

# **Analytic and Geometric Methods for Heat Kernel Applications in Finance**

**Ivan G. Avramidi**

*New Mexico Institute of Mining and Technology  
Socorro, NM 87801, USA*

March 15-16, 2007

**Lecture 7**

# Extensions and Applications in Finance

- Heat semigroup
- Time-dependent heat equation
- Path integrals
- Applications to stochastic volatility problems

# Heat Semi-group

## Time-Independent Operators

### Heat semi-group

$$U(t) = \exp(-tA) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k$$

Heat equation

$$(\partial_t + A)U(t) = 0, \quad U(0) = I,$$

Semi-group property

$$U(t_1 + t_2) = U(t_1)U(t_2).$$

## Volterra Series

Decomposition

$$A = A_0 + sA_1,$$

Integral equation

$$U(t) = U_0(t) - s \int_0^t d\tau U(\tau) A_1 U_0(t - \tau).$$

*Volterra series*

$$U(t) = U_0(t) + \sum_{k=1}^{\infty} (-1)^k s^k \int_0^t d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \\ \times U_0(t - \tau_k) A_1 U_0(\tau_k - \tau_{k-1}) \cdots U_0(\tau_2 - \tau_1) A_1 U_0(\tau_1).$$

Alternative form

$$U(t) = \left\{ I + \sum_{k=1}^{\infty} (-1)^k s^k \int_0^t d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \right. \\ \left. \times V(\tau_k - t) V(\tau_{k-1} - t) \cdots V(\tau_1 - t) \right\} U_0(t),$$

where

$$V(t) = e^{tA_0} A_1 e^{-tA_0}$$

Differential equation

$$\partial_t V = [A_0, V] = \text{Ad}_{A_0} V, \quad V(0) = A_1.$$

Solution

$$\begin{aligned} V(t) &= \exp[t\text{Ad}_{A_0}]A_1 = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\text{Ad}_{A_0})^k A_1 \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{[A_0, [A_0, \dots, [A_0, A_1] \dots]]}_k \\ &= A_1 + t[A_0, A_1] + \frac{1}{2}t^2[A_0, [A_0, A_1]] + O(t^3) \end{aligned}$$

Heat semi-group expansion

$$U(t) = \left\{ 1 - stA_1 + \frac{t^2}{2} (s^2 A_1^2 + s[A_0, A_1]) + O(t^3) \right\} U_0(t)$$

## Time-Dependent Operators

Differential equations

$$\partial_t U(t, t') = -A(t)U(t, t') \quad \partial_\tau U(t, \tau) = U(t, \tau)A(\tau).$$

and the initial condition

$$U(t', t') = I.$$

Semi-group property

$$U(t, t') = U(t, \tau)U(\tau, t').$$

Integral equations

$$U(t, t') = I - \int_{t'}^t d\tau A(\tau)U(\tau, t') = I + \int_{t'}^t d\tau U(t, \tau)A(\tau).$$

Perturbation series

$$U(t, t') = I + \sum_{k=1}^{\infty} (-1)^k \int_{t'}^t d\tau_k \cdots \int_{t'}^{\tau_3} d\tau_2 \int_{t'}^{\tau_2} d\tau_1 A(\tau_k) \cdots A(\tau_1),$$

If the operators  $A(t)$  commute at different times, then

$$U(t, t') = \exp \left( - \int_{t'}^t d\tau A(\tau) \right).$$

## Volterra Series

Decomposition

$$A(t, s) = A_0(t) + sA_1(t),$$

Integral equation

$$U(t, t') = U_0(t, t') - s \int_{t'}^t d\tau U(t, \tau) A_1(\tau) U_0(\tau, t').$$

Volterra series

$$U(t, t') = U_0(t, t') + \sum_{k=1}^{\infty} (-1)^k s^k \int_{t'}^t d\tau_k \int_{t'}^{\tau_k} d\tau_{k-1} \cdots \int_{t'}^{\tau_2} d\tau_1 \\ \times U_0(t, \tau_k) A_1(\tau_k) U_0(\tau_k, \tau_{k-1}) \cdots A_1(\tau_2) U_0(\tau_2, \tau_1) A_1(\tau_1) U_0(\tau_1, t')$$

Alternative form

$$U(t, t') = \left\{ I + \sum_{k=1}^{\infty} (-1)^k s^k \int_{t'}^t d\tau_k \int_{t'}^{\tau_k} d\tau_{k-1} \cdots \int_{t'}^{\tau_2} d\tau_1 \right. \\ \left. \times B(\tau_k, t) B(\tau_{k-1}, t) \cdots B(\tau_2, t) B(\tau_1, t) \right\} U_0(t, t').$$

where

$$B(\tau, t) = U_0(t, \tau) A_1(\tau) U_0(\tau, t) \\ = A_1(\tau) - \int_{\tau}^t d\tau_1 [A_0(\tau_1), A_1(\tau)] \\ + \int_{\tau}^t d\tau_2 \int_{\tau}^{\tau_2} d\tau_1 [A_0(\tau_2), [A_0(\tau_1), A_1(\tau)]] + O[(t - \tau)^3].$$

Volterra series

$$U(t, t') = \left\{ I - s \int_{t'}^t d\tau A_1(\tau) \right. \\ \left. + \int_{t'}^t d\tau_2 \int_{t'}^{\tau_2} d\tau_1 \left\{ s^2 A_1(\tau_2) A_1(\tau_1) + s[A_0(\tau_2), A_1(\tau_1)] \right\} \right. \\ \left. + O[(t - t')^3] \right\} U_0(t, t').$$

## Summary

Heat semi-group can be applied in various ways for calculation of an approximate heat kernel (Fourier method, algebraic method, Volterra series).

One has to single out an operator for which the heat semi-group is known exactly and treat the rest as a perturbation

There are algebraic techniques that enable one to compute the heat semi-group for operators that form a nice algebra (Campbell-Hausdorff formula)

The correction terms are usually expressed in terms of commutators of operators. If these commutators are small, then it gives a good approximation

# Heat Kernel of Time-Dependent Operator

## Heat Semi-Group Method

Separation of time-independent part

$$L(t) = L_0 + \varepsilon L_1(t).$$

Heat kernel up to the second order in  $\varepsilon$  and  $(t - t')$ ,

$$U(t, x|t', x') = \left\{ \begin{aligned} &1 - \varepsilon \int_{t'}^t d\tau L_1(\tau) \\ &+ \int_{t'}^t d\tau_2 \int_{t'}^{\tau_2} d\tau_1 \left\{ \varepsilon^2 L_1(\tau_2) L_1(\tau_1) + \varepsilon [L_0, L_1(\tau_1)] \right\} \\ &+ O[(t - t')^3, \varepsilon^3] \end{aligned} \right\} U_0(t - t'; x, x').$$

Time-independent part

$$L_0 = \frac{1}{(t - t')} \int_{t'}^t d\tau L(\tau),$$

Zero-order heat kernel

$$U_0(t, x|t', x') = U_0(\tau; x, x') \Big|_{\tau=t-t'},$$

where  $U_0(\tau; x, x')$  is the heat kernel of the operator  $L_0$  computed with both  $t$  and  $t'$  being fixed.

## Singular Perturbation Method

Elliptic time-dependent PDO

$$\begin{aligned} L &= -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q, \\ &= -g^{-1/2}(t, x)[\partial_i + \mathcal{A}_i(t, x)]g^{1/2}(t, x)g^{ij}(t, x)[\partial_j + \mathcal{A}_j(t, x)] \\ &\quad + Q(t, x), \end{aligned}$$

Singularly perturbed heat equation

$$[\varepsilon\partial_t + \varepsilon^2 L] U(t, x|t', x') = 0, \quad U(t', x|t', x') = \delta(x, x').$$

## Asymptotic ansatz

$$U(t, x|t', x') \sim \exp \left[ -\frac{1}{\varepsilon} S(t, x|t', x') \right] \sum_{k=0}^{\infty} \varepsilon^k b_k(t, x|t', x').$$

Leading asymptotics

$$U(t, x|t', x') \sim \exp \left[ -\frac{1}{\varepsilon} S(t, x|t', x') \right] b_0(t, x|t', x').$$

Initial conditions: as  $t \rightarrow t'$

$$S(t, x'|t', x') \sim \frac{1}{4(t-t')} \Phi(t', x, x'),$$

$$b_0(t, x|t', x') \sim [4\pi(t-t')]^{-n/2} g^{-1/4}(t', x') g^{-1/4}(t', x) \\ \times \left( \det [\partial_i \partial_{j'} \Phi(t', x, x')] \right)^{1/2},$$

Commutation formula

$$\exp\left(\frac{1}{\varepsilon}S\right) \left[\varepsilon\partial_t + \varepsilon^2L\right] \exp\left(-\frac{1}{\varepsilon}S\right) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2,$$

where  $T_0$  is a function,

$$T_0 = -\partial_t S - g^{ij} S_{;i} S_{;j},$$

$T_1$  is a first-order PDO

$$T_1 = \partial_t + 2g^{ij} S_{;j} (\nabla_i + \mathcal{A}_i) + g^{ij} S_{;ij},$$

and  $T_2$  is a second-order PDO

$$T_2 = L.$$

Recall that  $S_{;i} = \nabla_i S$  and  $S_{;ij} = \nabla_i \nabla_j S$ .

## Hamilton-Jacobi equation

$$\partial_t S + g^{ij} S_{;i} S_{;j} = 0,$$

## Recurrence relations (transport equations)

$$T_1 b_0 = 0, \quad T_1 b_{k+1} = -T_2 b_k,$$

## Hamiltonian system (with $x(t') = x'$ and $x(t) = x$ )

$$\frac{dx^i}{d\tau} = 2g^{ij}(\tau, x)p_j, \quad \frac{dp_k}{d\tau} = -\partial_k g^{ij}(\tau, x)p_i p_j.$$

## Action

$$S(t, x|t', x') = \int_{t'}^t d\tau \frac{1}{4} g_{ij}(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} \frac{dx^j(\tau)}{d\tau}$$

## Transport operator

$$\begin{aligned} T_1 &= \frac{d}{dt} + g^{ij} S_{;ij} + 2g^{ij} S_{;i} \mathcal{A}_j, \\ &= Z^{1/2} \left( \frac{d}{dt} + 2g^{ij} S_{;i} \mathcal{A}_j \right) Z^{-1/2}. \end{aligned}$$

where (total time derivative)

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx^i}{dt} \frac{\partial}{\partial x^i} .}$$

$$\boxed{Z(t, x|t', x') = g^{-1/2}(t, x) \det [-\partial_i \partial_{j'} S(t, x|t', x')] g^{-1/2}(t', x') .}$$

First coefficient

$$b_0(t, x|t', x') = (2\pi)^{-n/2} \mathcal{W}(t, x|t', x') Z^{1/2}(t, x|t', x'),$$

where

$$\mathcal{W}(t, x|t', x') = \exp \left\{ - \int_{t'}^t d\tau \frac{dx^i(\tau)}{d\tau} \mathcal{A}_i(\tau, x(\tau)) \right\}.$$

Leading asymptotics

$$U(t, x|t', x') \sim (2\pi)^{-n/2} Z^{1/2}(t, x|t', x') \mathcal{W}(t, x|t', x') \\ \times \exp \left\{ - \frac{1}{\varepsilon} S(t, x|t', x') \right\}.$$

## More General Setup

Singularly perturbed heat equation

$$\left[ \varepsilon \partial_t - \varepsilon^2 g^{ij} (\nabla_i + \mathcal{A}_i) (\nabla_j + \mathcal{A}_j) + Q \right] U(t, x | t', x') = 0,$$

Hamilton-Jacobi equation

$$\partial_t S + g^{ij}(t, x) S_{;i} S_{;j} - Q = 0,$$

Hamiltonian system

$$\frac{dx^i}{d\tau} = 2g^{ij}(\tau, x)p_j, \quad \frac{dp_k}{d\tau} = -\partial_k g^{ij}(\tau, x)p_i p_j + \partial_k Q.$$

Action

$$S(t, x | t', x') = \int_{t'}^t d\tau \left\{ \frac{1}{4} g_{ij}(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} \frac{dx^j(\tau)}{d\tau} + Q(\tau, x(\tau)) \right\},$$

## Summary

There are methods for approximate calculation of heat kernel for time-dependent operators:

Separation of time-independent part,  
Singular perturbation method

Singular perturbation method works almost exactly in the same way as for time-independent operators

To be able to use it one has to solve the Hamiltonian system (or a time-dependent geodesic flow)

# Path Integrals

## Definition

Idea: use semi-group property to express the heat semi-group for a finite time as a limit of a product of heat semi-groups for small times

Time-independent operators

Heat semi-group

$$U(t) = \lim_{N \rightarrow \infty} \left[ U\left(\frac{t}{N}\right) \right]^N,$$

where

$$U\left(\frac{t}{N}\right) = \exp\left(-\frac{t}{N}L\right) = I - \frac{t}{N}L + O\left(\frac{t}{N}\right)^2.$$

## Time-dependent operators

Partition of the interval  $(t', t)$

$$t_k = t' + k \frac{(t - t')}{N}. \quad k = 0, 1, \dots, N,$$

Factorization

$$U(t, t') = U(t, t_{N-1})U(t_{N-1}, t_{N-2}) \cdots U(t_2, t_1)U(t_1, t').$$

Short time heat semi-group

$$\begin{aligned} U(t, t') &= I - \int_{t'}^t d\tau L(\tau) + O[(t - t')^2] \\ &= \exp \left[ - \int_{t'}^t d\tau L(\tau) \right] + O[(t - t')^2]. \end{aligned}$$

Limit as  $N \rightarrow \infty$

$$U(t, t') = \lim_{N \rightarrow \infty} \exp \left[ - \int_{t_{N-1}}^t d\tau_N L(\tau_N) \right] \cdots \exp \left[ - \int_{t'}^{t_1} d\tau_1 L(\tau_1) \right] .$$

Heat kernel

$$U(t, x|t', x') = \lim_{N \rightarrow \infty} \int_{\mathbb{R}^{Nn}} dx_1 \dots dx_N U(t, x|t_{N-1}, x_{N-1}) \\ \times U(t_{N-1}, x_{N-1}|t_{N-2}, x_{N-2}) \cdots U(t_2, x_2|t_1, x_1) U(t_1, x_1|t', x') .$$

## Formal Expression

Elliptic PDO

$$L = -\alpha^{ij}(x, t)\partial_i\partial_j + \beta^j(x, t)\partial_j + \gamma(x, t).$$

Short time heat kernel for constant coefficients

$$U(t, x|t', x') = [4\pi(t - t')]^{-n/2}[\det A]^{-1/2} \exp [-(t - t')\gamma] ,$$
$$\times \exp \left\{ -\frac{\langle (x - x'), A^{-1}(x - x') \rangle}{4(t - t')} \right\}$$
$$\times \exp \left\{ \frac{1}{2} \langle (x - x'), A^{-1}\beta \rangle - (t - t')\frac{1}{4} \langle \beta, A^{-1}\beta \rangle \right\} ,$$

where  $A$  is the matrix  $A = (\alpha^{ij})$ .

Formal closed formula (Feynmann path integral)

$$U(t, x|t', x') = \int_{\mathcal{M}} \mathcal{D}x(\tau) \exp[-S(t, x|t', x')].$$

Integral is taken over all continuous paths  $x(\tau)$  starting at  $x'$  at  $\tau = t'$  and ending at  $x$  at  $\tau = t$ , that is,

$$x(t') = x', \quad x(t) = x,$$

Action functional

$$S(t, x|t', x') = \int_{t'}^t d\tau \left\{ \frac{1}{4} \left\langle \left( \frac{dx}{d\tau} - \beta \right), A^{-1} \left( \frac{dx}{d\tau} - \beta \right) \right\rangle + \gamma \right\}.$$

Pre-exponential factors are absorbed in the measure  $\mathcal{D}x(\tau)$

## Non-constant coefficients

Operator in invariant geometric form

$$\begin{aligned} L &= -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q \\ &= -g^{-1/2}(\partial_i + \mathcal{A}_i)g^{1/2}g^{ij}(\partial_j + \mathcal{A}_j) + Q \end{aligned}$$

Short time heat kernel

$$\begin{aligned} U(t, x|t', x') &\sim (2\pi)^{-n/2} Z^{1/2}(t, x|t', x') \\ &\times \exp \left\{ - \int_{t'}^t d\tau \left[ \frac{1}{4} g_{ij}(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} \frac{dx^j(\tau)}{d\tau} \right. \right. \\ &\left. \left. + \mathcal{A}_i(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} + Q(\tau, x(\tau)) \right] \right\}. \end{aligned}$$

Heat kernel composition

$$U(t, x|t', x') = \lim_{N \rightarrow \infty} \int_{\mathbb{R}^{Nn}} dx_1 g^{1/2}(x_1) \dots dx_N g^{1/2}(x_N)$$

$$U(t, x|t_{N-1}, x_{N-1})U(t_{N-1}, x_{N-1}|t_{N-2}, x_{N-2}) \cdots U(t_1, x_1|t', x')$$

Action

$$S(t, x|t', x') = \int_{t'}^t d\tau \left\{ \frac{1}{4} g_{ij}(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} \frac{dx^j(\tau)}{d\tau} + \mathcal{A}_i(\tau, x(\tau)) \frac{dx^i(\tau)}{d\tau} + Q(\tau, x(\tau)) \right\}.$$

## Perturbation Theory

The only practical method for computation of path integrals remains the perturbation theory (or a numerical simulation, which is very expensive).

One looks for critical points of the action, which represent classical trajectories.

Then one expands the action in a functional Taylor series near these trajectories. One leaves the quadratic terms in the exponent and expands the rest in a power series.

The only path integrals that appear are Gaussian path integrals for which very similar techniques are available as for finite-dimensional Gaussian integrals.

## Critical points (classical trajectories)

$$\frac{d^2 x^m}{d\tau^2} + \Gamma^m_{kj} \frac{dx^k}{d\tau} \frac{dx^j}{d\tau} + \left[ g^{im} (\partial_t g_{ik}) - 2g^{im} \mathcal{R}_{ik} \right] \frac{dx^k}{d\tau} + 2g^{im} \partial_t \mathcal{A}_i - 2g^{im} \partial_i Q = 0,$$

These equations are equivalent to the Hamiltonian system of the singular perturbation method

Classical trajectory

$$x_0(\tau), \quad x_0(t') = x', \quad x_0(t) = x$$

gives the main contribution to the path integral.

### Expansion near classical trajectory

$$x(\tau) = x_0(\tau) + y(\tau),$$

$$y(t') = y(t) = 0,$$

### Functional Taylor series

$$S(x(\tau)) = S(x_0(\tau)) + \frac{1}{2} \int_{t'}^t d\tau \langle y(\tau), H(\tau)y(\tau) \rangle + V(y(\tau)),$$

Here  $H$  is a second-order ordinary differential operator and  $V(y(\tau))$  is a functional of  $y(\tau)$  whose expansion in  $y$  begins with the terms of order  $y^3$ .

## Gaussian Path Integrals

$$G^{i_1 \dots i_k}(\tau_1, \dots, \tau_k) =$$

$$\times \int_{\mathcal{M}_y} \mathcal{D}y(\tau) \exp \left[ -\frac{1}{2} \int_{t'}^t d\tau \langle y(\tau), H(\tau)y(\tau) \rangle \right] y^{i_1}(\tau_1) \dots y^{i_k}(\tau_k).$$

Green function of the operator  $H_{ij}$

$$H_{ij}(\tau)G^{jk}(\tau, \tau') = \delta^k_i \delta(\tau - \tau').$$

Then

$$G^{i_1 \dots i_{2k+1}}(\tau_1, \dots, \tau_{2k+1}) = 0$$

$$G^{i_1 \dots i_{2k}}(\tau_1, \dots, \tau_{2k}) = (\text{Det } H)^{-1/2} \frac{(2k)!}{2^k k!}$$

$$\times \text{Sym } G^{i_1 i_2}(\tau_1, \tau_2) \dots G^{i_{2k-1} i_{2k}}(\tau_{2k-1}, \tau_{2k}).$$

### Symmetrization

$$\text{Sym } G^{i_1 i_2}(\tau_1, \tau_2) G^{i_3 i_4}(\tau_3, \tau_4) = \frac{1}{3} \left\{ G^{i_1 i_2}(\tau_1, \tau_2) G^{i_3 i_4}(\tau_3, \tau_4) \right. \\ \left. + G^{i_1 i_3}(\tau_1, \tau_3) G^{i_2 i_4}(\tau_2, \tau_4) + G^{i_1 i_4}(\tau_1, \tau_4) G^{i_2 i_3}(\tau_2, \tau_3) \right\}.$$

### Functional determinant

$$\text{Det } H = \exp \left( -\zeta'(0) \right).$$

### Zeta-function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty du u^{s-1} \text{Tr} \exp(-uH) = \frac{1}{\Gamma(s)} \int_0^\infty du u^{s-1} \int_{t'}^t d\tau K(u; \tau, \tau)$$

**Constant Coefficients** ( $\alpha_{ij} = \text{const}$ ,  $\beta = \gamma = 0$ )

Classical trajectories

$$x_0^i(\tau) = x'^i + \frac{\tau - t'}{t - t'}(x - x')^i,$$

Classical action

$$S_0(t, x|t', x') = S(x_0(\tau)) = \frac{1}{4(t - t')} \langle (x - x'), A^{-1}(x - x') \rangle .$$

Normalization of the Gaussian measure

$$\int_{\mathcal{M}_y} \mathcal{D}y(\tau) \exp \left[ -\frac{1}{2} \int_{t'}^t d\tau \langle y(\tau), H(\tau)y(\tau) \rangle \right] = \frac{(\det A)^{-1/2}}{[4\pi(t - t')]^{n/2}}$$

where

$$H_{ij} = -\frac{1}{2} \alpha_{ij} \frac{d^2}{d\tau^2},$$

## Summary

Path integrals provide a valuable tool for general theoretical considerations

The application of path integrals for solutions of practical problems is limited to Gaussian integrals

Perturbation theory for path integrals is equivalent to variations of singular perturbation method

Numerical schemes for calculation of path integrals (Monte-Carlo etc): i) are very expensive, ii) do not give an analytical formula.

If one is ready to apply numerical methods then it is *much easier to solve the heat equation directly numerically* than to compute the corresponding path integral

# Applications to Stochastic Volatility Problems

**SABR model** (Hagan Formula)

Parabolic PDE for the option price  $G(t, f, \sigma; T, F, \Sigma)$

$$\left( \frac{\partial}{\partial t} + \tilde{L} \right) G = 0,$$

where

$$\tilde{L} = \frac{1}{2} \sigma^2 \left( C(f)^2 \frac{\partial^2}{\partial f^2} + 2v\rho C(f) \frac{\partial^2}{\partial f \partial \sigma} + v^2 \frac{\partial^2}{\partial \sigma^2} \right),$$

Terminal condition

$$G(T, f, \sigma; T, F, \Sigma) = \delta(f - F) \delta(\sigma - \Sigma).$$

Here  $v$  is the volatility of volatility,  $\rho$  is the correlation between two Winer processes.

The function  $C(f)$  is supposed to be positive monotone non-decreasing and smooth and such that

$$C(-f) = -C(f), \quad \partial_f C(f) \Big|_{f=0} = 0.$$

Boundary conditions

$$\lim_{F, \Sigma \rightarrow \infty} G(t, f, \sigma; T, F, \Sigma) = 0.$$

Change of variables

$$\tau = T - t, \quad x^1 = x = f, \quad x^2 = y = \frac{\sigma}{v}.$$

Heat equation

$$(\partial_\tau + L)U(\tau; x, x') = 0,$$

where

$$L = -\frac{v^2}{2}y^2 \left[ C^2(x)\partial_x^2 + 2\rho C(x)\partial_x\partial_y + \partial_y^2 \right].$$

Riemannian metric

$$g^{11} = \frac{v^2}{2}y^2C^2, \quad g^{12} = \frac{v^2}{2}\rho y^2C, \quad g^{22} = \frac{v^2}{2}y^2.$$

$$g_{11} = \frac{2}{v^2(1-\rho^2)}\frac{1}{y^2C^2}, \quad g_{12} = -\frac{2\rho}{v^2(1-\rho^2)}\frac{1}{y^2C},$$

$$g_{22} = \frac{2}{v^2(1-\rho^2)}\frac{1}{y^2}.$$

### Riemannian volume element

$$\sqrt{g} \, dx \, dy = \frac{2}{v^2 \sqrt{1 - \rho^2}} \frac{1}{y^2 C(x)} \, dx \, dy .$$

### Gaussian curvature

$$K = R^{12}{}_{12} = -\frac{v^2}{2} .$$

Manifold is diffeomorphic to the hyperbolic plane  $H^2$ , a space of constant negative curvature.

By solving the equations of geodesics one can find the relation of the coordinates  $x$  and  $y$  to the standard geodesic coordinates.

## Perturbation Theory

Decomposition

$$L = L_0 + L_1$$

where  $L_0$  is the scalar Laplacian,

$$L_0 = -g^{-1/2} \partial_i g^{1/2} g^{ij} \partial_j ,$$

and  $L_1$  is a first order operator,

$$L_1 = \frac{v^2}{2} y^2 C(x) C'(x) \partial_x .$$

Heat kernel of the operator  $L_0$

$$U_0(t; x, x') = \frac{1}{4\pi t} \sqrt{\frac{ar}{\sinh(ar)}} \exp\left(-\frac{r^2}{4t}\right) \times \left\{ 1 - \frac{t}{4r^2} [a^2 r^2 + ar \coth(ar) - 1] + O(t^2) \right\},$$

where  $r$  is the geodesic distance between  $x$  and  $x'$  and  $a = \frac{v}{\sqrt{2}}$

By treating the operator  $L_1$  as a perturbation, we get

$$U(t; x, x') = \left\{ 1 - tL_1 + \frac{t^2}{2} (L_1^2 + [L_0, L_1]) + O(t^3) \right\} U_0(t; x, x').$$

Hagan formula is obtained by restricting ourselves to the first order in  $L_1$

## Affine Model (*Heston Formula*)

Parabolic PDE

$$\left(-\frac{\partial}{\partial t} + \tilde{L}\right) V = 0,$$

where

$$\tilde{L} = -\frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} - \rho\eta vS\frac{\partial^2}{\partial S\partial v} - \frac{1}{2}\eta^2v\frac{\partial^2}{\partial v^2} - rS\frac{\partial}{\partial S} + \lambda(v - \bar{v})\frac{\partial}{\partial v} + r.$$

Here  $S$  is the *stock price*,  $v$  is its *variance*,  $\eta$  is the *volatility of volatility*,  $r$  is the risk-free interest rate, and  $\rho$ ,  $\lambda$  and  $\bar{v}$  are some real parameters.

Terminal condition at  $t = T$

$$V(T, S, v) = V_0(S, v).$$

Change of variables

$$x = \log S, \quad u = \frac{v}{\eta}, \quad \tau = \frac{\eta}{2}(T - t).$$

Heat equation

$$(\partial_\tau + L)V = 0,$$

where

$$L = -u \left( \partial_x^2 + 2\rho \partial_x \partial_u + \partial_u^2 \right) + \left( u - 2\frac{r}{\eta} \right) \partial_x + 2\frac{\lambda}{\eta} \left( u - \frac{\bar{v}}{\eta} \right) \partial_u + 2\frac{r}{\eta}.$$

Heat kernel  $U(\tau; x, u, x', u')$

$$(\partial_\tau + L)U = 0,$$

with the initial condition

$$U(0; x, u, x', u') = \delta(x - x')\delta(u - u'),$$

Riemannian metric

$$(g^{ij}) = u \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

$$(g_{ij}) = \frac{1}{(1 - \rho^2)u} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}.$$

Riemannian volume element

$$g^{1/2} dx du = \frac{1}{\sqrt{1 - \rho^2}} \frac{dx du}{u}.$$

Gaussian curvature

$$K = -\frac{1}{2u}.$$

Manifold is that of negative curvature with true singularity at the boundary  $u = 0$ .

Fourier transform in  $x$  (with  $a$  a real constant)

$$U(\tau; x, u, x', u') = \int_{ia-\infty}^{ia+\infty} \frac{dp}{2\pi} e^{ip(x-x')} \hat{U}(\tau, p; u, u'),$$

Transformed equation

$$\boxed{[\partial_\tau - u\partial_u^2 + (2\beta_1 u + \beta_0)\partial_u + \gamma_1 u + \gamma_0] \hat{U} = 0,}$$

where

$$\begin{aligned} \beta_1 &= \frac{\lambda}{\eta} - i\rho p, & \beta_0 &= -2\frac{\lambda}{\eta^2}\bar{v}, \\ \gamma_1 &= p^2 + ip, & \gamma_0 &= 2\frac{r}{\eta}(1 - ip), \end{aligned}$$

Initial condition

$$\hat{U}(0, p; u, u') = \delta(u - u').$$

Laplace transform in  $u$

$$U(\tau; x, u, x', u') = \int_{ia-\infty}^{ia+\infty} \frac{dp}{2\pi} \int_{b-i\infty}^{b+i\infty} \frac{dq}{2\pi i} e^{ip(x-x') + qu} F(\tau, p, q; u'),$$

where  $b$  is a sufficiently large positive constant.

First-order PDE

$$[\partial_\tau + f(q)\partial_q + \varphi(q)] F = 0,$$

where

$$f(q) = q^2 - 2\beta_1 q - \gamma_1$$

$$\varphi(q) = (\beta_0 + 2)q + \gamma_0 - 2\beta_1.$$

Initial condition

$$F(0, p, q; u') = e^{-qu'}.$$

## Hamiltonian system

$$\frac{d\hat{q}}{d\tau} = f(\hat{q}), \quad \frac{dF}{d\tau} = -\varphi(\hat{q})F.$$

Integral

$$F(\tau, p, q; u') = e^{-q_0 u'} \left( \frac{q_0 - q_1}{q - q_1} \right)^{a_1} \left( \frac{q_0 - q_2}{q - q_2} \right)^{a_2}$$

where

$$q_0 = \beta_1 + D \frac{1 + R}{1 - R}, \quad R = \frac{q - \beta_1 - D}{q - \beta_1 + D} e^{-2D\tau}.$$

$$q_{1,2} = \beta_1 \pm D, \quad D = \sqrt{\beta_1^2 + \gamma_1}.$$

$$a_1 = \frac{1}{2D} [(\beta_0 + 2)D + \gamma_0 + \beta_0\beta_1]$$

$$a_2 = \frac{1}{2D} [(\beta_0 + 2)D - \gamma_0 - \beta_0\beta_1]$$

Heat kernel

$$U(\tau; x, u, x', u') = \int_{ia-\infty}^{ia+\infty} \frac{dp}{2\pi} \int_{b-i\infty}^{b+i\infty} \frac{dq}{2\pi i} e^{ip(x-x') + qu - q_0 u'} \\ \times \left( \frac{q_0 - q_1}{q - q_1} \right)^{a_1} \left( \frac{q_0 - q_2}{q - q_2} \right)^{a_2} .$$

Initial conditions that do not depend on  $u$

$$P_n(0; x, u) = e^{nx} \theta(x),$$

where  $\theta(x)$  is the step-function

Fourier transform (with  $a < -n$ )

$$P_n(\tau; x, u) = \int_{ia-\infty}^{ia+\infty} \frac{dp}{2\pi} e^{ipx} \hat{P}_n(\tau, p; u),$$

Transformed equation

$$\left[ \partial_\tau - u \partial_u^2 + (2\beta_1 u + \beta_0) \partial_u + \gamma_1 u + \gamma_0 \right] \hat{P}_n = 0,$$

Initial condition

$$\hat{P}_n(0, p; u) = \frac{1}{ip - n}.$$

## Singular perturbation technique

$$\left[ \varepsilon \partial_\tau - \varepsilon^2 u \partial_u^2 + (2\beta_1 u + \beta_0) \varepsilon \partial_u + \gamma_1 u + \gamma_0 \right] \hat{P}_n = 0,$$

Ansatz

$$\hat{P}_n = \exp\left(\frac{\Phi}{\varepsilon}\right) \Omega, \quad \Omega = \sum_{k=0}^{\infty} \varepsilon^k \Omega_k.$$

Initial conditions

$$\Phi(0, p; u) = 0, \quad \Omega(0, p; u) = \frac{1}{ip - n}.$$

Hamilton-Jacobi equation

$$\partial_\tau \Phi - u(\partial_u \Phi)^2 + (2\beta_1 u + \beta_0) \partial_u \Phi + \gamma_1 u + \gamma_0 = 0$$

Recursive system

$$\left\{ \partial_\tau + [2\beta_1 u + \beta_0 - 2u(\partial_u \Phi)] \partial_u - u(\partial_u^2 \Phi) \right\} \Omega_k = u \partial_u^2 \Omega_{k-1}.$$

Action

$$\Phi(\tau, p, u) = uA(\tau, p) + B(\tau, p),$$

where

$$\partial_\tau A = A^2 - 2\beta_1 A - \gamma_1, \quad \partial_\tau B = -\beta_0 A - \gamma_0.$$

Initial conditions

$$A(0) = B(0) = 0$$

Solution

$$A = -\gamma_1 \frac{\sinh(D\tau)}{\beta_1 \sinh(D\tau) + D \cosh(D\tau)},$$

$$B = -(\gamma_0 + \beta_0\beta_1)\tau + \beta_0 \log \left[ \cosh(D\tau) + \beta_1 \frac{\sinh(D\tau)}{D} \right].$$

Coefficients

$$\Omega_0 = \frac{1}{ip - n} \quad \Omega_k = 0, \quad \text{for } k \geq 1$$

Solution

$$\begin{aligned} P_n(\tau; x, u) &= \int_{ia-\infty}^{ia+\infty} \frac{dp}{2\pi} \frac{1}{(ip - n)} e^{ipx - (\gamma_0 + \beta_0\beta_1)\tau} \\ &\times \exp\left(-\frac{\gamma_1 \sinh(D\tau)}{\beta_1 \sinh(D\tau) + D \cosh(D\tau)} u\right) \\ &\times \left(\cosh(D\tau) + \beta_1 \frac{\sinh(D\tau)}{D}\right)^{\beta_0}, \end{aligned}$$

This is the basis of the well-known Heston formula.