_{\gamma\beta} \chi^\beta{}_k \tag{3.128}$$

it follows that on mass shell, (i.e., for $S_{,j} = 0$), any zero-mode h^i_0,

$$\Delta_{ik} h^k_0 = 0, \tag{3.129}$$

must satisfy the equation

$$0 = R^i{}_\alpha \Delta_{in} h^k_0 = F^\gamma{}_\alpha \beta_{\gamma\beta} \chi^\beta{}_k h^k_0. \tag{3.130}$$

Therefore

$$\chi^\beta{}_k h^k_0 = 0, \tag{3.131}$$

and, further,

$$0 = \Delta_{ik} h^k_0 = S_{,ik} h^k_0. \tag{3.132}$$

Therefore, on mass shell the zero-modes of the operator Δ are the zero-modes of the operator $S_{,ik}$. But we know from the completeness condition that all zero-modes of $S_{,ik}$ have the form

$$h_0^k = R^k{}_\alpha \xi^\alpha. \tag{3.133}$$

Substituting this into the equation (3.131) we get

$$0 = \chi^\beta{}_k R^k{}_\alpha \xi^\alpha = F^\beta{}_\alpha \xi^\alpha. \tag{3.134}$$

The operator F^α_β is nonsingular by construction. Therefore, we find that

$$\xi^\alpha = 0. \tag{3.135}$$

This means that there are no functions of compact support that satisfy the equation (3.129). This proves that the operator Δ is non-singular on mass-shell. By analyticity this means also that it is nonsingular in the neighbourhood of the mass shell.

3.6 De Witt gauge conditions

A natural and very convenient choice of the functional χ^α_i is

$$\chi^\alpha_i = \beta^{\alpha\beta} R^k{}_\beta E_{ki} \tag{3.136}$$

where $\beta^{\alpha\beta}$ is the inverse of the matrix $\beta_{\alpha\beta}$. The supplementary condition

$$\chi^\alpha_i \delta\varphi^i = \beta^{\alpha\beta} R^k{}_\beta E_{ki} \delta\varphi^i = 0 \tag{3.137}$$

means then that small disturbance $\delta\varphi^i$ is orthogonal to the orbit.

Any field disturbance $\delta\varphi^i$ can be decomposed in the tangent and orthogonal components

$$\delta\varphi^i = \delta_\perp \varphi^i + \delta_\xi \varphi^i \tag{3.138}$$

where

$$\delta_\xi \varphi^i = R^i{}_\alpha \xi^\alpha \tag{3.139}$$

$$\delta_\perp \varphi^i = \Pi^i{}_k \delta\varphi^k. \tag{3.140}$$

Since $\mathrm{rank} R^i{}_\alpha = p \times N$ we have the rank of the projector Π

$$\mathrm{rank}\Pi^i{}_k = (D - p) \times N. \tag{3.141}$$

This means that there are $(D - p) \times N$ independent orthogonal disturbances $\delta_\perp \varphi^i$. This orthogonal disturbances are nothing but the linearized invariant variables $I^A(\varphi)$,

$$I^A(\phi + \delta\varphi) = I^A(\phi) + I^A{}_{,i}(\phi) \delta_\perp \varphi^i + \cdots \tag{3.142}$$

Whereas $I^A(\varphi)$ are very complicated nonlocal nonlinear functionals of the fields φ, the orthogonal disturbances are also nonlocal but linear.

In De Witt gauge conditions the operators F and Δ take especialy simple form

$$F^{\alpha}_{\ \beta} = \beta^{\alpha\gamma} R^i_{\ \gamma} E_{ik} R^k_{\ \beta} = \beta^{\alpha\gamma} N_{\gamma\beta} \tag{3.143}$$

$$\Delta_{ik} = S_{,ik} + E_{im} R^m_{\ \alpha} \beta^{\alpha\beta} R^n_{\ \beta} E_{nk} \tag{3.144}$$

where the operator N is defined in (3.87). Both operators, N and Δ, are symmetric non-degenerate operators.

The De Witt gauge conditions maintain the manifest covariance automatically, because all the quantities transform according to the sort and the position of their indices.

Indeed, we have

$$\begin{aligned} \delta_\xi \left(R^k_{\ \beta} E_{ki}\right) &= (R^k_{\ \beta} E_{ki})_{,j} R^j_{\ \gamma} \xi^\gamma \\ &= \left(R^k_{\ \beta,j} R^j_{\ \gamma} E_{ki} + R^k_{\ \beta} E_{ki,j} R^j_{\ \gamma}\right) \xi^\gamma \\ &= \left\{R^k_{\ \beta,j} R^j_{\ \gamma} E_{ki} + R^k_{\ \beta} \left(-E_{kj} R^j_{\ \gamma,i} - E_{ji} R^j_{\ \gamma,k}\right)\right\} \xi^\gamma \\ &= \left\{-R^j_{\ \gamma,i} \left(R^k_{\ \beta} E_{kj}\right) + \left(R^k_{\ \beta,j} R^j_{\ \gamma} - R^k_{\ \gamma,j} R^j_{\ \beta}\right) E_{ki}\right\} \xi^\gamma \\ &= R^j_{\ \gamma,i} \left(R^k_{\ \beta} E_{kj}\right) \xi^\gamma + C^\delta_{\ \gamma\beta} \left(R^k_{\ \delta} E_{ki}\right) \xi^\gamma. \end{aligned} \tag{3.145}$$

Then one chooses the matrix $\beta_{\alpha\beta}$ to transform as its indices indicate

$$\delta_\xi \beta_{\alpha\beta} = \beta_{\alpha\beta,i} R^j_{\ \gamma} \xi^\gamma = -C^\delta_{\ \gamma\beta} \beta_{\alpha\delta} \xi^\gamma - C^\delta_{\ \gamma\alpha} \beta_{\delta\beta} \xi^\gamma \tag{3.146}$$

$$\delta_\xi \beta^{\alpha\beta} = \beta^{\alpha\beta}_{\ \ ,i} R^j_{\ \gamma} \xi^\gamma = C^\alpha_{\ \gamma\delta} \beta^{\alpha\beta} \xi^\gamma + C^\beta_{\ \gamma\delta} \beta^{\alpha\delta} \xi^\gamma. \tag{3.147}$$

Therefrom and using the eqs.(3.143)-(3.145) we obtain easily

$$\delta_\xi \chi^\alpha_{\ i} = \delta_\xi \left(\beta^{\alpha\beta} R^k_{\ \beta} E_{ki}\right) = C^\alpha_{\ \gamma\delta} \chi^\delta_{\ i} \xi^\gamma - R^k_{\ \gamma,i} \chi^\alpha_{\ k} \xi^\gamma. \tag{3.148}$$

$$\delta_\xi F^\alpha_{\ \beta} = C^\alpha_{\ \gamma\delta} F^\delta_{\ \beta} \xi^\gamma - C^\delta_{\ \gamma\beta} F^\alpha_{\ \delta} \xi^\gamma \tag{3.149}$$

$$\delta_\xi \Delta_{ik} = -R^n_{\ \gamma,i} \Delta_{nk} \xi^\gamma - R^n_{\ \gamma,k} \Delta_{in} \xi^\gamma. \tag{3.150}$$

We see that the operator Δ_{in} is manifestly covariant, i.e., if transform like the invariant metric E_{ik}, or second derivative of an absolute invariant $S_{,ik}$.

Thus we have defined a non-singular operator Δ. Using this operator one can solve the equations of small disturbances (3.114)

$$S_{,ij} \delta\varphi^j = -\delta J_i. \tag{3.151}$$

When the gauge conditions (3.123) are satisfied they can be replaced by

$$\Delta_{ij} \delta\varphi^j = -\delta J_i \tag{3.152}$$

and can be solved in terms of the Green functions of the operator Δ with given boundary conditions. The general solution is then given by adding an arbitrary field of the form $R^i_{\ \alpha} \xi^\alpha$.

3.7 Functional integral in gauge theories

Now we are going to quantize the gauge theories by means of the Feyman functional integral. In the same way as in non-gauge theories we consider the in– and out– regions, define some $|\text{in}>$ and $|\text{out}>$ states in these region and study the amplitude $<\text{out}|\text{in}>$. In analogy with non-gauge theories we write the amplitude in form of a funcional integral

$$<\text{out}|\text{in}>= \int \mathcal{D}\varphi f(\varphi) e^{iS(\varphi)}, \tag{3.153}$$

where $S(\varphi)$ is the action and $f(\varphi)$ is some unknown functional.

The problem with this integral is that it is defined only formally even in the non-gauge theories. In gauge theories there is an *additional difficulty* caused by the gauge invariance of the action. The formal convergence of this integral was guaranteed by the exponential $\exp(iS(\varphi))$. The main contribution came from the critical points, i.e., the solutions of the equation of motion.

The contributions of the field configurations that lie far away from the mass shell were suppressed by the oscillations of the integrand. Therefore, the functional integral could be defined in perturbation theory, where it just takes into account the small fluctuations around the mass shell. It turned out to be possible to define this integral by means of the diagrammatic technique (see the previous lecture).

In gauge gield theories the action $S(\varphi)$ is invariant along the orbits. This means that the large fluctuations along the orbits are not suppressed because there is no fast oscillation of $\exp(iS(\varphi))$ — it remains constant along the orbits. Thus the convergence of the functional integral along the orbits must be guaranteed by the functional $f(\varphi)$. As we have seen all field configurations on an orbit are physically equivalent. Therefore, we actually do not have to integrate along the orbits at all! We only have to integrate over the orbit space $\bar{\mathcal{M}} = \mathcal{M}/G$.

To give a concrete meaning to these intuitive ideas let us consider a reparametrization of the configuration space $\mathcal{M}$ by the coordinates $\bar{\varphi}^i = (I^A, \chi^\alpha)$, where I^A label the orbits and χ^α the points in the orbit. From the invariance of the action functional it follows that it depends only on I,

$$S(\varphi) = \bar{S}(I). \tag{3.154}$$

Therefore, it defines an action functional on the orbit space $\bar{\mathcal{M}}$. This functional is an usual non-gauge functional, however, extremely nonlocal. Therefore, we can write

$$<\text{out}|\text{in}>= \int\limits_{\bar{\mathcal{M}}} \mathcal{D}I \bar{\mu}(I) e^{i\bar{S}(I)} \tag{3.155}$$

where $\bar{\mu}(I)$ is some measure. This integral can be obviously rewritten as an integral over $\mathcal{M}$ by introducing a δ-functional

$$<\text{out}|\text{in}>= \int\limits_{\mathcal{M}} \mathcal{D}I\mathcal{D}\chi \bar{\mu}(I) \delta(\chi - \zeta) e^{i\bar{S}(I)} \tag{3.156}$$

where ζ^α are some constants.

The trick consists now in changing the integration variables and going back to the initial field variables φ. Using

$$\mathcal{D}I\mathcal{D}\chi = \mathcal{D}\varphi J(\varphi), \tag{3.157}$$

where

$$J(\varphi) = \det B \tag{3.158}$$

$$B^i{}_k = \bar{\varphi}^i{}_{,k} = \begin{pmatrix} I^A{}_{,i} \\ \chi^\alpha{}_{,i} \end{pmatrix} \tag{3.159}$$

is the Jacobian, we obtain

$$< \text{out}|\text{in} >= \int \mathcal{D}\varphi J(\varphi)\bar{\mu}(I(\varphi))\delta(\chi(\varphi)-\xi)e^{iS(\varphi)} \tag{3.160}$$

Thus we have found the functional $f(\varphi)$ in equation (3.153)

$$f(\varphi) = J(\varphi)\bar{\mu}(I(\varphi))\delta(\chi(\varphi)-\xi). \tag{3.161}$$

Let us calculate the Jacobian $J(\varphi)$. We have

$$(\log J)_{,j} = \operatorname{tr}(B^{-1}B_{,j}). \tag{3.162}$$

The matrix B^{-1} reads

$$B^{-1k}{}_i = \left(\frac{\delta\varphi^k}{\delta I^A}, \frac{\delta\varphi^k}{\delta\chi^\alpha}\right). \tag{3.163}$$

Therefore

$$(\log J)_{,j} = \left(\frac{\delta\varphi^i}{\delta I^A}, \frac{\delta\varphi^i}{\delta\chi^\alpha}\right)\begin{pmatrix} I^A{}_{,ij} \\ \chi^\alpha{}_{,ij} \end{pmatrix} = \frac{\delta\varphi^i}{\delta I^A}I^A{}_{,ij} + \frac{\delta\varphi^i}{\delta\chi^\alpha}\chi^\alpha{}_{,ij}. \tag{3.164}$$

Remembering eq. (3.79) we have

$$\frac{\delta\varphi^i}{\delta\chi^\alpha} = X^i{}_\alpha = R^i{}_\beta F^{-1\beta}{}_\alpha. \tag{3.165}$$

Therefore,

$$\begin{aligned} \frac{\delta\varphi^i}{\delta\chi^\alpha}\chi^\alpha{}_{,ij} &= F^{-1\beta}{}_\alpha\chi^\alpha{}_{,ij}R^i{}_\beta \\ &= F^{-1\beta}{}_\alpha\left(\chi^\alpha{}_{,i}R^i{}_\beta\right)_{,j} - F^{-1\beta}{}_\alpha\chi^\alpha{}_{,i}R^i{}_{\beta,j} \\ &= F^{-1\beta}{}_\alpha F^\alpha{}_{\beta,j} - F^{-1\beta}{}_\alpha\chi^\alpha{}_{,i}R^i{}_{\beta,j} \\ &= (\log\det F)_{,j} - F^{-1\beta}{}_\alpha\chi^\alpha{}_{,i}R^i{}_{\beta,j}. \end{aligned} \tag{3.166}$$

Therefrom

$$(\log J)_{,j} = (\log\det F)_{,j} - F^{-1\beta}{}_\alpha\chi^\alpha{}_{,i}R^i{}_{\beta,j} + \frac{\delta\varphi^i}{\delta I^A}I^A{}_{,ij}. \tag{3.167}$$

If we factorize out the $\det F$

$$J(\varphi) = \tilde{\mu}(\varphi)\det F(\varphi) \tag{3.168}$$

then $f(\varphi)$ can be rewritten as

$$f(\varphi) = \mu(\varphi)\det F(\varphi)\delta(\chi(\varphi) - \xi) \tag{3.169}$$

where

$$\mu(\varphi) = \tilde{\mu}(\varphi)\bar{\mu}(I(\varphi)) \tag{3.170}$$

is the measure on the configuration space $\mathcal{M}$. The measure $\mu(\varphi)$ transforms as

$$(\log\mu)_{,j}R^j{}_\gamma = -F^{-1\beta}{}_\alpha\chi^\alpha{}_{,i}R^i{}_{\beta,j}R^j{}_\gamma + \frac{\delta\varphi^i}{\delta I^A}I^A{}_{,ij}R^j{}_\gamma + (\log\bar{\mu})_{,j}R^j{}_\gamma \tag{3.171}$$

$\bar{\mu}(I)$ is invariant since it depends only on the invariants I

$$(\log\bar{\mu})_{,i}R^j{}_\gamma = 0. \tag{3.172}$$

Further using the identity

$$I^A{}_{,j}R^j{}_\gamma = 0 \tag{3.173}$$

we have

$$I^A{}_{,ji}R^i{}_\gamma + I^A{}_{,j}R^j{}_{\gamma,i} = 0 \tag{3.174}$$

and, hence,

$$\frac{\delta\varphi^i}{\delta I^A}I^A{}_{,ij}R^j{}_\gamma = -\frac{\delta\varphi^i}{\delta I^A}I^A{}_{,j}R^j{}_{\gamma,i}. \tag{3.175}$$

Using another identity

$$\delta^i{}_j = \frac{\delta\varphi^i}{\delta\bar{\varphi}^k}\frac{\delta\bar{\varphi}^k}{\delta\varphi^j} = \frac{\delta\varphi^i}{\delta I^A}I^A{}_{,j} + \frac{\delta\varphi^i}{\delta\chi^\alpha}\delta\chi^\alpha{}_{,j} \tag{3.176}$$

and remembering equation (3.165) we get from (3.175)

$$\begin{aligned}
\frac{\delta\varphi^i}{\delta I^A}I^A{}_{,ij}R^j{}_\gamma &= -\left(\delta^i{}_j - \frac{\delta\varphi^i}{\delta\chi^\alpha}\delta\chi^\alpha{}_{,j}\right)R^j{}_{\gamma,i} = -R^i{}_{\gamma,i} + F^{-1\beta}{}_\alpha\chi^\alpha{}_{,j}R^j{}_{\gamma,i}R^i{}_\beta \\
&= -R^i{}_{\gamma,i} + F^{-1\beta}{}_\alpha\chi^\alpha{}_{,j}C^\delta{}_{\gamma\beta}R^j{}_\delta + F^{-1\beta}{}_\alpha\chi^\alpha{}_{,j}R^j{}_{\beta,i}R^j{}_\gamma \\
&= -R^i{}_{\gamma,i} + C^\alpha{}_{\gamma\alpha} + F^{-1\beta}{}_\alpha\chi^\alpha{}_{,j}R^j{}_{\beta,i}R^j{}_\gamma.
\end{aligned} \tag{3.177}$$

Substituting equation (3.177) in (3.171) we obtain finally

$$(\log\mu)_{,j}R^j{}_\gamma = -R^i{}_{\gamma,i} + C^\alpha{}_{\gamma\alpha}. \tag{3.178}$$

The quantities $R^i{}_{\gamma,i}$ and $C^\alpha{}_{\gamma\alpha}$ contain a combined summation-integration of δ-functions with coincident arguments and are purely formal constant objects. These objects can be given a practical sence within the framework of the renormalization theory. In the renormalizable theories these objects can be made to vanish. Up to such objects the measure $\mu(\varphi)$ is gauge-invariant

$$(\log\mu)_{,j}R^j{}_\alpha = 0. \tag{3.179}$$

One can simply make a conjecture, so called path integral *quantization conjecture*, that the local measure $\mu(\varphi)$ is constant and, hence, can be normalized to unity

$$\mu(\varphi) = 1. \tag{3.180}$$

A bit more elegant is to put

$$\mu(\varphi) = (\det E)^{1/2}, \tag{3.181}$$

which cancels a part of most strong ultra-violet divergences, so called volume divergences, in the perturbation theory.

Thus we have finally

$$< \mathrm{out}|\mathrm{in} >= \int \mathcal{D}\varphi\mu(\varphi)\det F(\varphi)\delta(\chi(\varphi) - \xi)\exp[iS(\varphi)]. \tag{3.182}$$

One can transform the functional integral further by introducing some additional field variables and functional integrations. First, one can use the Fourier integral representation of the functional delta functional to get

$$< \mathrm{out}|\mathrm{in} >= \int \mathcal{D}\varphi\mathcal{D}\lambda\mu(\varphi)\det F(\varphi)\exp\left\{i[S(\varphi) + \lambda_\alpha(\chi^\alpha(\varphi) - \xi^\alpha)]\right\}. \tag{3.183}$$

The new field λ_α plays the role of a Lagrange multiplier. It is assumed to satisfy the appropriate boundary conditions in the in- and out- regions coherent to those of the fields φ^i. The total functional in the exponent

$$S(\varphi) + \lambda(\chi(\varphi) - \zeta) \tag{3.184}$$

is not gauge invariant any longer. Therefore, its second derivative is a nondegenerate operator and has well defined Green functions.

Remembering that the constants ζ^α were arbitrary one can go a bit further and integrate eq. (3.182) over ζ with a Gaussian measure

$$\int d\zeta(\det\beta)^{1/2}\exp\left(\frac{i}{2}\zeta^\mu\beta_{\ \mu\nu}\zeta^\nu\right) \tag{3.185}$$

with a nondegenerate matrix β. As a result we get finally

$$< \mathrm{out}|\mathrm{in} >= \int \mathcal{D}\varphi\mu(\varphi)(\det\beta)^{1/2}\det F(\varphi)\exp\left\{i[S(\varphi) + \frac{1}{2}\chi^\mu(\varphi)\beta_{\mu\nu}\chi^\nu(\varphi)]\right\}. \tag{3.186}$$

The second term in the total functional in the exponent breaks down the gauge invariance. It is called the *gauge-breaking term*. Therefore, the exponential is not gauge invariant and guaranties the convergence of the functional integral for large φ. Its second derivative determines a non-singular operator of small disturbances. It has a well defined Green function (propagator) and gives a basis for the perturbation theory similar to that constructed in the previous lecture.

Often it is convenient to go further and to represent the determinants arising in equation (3.186) in terms of functional integrals over auxillary anticommuting Grassmanian field variables, so called *ghost fields*. Remembering the formulas of the previous lecture one can write

$$\det F(\varphi) = \int \mathcal{D}\theta\mathcal{D}\psi \exp\left\{i\theta_\alpha F^\alpha_{\ \beta}\psi^\beta\right\}, \qquad (3.187)$$

$$\det \beta^{1/2} = \int \mathcal{D}\omega \exp\left\{\frac{i}{2}\omega^\mu\beta_{\mu\nu}\omega^\nu\right\}, \qquad (3.188)$$

where $\psi^\beta\theta_\alpha, \omega^\mu$ are the ghost fields satisfying appropriate boundary conditions in in– out– regions coherent with those of the fields φ.

Therefore, the $<\mathrm{out}|\mathrm{in}>$ amplitude takes the form

$$<\mathrm{out}|\mathrm{in}>= \int \mathcal{D}\varphi\mathcal{D}\psi\mathcal{D}\theta\mathcal{D}\omega \exp(iS_{\mathrm{tot}}(\varphi,\psi,\theta,\omega)) \qquad (3.189)$$

where

$$S_{\mathrm{tot}}(\varphi,\psi,\theta,\omega) = S(\varphi) + \frac{1}{2}\chi^\mu(\varphi)\beta_{\mu\nu}\chi^\nu(\varphi) + \theta_\alpha F^\alpha_{\ \beta}\psi^\beta + \frac{1}{2}\omega^\mu\beta_{\mu\nu}\omega^\nu. \qquad (3.190)$$

Thus a system of gauge fields φ^i described by the action $S(\varphi)$ is equivalent to an auxillary system of the fields $\varphi^i, \psi_\alpha, \theta^\beta, \omega^\mu$ described by the non-gauge action $S_{\mathrm{tot}}(\varphi,\psi,\theta,\omega)$. Therefore, by introducing the sources one can use now the whole apparatus of the generating functionals and construct the effective action and the S- matrix. Since the total action $S_{\mathrm{tot}}(\varphi,\psi,\theta,\omega)$ is not gauge invariant its second derivative is a nonsingular operator and has a well defined propagator. All the material of previous lecture is applicable to the total action. The only difference is that the ghost fields are purely formal and should not appear in the physical states in in- and out- regions.

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