1 Remarks on Lebesgue Integral

Definition 1 **Characteristic function** *of a set* $A \subset X$ *is a mapping* $\chi_A : X \to \{0, 1\}$ *defined by*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{1.1}$$

Definition 2 *For a non-zero function* $f : \mathbb{R}^n \to \mathbb{R}$*, the set,* $\operatorname{supp} f$*, of all points* $x \in \mathbb{R}^n$ *for which* $f(x) \neq 0$ *is called the* **support of** f*, i.e.*

$$\operatorname{supp} f = \{x \in \mathbb{R}^n | f(x) \neq 0\}\,. \tag{1.2}$$

Clearly, $\operatorname{supp} \chi_A = A$.

Definition 3 *Let* I *be a semi-open interval in* $\mathbb{R}^n$ *defined by*

$$I = \{x \in \mathbb{R}^n \mid a_k \leq x_k < b_k,\ k = 1, \dots, n\} \tag{1.3}$$

for some $a_k < b_k$*. The* **measure** *of the set* I *is defined to be*

$$\mu(I) = (b_1 - a_1) \cdots (b_n - a_n)\,. \tag{1.4}$$

The **Lebesgue integral** *of a characteristic function of the set* I *is defined by*

$$\int \chi_I dx = \mu(I)\,. \tag{1.5}$$

Definition 4 *A finite linear combination of characteristic functions of semi-open intervals*

$$f = \sum_{k=1}^{N} \alpha_k \chi_{I_k} \tag{1.6}$$

is called a **step function**.

Definition 5 *The* **Lebesgue integral** *of a step function is defined by linearity*

$$\int \sum_{k=1}^{N} \alpha_k \chi_{I_k} dx = \sum_{k=1}^{N} \alpha_k \mu(I_k)\,. \tag{1.7}$$

Definition 6 *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* **Lebesgue integrable** *if* $\exists$ *a sequence of step functions* $\{f_k\}$ *such that*

$$f \simeq \sum_{k=1}^{\infty} f_k\,, \tag{1.8}$$

which means that two conditions are satisfied

$$a) \quad \sum_{k=1}^{\infty} \int |f_k|\, dx < \infty \tag{1.9}$$

$$b) \quad f(x) = \sum_{k=1}^{\infty} f_k(x) \quad \forall x \in \mathbb{R}^n \text{ such that } \quad \sum_{k=1}^{\infty} |f_k(x)| < \infty\,. \tag{1.10}$$

The **Lebesgue integral** *of f is then defined by*

$$\int f\, dx = \sum_{k=1}^{\infty} \int f_k\, dx \tag{1.11}$$

Proposition 1 *The space, $L^1(\mathbb{R}^n)$, of all Lebesgue integrable functions on $\mathbb{R}^n$ is a vector space and $\int$ is a linear functional on it.*

Theorem 1 *a) If $f, g \in L^1(\mathbb{R}^n)$ and $f \le g$, then $\int f dx \le \int g dx$.*

b) If $f \in L^1(\mathbb{R}^n)$, then $|f| \in L^1(\mathbb{R}^n)$ and $|\int f\, dx| \le \int |f| dx$.

Theorem 2 *If $\{f_k\}$ is a sequence of integrable functions and*

$$f \simeq \sum_{k=1}^{\infty} f_k\,, \tag{1.12}$$

then

$$\int f = \sum_{k=1}^{\infty} \int f_k\,, \tag{1.13}$$

Definition 7 *The* **L^1-norm** *in $L^1(\mathbb{R}^n)$ is defined by*

$$||f|| = \int |f| dx \tag{1.14}$$

Definition 8 *A function f is called a* **null function** *is it is integrable and $||f|| = 0$. Two functions f and g are said to be* **equivalent** *if $f - g$ is a null function.*

Definition 9 *The* **equivalence class of** *$f \in L^1(\mathbb{R}^n)$, denoted by $[f]$, is the set of all functions equivalent to f.*

Remark. Strictly speaking, to make $L^1(\mathbb{R}^n)$ a normed space one has to consider instead of functions the classes of equivalent functions.

Definition 10 *A set $X \subset \mathbb{R}^n$ is called a* **null set** *(or a* **set of measure zero***) if its characteristic function is a null function.*

Theorem 3 *a) Every countable set is a null set.*

b) A countable union of null sets is a null set.

c) Every subset of a null set is a null set.

Definition 11 *Two integrable functions, $f, g \in L^1(\mathbb{R}^n)$, are said to be* **equal almost everywhere**, *$f = g$ a.e., if the set of all $x \in \mathbb{R}^n$ for which $f(x) \neq g(x)$ is a null set.*

Theorem 4

$$f = g \text{ a.e} \qquad \Longleftrightarrow \qquad ||f - g|| = \int |f - g| = 0 \tag{1.15}$$

Theorem 5 *The space $L^1(\mathbb{R}^n)$ is complete.*