

Analytic and Geometric Methods for Heat Kernel Applications in Finance

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Lecture 6

Non-Perturbative Methods for Heat Kernel

Operator Method

Elliptic PDO

$$\begin{aligned} L(x, \partial) &= -g^{ij} (\nabla_i + \mathcal{A}_i) (\nabla_j + \mathcal{A}_j) + Q \\ &= -g^{-1/2} \nabla_i^{\mathcal{A}} g^{1/2} g^{ij} \nabla_j^{\mathcal{A}} + Q, \end{aligned}$$

where

$$\nabla_i^{\mathcal{A}} = \partial_i + \mathcal{A}_i$$

Commutators

$$\boxed{[\nabla_i^{\mathcal{A}}, \nabla_j^{\mathcal{A}}] = \mathcal{R}_{ij},}$$

where

$$\mathcal{R}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i.$$

Operators

Coordinates and momenta

$$[\hat{x}^i, \hat{x}^j] = 0 \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}^k, \hat{p}_j] = i\delta_j^k$$

$$\hat{\Pi}_k = \hat{p}_k - i\mathcal{A}_k(\hat{x}).$$

$$[\hat{x}^j, \hat{\Pi}_k] = i\delta_j^k, \quad [\hat{\Pi}_i, \hat{\Pi}_j] = -\mathcal{R}_{ij}(\hat{x}),$$

Eigenvectors and eigenvalues

$$\hat{x}^i|x'\rangle = x'^i|x'\rangle, \quad \langle x''|x'\rangle = \delta(x'' - x').$$

$$\langle x''|\hat{p}_j|x'\rangle = -i\partial_{j'}\delta(x'' - x').$$

Hamiltonian ($x^i \mapsto \hat{x}^i, \partial_j \mapsto i\hat{p}_j$)

$$H(\hat{x}, \hat{p}) = g^{-1/4}(\hat{x})\hat{\Pi}_i g^{1/2}(\hat{x})g^{ij}(\hat{x})\hat{\Pi}_j g^{-1/4}(\hat{x}) + Q(\hat{x}),$$

Time-dependent operators

$$\hat{x}^i(t) = \exp(tH)\hat{x}^i \exp(-tH), \quad \hat{\Pi}_j(t) = \exp(tH)\hat{\Pi}_j \exp(-tH).$$

Operator equations

$$\frac{dx^i(t)}{dt} = [H, x^i(t)], \quad \frac{d\hat{\Pi}_j(t)}{dt} = [H, \hat{\Pi}_j(t)]$$

Initial conditions

$$\hat{x}^i(0) = \hat{x}^i, \quad \hat{\Pi}_j(0) = \hat{\Pi}_j.$$

Time-dependent vectors

$$|x'(t)\rangle = \exp(tH)|x'\rangle, \quad \langle x'(t)| = \langle x''| \exp(-tH).$$

Heat kernel

Heat equation

$$[\partial_t + L(x'', \partial_{x''})]U(t; x'', x') = 0$$

Initial condition

$$U(0; x'', x') = \delta(x'', x').$$

Covariant delta-function

$$\delta(x'', x') = g^{-1/4}(x'')\delta(x' - x')g^{-1/4}(x').$$

Operator solution

$$\begin{aligned} U(t; x'', x') &= \exp[-tL(x'', \partial_{x''})]\delta(x'', x') \\ &= g^{-1/4}(x'')g^{-1/4}(x')\langle x'' | \exp(-tH) | x' \rangle. \end{aligned}$$

Idea: express all operators in terms of $\hat{x}(t)$ and \hat{x} and put all operators $\hat{x}(t)$ to the left and the operators \hat{x} to the right

Solution of operator equations

$$\hat{x}(t) = f_1(t; \hat{x}, \hat{\Pi}), \quad \hat{\Pi}(t) = f_2(t; \hat{x}, \hat{\Pi}),$$

Initial momenta

$$\hat{\Pi} = f_3(t; \hat{x}(t), \hat{x}),$$

Hamiltonian

$$H(\hat{x}, \hat{\Pi}) = h_1(t; \hat{x}(t))h_2(t; \hat{x}),$$

Heat equation

$$\begin{aligned}\partial_t U(t; x'', x') &= -\langle x''(t) | H(\hat{x}, \hat{\Pi}) | x' \rangle \\ &= -h(t; x'', x') U(t; x'', x').\end{aligned}$$

where

$$h(t; x'', x') = h_1(t; x'') h_2(t; x')$$

$$U(t; x'', x') = C(x'', x') \exp\left(-\int dt h(t; x'', x')\right),$$

where $C(x'', x')$ is determined from the initial conditions

Example: Linear Connection

Elliptic PDO

$$L = -\delta^{jk} (\partial_j + \mathcal{A}_j) (\partial_k + \mathcal{A}_k) ,$$

where

$$\mathcal{A}_i = -\frac{1}{2} \mathcal{R}_{ij} x^j ,$$

Hamiltonian

$$H = \hat{\Pi}_j \hat{\Pi}^j ,$$

Operator equations

$$\frac{d\hat{x}^j}{dt} = -2i\hat{\Pi}^j , \quad \frac{d\hat{\Pi}_j}{dt} = 2\mathcal{R}_{jk}\hat{\Pi}^k .$$

Solution

$$\hat{\Pi}^j(t) = [\exp(2t\mathcal{R})]^j_k \hat{\Pi}^k,$$

$$\hat{x}^j(t) = \hat{x}^j - i \left[\frac{\exp(2t\mathcal{R}) - 1}{\mathcal{R}} \right]^j_k \hat{\Pi}^k,$$

Initial momenta

$$\hat{\Pi}^j = i \left[\frac{\mathcal{R}}{\exp(2t\mathcal{R}) - 1} \right]^j_k [\hat{x}(t) - \hat{x}]^k.$$

Hamiltonian

$$H = -\frac{1}{4} [\hat{x}(t) - \hat{x}]^j \left(\frac{\mathcal{R}^2}{\sinh^2(t\mathcal{R})} \right)_{jk} [\hat{x}(t) - \hat{x}]^k.$$

Commutator

$$[\hat{x}^j(t), \hat{x}^k] = - \left[\frac{\exp(2t\mathcal{R}) - 1}{\mathcal{R}} \right]^{jk}.$$

Time-ordered Hamiltonian

$$\begin{aligned}
 h(t; x'', x') &= \langle x''(t) | H | x' \rangle \\
 &= -\frac{1}{4} (x'' - x')^j \left(\frac{\mathcal{R}^2}{\sinh^2(t\mathcal{R})} \right)_{jk} (x'' - x')^k + \frac{1}{2} \text{tr} [\mathcal{R} \coth(t\mathcal{R})] ,
 \end{aligned}$$

Heat kernel

$$\begin{aligned}
 U(t; x'', x') &= (4\pi t)^{-n/2} \mathcal{P}(x'', x') \det \left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} \\
 &\quad \times \exp \left(-\frac{1}{4} (x'' - x')^i [\mathcal{R} \coth(t\mathcal{R})]_{ij} (x'' - x')^j \right)
 \end{aligned}$$

Function $\mathcal{P}(x'', x')$ satisfies the equation

$$(x - x')^j \left[\partial_j - \frac{1}{2} \mathcal{R}_{jm} (x - x')^m \right] \mathcal{P}(x, x') = 0$$

and the initial condition

$$\mathcal{P}(x, x) = 1.$$

Therefore,

$$\mathcal{P}(x'', x') = \exp \left(\frac{1}{2} \mathcal{R}_{jk} x''^j x'^k \right).$$

Harmonic oscillator

$$L = -\delta^{ij} \partial_i \partial_j + \frac{1}{2} P_{ij} x^i x^j,$$

where P_{ij} is a constant symmetric tensor, can be treated similarly. Both these cases are solvable because the operator equations are linear.

Covariant Fourier Transform Method

Elliptic PDO

$$L = -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q,$$

Covariant Fourier integral representation of the δ -function

$$\delta(x, x') = \Delta^{1/2}(x, x') \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') \exp \left[ik_{j'} \sigma^{j'}(x, x') \right]$$

Heat kernel

$$\begin{aligned} U(t; x, x') &= \exp(-tL) \delta(x, x') \\ &= \Delta^{1/2}(x, x') \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') \exp \left[ik_{j'} \sigma^{j'}(x, x') \right] \exp(-tA) \cdot 1, \end{aligned}$$

where

$$A = \exp \left(-ik_{j'} \sigma^{j'} \right) \mathcal{P}^{-1} \Delta^{-1/2} L \Delta^{1/2} \mathcal{P} \exp \left(ik_{l'} \sigma^{l'} \right),$$

Separation of the free case

$$X^{i'j'} = g^{i'j'}(x') + \tilde{X}^{i'j'}.$$

Then

$$A = A_0 + A_1 + A_2 + k_{i'}k_{j'}g^{i'j'}(x')$$

where

$$A_0 = -X^{i'j'}\mathcal{D}_{i'}\mathcal{D}_{j'} - Y^{i'}\mathcal{D}_{i'} + Z,$$

$$A_1 = -ik_{i'}(2X^{i'j'}\mathcal{D}_{j'} + Y^{i'}),$$

$$A_2 = k_{i'}k_{j'}\tilde{X}^{i'j'}.$$

Definitions

Operators

$$\mathcal{D}_{i'} = \gamma^j_{i'} \nabla_j,$$

where $\gamma^i_{j'}$ is the inverse of the matrix $\eta^{i'j} = \nabla_j \sigma^{i'}$, so that

$$[\mathcal{D}_{i'}, \mathcal{D}_{j'}] = 0, \quad [\mathcal{D}_{i'}, \sigma^{j'}] = \delta^{j'}_{i'}.$$

Two-points functions

$$X^{i'j'} = \eta^{i'k} \eta^{j'k}, \quad Y^{i'} = \mathcal{D}_{j'} X^{i'j'} + 2X^{i'j'} \hat{\mathcal{A}}_{j'}$$

$$Z = X^{i'j'} (\zeta_{i'} \zeta_{j'} - \hat{\mathcal{A}}_{i'} \hat{\mathcal{A}}_{j'}) - \mathcal{D}_{j'} [X^{i'j'} (\zeta_{i'} + \hat{\mathcal{A}}_{i'})] + Q,$$

$$\hat{\mathcal{A}}_{i'} = \mathcal{P}^{-1} (\mathcal{D}_{i'} + \mathcal{A}_{i'}) \mathcal{P}, \quad \zeta_{i'} = \mathcal{D}_{i'} \zeta, \quad \zeta = \log \Delta^{1/2}$$

By scaling $k \mapsto \frac{k}{\sqrt{t}}$ we obtain

Heat Kernel

$$U(t; x, x') = (4\pi t)^{-n/2} \Delta^{1/2}(x, x') \\ \times \left\langle \exp \left[it^{-1/2} k_{j'} \sigma^{j'} \right] \exp \left(-A_2 - \sqrt{t} A_1 - t A_0 \right) \cdot \mathbf{1} \right\rangle$$

Gaussian average

$$\langle f(k) \rangle = \lim_{x \rightarrow x'} \int \frac{dk}{\pi^{n/2}} g^{-1/2} \exp \left[-g^{i'j'}(x') k_{i'} k_{j'} \right] f(k),$$

By expanding in a powers of t and σ^i we get an asymptotic expansion as $t \rightarrow 0$ with coefficients expressed in terms of Taylor coefficients of the functions $X^{i'j'}$, $Y^{i'}$ and Z .

Heat kernel diagonal

$$U(t; x, x) = (4\pi t)^{-n/2} \Omega(t; x, x),$$

where

$$\Omega(t; x, x) = \langle \exp(-A_2 - \sqrt{t}A_1 - tA_0) \cdot \mathbf{1} \rangle.$$

Asymptotic series as $t \rightarrow 0$

$$\Omega(t; x, x) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} [b_k],$$

Heat kernel coefficients

$$[b_k] = \sum_{N=0}^{\infty} \frac{k!}{N!} \sum_{\substack{0 \leq k_1, \dots, k_N \leq 2 \\ k_1 + \dots + k_N = k}} \langle A_{k_1} \cdots A_{k_N} \cdot \mathbf{1} \rangle,$$

Summation goes over all integers k_1, \dots, k_N taking the values 0, 1 and 2 and such that their sum is equal to k .

Algebraic Methods

Basic idea

Elliptic PDO

$$L = -g^{ij} \nabla_i^{\mathcal{A}} \nabla_j^{\mathcal{A}} + Q = -g^{ij} (\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q,$$

where $\nabla_i^{\mathcal{A}}$ are some first-order differential operators

Curvatures (background fields)

$$\mathfrak{R} = \{R^i_{jkl}, \mathcal{R}_{ij}, Q\}$$

where

$$\mathcal{R}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

The long-wave approximation corresponds to slowly changing background fields

$$\nabla\nabla\mathcal{R} \ll \mathcal{R}\mathcal{R}$$

In the zero order of this approximation the fields are assumed to be covariantly constant.

$$\nabla\mathcal{R} = 0$$

A covariantly constant curvature is not necessarily zero but rather can be arbitrarily large.

Examples: homogeneous and symmetric spaces (sphere S^n , hyperbolic space H^n , etc).

It is possible to construct the heat kernel without solving the heat equation but using only the algebraic commutation relations of some first order differential operators.

Zero curvature

$$R^i{}_{jkl} = \mathcal{R}_{ij} = 0, \quad Q = \text{const}$$

Abelian algebra

$$[\nabla_i, \nabla_j] = 0, \quad [\nabla_i, Q] = 0, \quad [\nabla_i, g_{jk}] = 0.$$

Heat-semigroup

$$\begin{aligned} \exp(-tL) &= (4\pi t)^{-n/2} \exp(-tQ) \\ &\times \int_{\mathbb{R}^n} dk g^{1/2} \exp\left(-\frac{1}{4t} k^i g_{ij} k^j\right) \exp(k^j \nabla_j), \end{aligned}$$

Action on the delta-function $\delta(x, x')$

$$\exp(k^j \nabla_j) \delta(x, x') = \delta(x - x' + k)$$

Heat kernel

$$U(t; x, x') = (4\pi t)^{-n/2} \exp \left[-\frac{1}{4t} (x - x')^i g_{ij} (x - x')^j - tQ \right] .$$

Non-zero curvature

Covariant derivatives ∇ do not commute

We want to find an integral representation of the heat semi-group

$$\exp(-tL) = \int dk \Psi(t, k) \exp \left\{ -\frac{1}{4t} k^A \Phi_{AB}(t) k^B \right\} \exp(k^A \xi_A),$$

where $\xi_A = \{X_a, Y_i\}$, $X_a = X_a^\mu \nabla_\mu$ are some first-order differential operators and Y_i are some functions, and the functions $\Phi(t)$ and $\Psi(t, k)$ are expressed in terms of the commutators of these operators, i.e. curvatures.

The main point is that it is much easier to calculate the exponential of the first-order operator than the exponential of the second-order operator.

Linear Connection and Constant Potential in Flat Space

Flat metric

$$R^i{}_{jkl} = 0$$

Linear connection

$$\nabla_j^{\mathcal{A}} = \partial_j - \frac{1}{2} \mathcal{R}_{jk} x^k,$$

Constant potential

$$Q = \text{const}$$

Nilpotent Lie algebra

$$[\nabla_j^{\mathcal{A}}, \nabla_k^{\mathcal{A}}] = \mathcal{R}_{jk}, \quad [\nabla_j^{\mathcal{A}}, \mathcal{R}_{kl}] = [\nabla_j^{\mathcal{A}}, Q] = 0.$$

Heat semi-group (non-trivial)

$$\begin{aligned} \exp(-tL) &= (4\pi t)^{-n/2} \det \left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} \exp(-tQ) \\ &\quad \times \int_{\mathbb{R}^n} dk \exp \left\{ -\frac{1}{4t} k^l [t\mathcal{R} \coth(t\mathcal{R})]_{lm} k^m \right\} \exp(k^j \nabla_j^{\mathcal{A}}), \end{aligned}$$

Action on the delta-function

$$\exp(k^j \nabla_j^{\mathcal{A}}) \mathcal{P}(x, x') \delta(x - x') = \mathcal{P}(x, x') \delta(x - x' + k),$$

where

$$\mathcal{P}(x, x') = \exp \left(\frac{1}{2} \mathcal{R}_{ij} x^i x'^j \right)$$

Heat kernel

$$U(t; x, x') = (4\pi t)^{-n/2} \det \left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} \exp(-tQ) \\ \times \mathcal{P}(x, x') \exp \left\{ -\frac{1}{4t} (x - x')^j [t\mathcal{R} \coth(t\mathcal{R})]_{lm} (x - x')^m \right\} .$$

Linear Connection and Quadratic Potential in Flat Space

Take into account first and second derivatives of potential

$$Q_{;i} = \nabla_i Q, \quad P_{ij} = \frac{1}{2} \nabla_i \nabla_j Q.$$

Nilpotent Lie algebra

$$[\nabla_j^A, \nabla_k^A] = \mathcal{R}_{jk}, \quad [\nabla_i^A, Q] = Q_{;i}, \quad [\nabla_j^A, Q_{;i}] = 2P_{ij}$$

Parametrization

$$Q = M - \beta^{\mu\nu} L_\mu L_\nu,$$

where $(\mu = 1, \dots, p)$, $p \leq n$, and $\beta^{\mu\nu}$ is some constant symmetric nondegenerate $p \times p$ matrix, M is a constant and L_μ are some *linear* functions.

Generators

$$\xi_A = (\nabla_i^A, L_\mu)$$

where $(A = 1, \dots, N)$, with $N = n + p$.

Commutation relations

$$\boxed{[\xi_A, \xi_B] = \mathcal{F}_{AB}, \quad [\xi_A, \mathcal{F}_{CD}] = 0,}$$

where

$$(\mathcal{F}_{AB}) = \begin{pmatrix} \mathcal{R}_{kj} & L_{\mu;k} \\ -L_{\mu;j} & 0 \end{pmatrix}.$$

Operator

$$\boxed{L = -\gamma^{AB} \xi_A \xi_B + M,}$$

where

$$(\gamma^{AB}) = \begin{pmatrix} \delta^{jk} & 0 \\ 0 & \beta^{\mu\nu} \end{pmatrix}.$$

Heat semi-group

$$\begin{aligned} \exp(-tL) &= (4\pi t)^{-D/2} \det \left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}} \right)^{-1/2} \exp(-tM) \\ &\times \int_{\mathbb{R}^D} dk \gamma^{1/2} \exp \left\{ -\frac{1}{4t} k^A [t\mathcal{F} \coth(t\mathcal{F})]_{AB} k^B \right\} \exp(k^C \xi_C), \end{aligned}$$

where $\gamma = (\det \gamma^{AB})^{-1}$

Next, we split the integration variables $(k^A) = (q^i, \omega^\mu)$, use the Campbell-Hausdorff formula to single out the non-commutative part, and integrate over ω^μ .

After long but straightforward calculation we obtain the heat semi-group

$$\begin{aligned} \exp(-tL) &= (4\pi t)^{-n/2} \exp \left\{ -tQ + \Phi(t) + \frac{1}{4}t^3 Q_{;i} \Psi^{ij}(t) Q_{;j} \right\} \\ &\times \int_{\mathbb{R}^n} dq \exp \left\{ -\frac{1}{4t} q^i D_{ij}(t) q^j - \frac{t}{2} Q_{;i} [\delta^i_j + A^i_j(t)] q^j \right\} \exp(q^k \nabla_k A), \end{aligned}$$

where $\Phi(t)$ is a function and $\Psi(t)$, $D(t)$ and $A(t)$ are matrices depending on \mathcal{R} and P (defined below)

To describe the result we define auxiliary matrices

$$B(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z^2} \coth(tz^{-1})G(z),$$

$$A(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z} \coth(tz^{-1})G(z),$$

$$C(t) = \oint_C \frac{dz}{2\pi i} t \coth(tz^{-1})G(z),$$

$$K(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z^2} \sinh(tz^{-1})G(z),$$

$$S(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z} \sinh(tz^{-1})G(z),$$

$$N(t) = \oint_C \frac{dz}{2\pi i} t \sinh(tz^{-1})G(z),$$

where the integral is taken over a sufficiently small counterclockwise oriented circle C around the origin and

$$G(z) = [1 - z\mathcal{R} - z^2P]^{-1}$$

Then

$$D(t) = B(t) + t^2[1 - A(t)]P[1 + t^2C(t)P]^{-1}[1 + A(t)],$$

$$\begin{aligned} \Phi(t) = & -\frac{1}{2} \log \det [1 + t^2N(t)P] \\ & -\frac{1}{2} \log \det \left\{ K(t) - t^2S(t)P[1 + t^2N(t)P]^{-1}S(t) \right\} \\ & -\frac{1}{2} \log \det [1 + t^2C(t)P], \end{aligned}$$

$$\Psi(t) = [1 + t^2C(t)P]^{-1}C(t).$$

Heat kernel

$$\begin{aligned} U(t; x, x') &= (4\pi t)^{-n/2} \mathcal{P}(x, x') \exp \{-tQ(x) + \Phi(t)\} \\ &\times \exp \left\{ \frac{1}{4} t^3 Q_{;i}(x) \Psi^{ij}(t) Q_{;j}(x) \right\} \\ &\times \exp \left\{ -\frac{1}{4t} (x - x')^i D_{ij}(t) (x - x')^j \right\} \\ &\times \exp \left\{ \frac{t}{2} Q_{;i}(x) [\delta^i_j + A^i_j(t)] (x - x')^j \right\}. \end{aligned}$$

Heat kernel diagonal

$$U(t; x, x) = (4\pi t)^{-n/2} \exp \left\{ -tQ + \Phi(t) + \frac{1}{4} t^3 Q_{;i} \Psi^{ij}(t) Q_{;j} \right\}.$$

Particular case (when the matrices \mathcal{R} and P commute)

$$\mathcal{R}^i_j P^j_k = P^i_j \mathcal{R}^j_k.$$

Heat kernel diagonal

$$[U(t)] = (4\pi t)^{-n/2} \det \left(\frac{\sinh(t\Delta)}{t\Delta} \right)^{-1/2} \exp(-tQ) \\ \times \exp \left\{ \frac{1}{4} t Q_{;i} \left[\frac{1}{P} \left(\frac{\Delta \cosh(t\mathcal{R}) - \cosh(t\Delta)}{2tP \sinh(t\Delta)} + 1 \right) \right]^i {}_j Q^{;j} \right\},$$

where

$$\Delta = (4P + \mathcal{R}^2)^{1/2}.$$

If potential is linear, that is, $P_{ij} = \frac{1}{2} \nabla_i \nabla_j Q = 0$, then

$$[U(t)] = (4\pi t)^{-n/2} \det \left(\frac{\sinh(t\mathcal{R})}{t\mathcal{R}} \right)^{-1/2} \exp(-tQ) \\ \times \exp \left\{ \frac{1}{4} t Q_{;i} \left(\frac{t\mathcal{R} \coth(t\mathcal{R}) - 1}{\mathcal{R}^2} \right)^i {}_j Q^{;j} \right\}.$$

If $\mathcal{R}_{ij} = 0$, then

$$[U(t)] = (4\pi t)^{-n/2} \det \left(\frac{\sinh(2t\sqrt{P})}{2t\sqrt{P}} \right)^{-1/2} \exp(-tQ) \\ \times \exp \left\{ -\frac{1}{4} Q_{;i} \left(\frac{\tanh(t\sqrt{P}) - t\sqrt{P}}{P^{3/2}} \right)^i {}_j Q^{;j} \right\}.$$

Heat Kernel on Symmetric Spaces

Algebraic Constraints

Consider pure scalar Laplacian

$$L = -g^{ij} \nabla_i \nabla_j .$$

Covariantly constant curvature (locally symmetric spaces)

$$\nabla_i R^j{}_{klm} = 0 .$$

Constraints

$$R^{ij}{}_{km} R^k{}_{nlp} - R^{ij}{}_{kn} R^k{}_{mlp} + R^{ij}{}_{kl} R^k{}_{pmn} - R^{ij}{}_{kp} R^k{}_{lmn} = 0 ,$$

Parallel Orthonormal Frame

Extend an orthonormal basis, $e_a^{i'}$ and $e^{a_{i'}}$, at a fixed point x' to a parallel local orthonormal frame, e_a^i and e^{a_i} , at the point x by parallel transport along the geodesic connecting the points x and x'

The frame components of tensors are defined by projecting them onto the orthonormal frame

$$R^a_{bcd} = R^i_{jkl} e^a_i e_b^j e_c^k e_d^l .$$

The frame components of a covariantly constant tensor are constant

Normal coordinates

$$y^a = e^a_i \sigma^i(x, x') = -e^a_{j'} \sigma^{j'}(x, x') ,$$

Global structure of symmetric spaces

A symmetric space is a simply connected and complete manifold with covariantly constant curvature.

This means that each closed loop can be continuously deformed to a point and that the manifold does not have a boundary.

A symmetric space is compact, non-compact or Euclidean if for any two non-zero non-parallel vectors u and v , the quantity $R_{ijkl}u^i u^k v^j v^l$ (so-called sectional curvature) is positive, negative or zero.

Every symmetric space is a product of a Euclidean, a compact and a non-compact symmetric spaces

$$M = M_0 \times M_+ \times M_- ,$$

A semi-simple symmetric space is a product of a compact and a non-compact symmetric spaces

$$M_s = M_+ \times M_- ,$$

The contribution of the Euclidean factor to the heat kernel is trivial. So, we restrict ourselves to semi-simple symmetric spaces.

Curvature Tensor and Holonomy Algebra

Curvature tensor

$$R_{abcd} = \beta_{\mu\nu} E^\mu{}_{ab} E^\mu{}_{cd},$$

where $\beta_{\mu\nu}$, $\mu, \nu = 1, 2, \dots, p$, $p \leq n(n-1)/2$ is a symmetric non-degenerate matrix

Holonomy algebra, \mathcal{H} ,

$$D^a{}_{\mu b} = -\beta_{\mu\nu} E^\nu{}_{cb} \delta^{ca}, \quad [D_\mu, D_\nu] = F^\alpha{}_{\mu\nu} D_\alpha,$$

where $F^\alpha{}_{\mu\nu}$ are the structure constants

Adjoint representation of the holonomy algebra

$$(F_\mu)^\alpha{}_\beta = F^\alpha{}_{\mu\beta}, \quad [F_\mu, F_\nu] = F^\alpha{}_{\mu\nu} F_\alpha.$$

In symmetric spaces there is a larger algebra, \mathcal{G} ,

$$[C_A, C_B] = C^C{}_{AB} C_C,$$

where

$$(C_A)^B{}_C = C^B{}_{AC} = -C^B{}_{CA}$$

$A, B, C = 1, 2, \dots, N, N = p + n$, and

$$C^\mu{}_{ab} = E^\mu{}_{ab}, \quad C^a{}_{\mu b} = D^a{}_{\mu b}, \quad C^\mu{}_{\alpha\beta} = F^\mu{}_{\alpha\beta},$$

all other components being zero.

Killing Vectors Fields

Killing vector fields are defined by

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

Basis of Killing vectors

$$P_a^i = e_b^i (\cos \sqrt{K})^b{}_a,$$
$$L_\mu^i = -e_b^i \left(\frac{\sin \sqrt{K}}{\sqrt{K}} \right)^b{}_a y^c D^a{}_{\mu c}.$$

where the matrix $K = (K^a{}_b)$ is

$$K^a{}_b = R^a{}_{cbd} y^c y^d$$

Lie Derivatives

In normal coordinates

$$\mathcal{L}_a = P_a^i \partial_i = \left(\sqrt{K} \cot \sqrt{K} \right)^b{}_a \frac{\partial}{\partial y^b},$$

$$\mathcal{L}_\mu = L_\mu^i \partial_i = -D^b{}_{\mu a} y^a \frac{\partial}{\partial y^b}$$

Isometry algebra

$$[\mathcal{L}_A, \mathcal{L}_B] = C^C{}_{AB} \mathcal{L}_C,$$

Isotropy subalgebra

$$[\mathcal{L}_\mu, \mathcal{L}_\nu] = F^\alpha{}_{\mu\nu} \mathcal{L}_\alpha,$$

Laplacian

Identity

$$\gamma^{AB} \xi_A^i \xi_B^j = g^{ij} .$$

where

$$\gamma = (\gamma_{AB}) = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix} .$$

Laplacian

$$L = -g^{ij} \nabla_i \nabla_j = -\gamma^{AB} \mathcal{L}_A \mathcal{L}_B .$$

Laplacian commutes with Lie derivatives

$$[\mathcal{L}_A, L] = 0 .$$

Heat Semi-group (non-trivial)

$$\begin{aligned} \exp(-tL) &= (4\pi t)^{-N/2} \exp\left(\frac{1}{6}R_G t\right) \\ &\times \int_{\mathbb{R}_{\text{reg}}^N} dk |\gamma|^{1/2} \det\left(\frac{\sinh[C(k)/2]}{C(k)/2}\right)^{1/2} \exp\left\{-\frac{1}{4t}\langle k, \gamma k \rangle\right\} \\ &\times \exp[\mathcal{L}(k)], \end{aligned}$$

where $|\gamma| = \det \gamma_{AB}$ and

$$\langle k, \gamma k \rangle = \gamma_{AB} k^A k^B, \quad R_G = -\frac{1}{4} \gamma^{AB} C^C{}_{AD} C^D{}_{BC},$$

$$C(k) = k^A C_A, \quad \mathcal{L}(k) = k^A \mathcal{L}_A$$

Here $\mathbb{R}_{\text{reg}}^N$ means that the integral over k^A needs to be regularized.

Calculation of Isometries

Canonical coordinates on the isometry group $k^A = (p^a, \omega^\mu)$

Action on the delta-function

$$\exp[\mathcal{L}(k)]\delta(x, x') = J(\omega, x, x')\delta[p - \bar{p}(\omega, x, x')],$$

where

$$J(\omega, x, x') = \det \left(\frac{\sinh [D(\omega)/2]}{D(\omega)/2} \right)^{-1} + O(y).$$

$$\bar{p}^a = - \left(D(\omega) \frac{\exp[-D(\omega)]}{1 - \exp[-D(\omega)]} \right)^a {}_b y^b + O(y^2).$$

and

$$D(\omega) = \omega^\mu D_\mu.$$

Heat Kernel

$$\begin{aligned}
 U(t; x, x') &= (4\pi t)^{-n/2} \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H \right) t \right\} \\
 &\times \int_{\mathbb{R}_{\text{reg}}^n} \frac{d\omega}{(4\pi t)^{p/2}} |\beta|^{1/2} \exp \left\{ -\frac{1}{4t} [\langle \omega, \beta \omega \rangle + \langle y, B(\omega)y \rangle] \right\} \\
 &\times \det \left(\frac{\sinh [F(\omega)/2]}{F(\omega)/2} \right)^{1/2} \det \left(\frac{\sinh [D(\omega)/2]}{D(\omega)/2} \right)^{-1/2},
 \end{aligned}$$

where $|\beta| = \det \beta_{\mu\nu}$, $\langle \omega, \beta \omega \rangle = \beta_{\mu\nu} \omega^\mu \omega^\nu$,

$$B(\omega) = \left(\frac{\sinh [D(\omega)/2]}{D(\omega)/2} \right)^{-2}, \quad \langle y, B(\omega)y \rangle = y^a B_{ab}(\omega) y^b,$$

$$F(\omega) = \omega^\mu F_\mu, \quad R_H = -\frac{1}{4} \beta^{\alpha\beta} F^\mu{}_{\alpha\gamma} F^\gamma{}_{\beta\mu},$$

Heat Kernel Diagonal

$$\begin{aligned} U(t; x, x) &= (4\pi t)^{-n/2} \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H \right) t \right\} \\ &\times \int_{\mathbb{R}_{\text{reg}}^n} \frac{d\omega}{(4\pi t)^{p/2}} |\beta|^{1/2} \exp \left\{ -\frac{1}{4t} \langle \omega, \beta \omega \rangle \right\} \\ &\times \det \left(\frac{\sinh [F(\omega)/2]}{F(\omega)/2} \right)^{1/2} \det \left(\frac{\sinh [D(\omega)/2]}{D(\omega)/2} \right)^{-1/2} \end{aligned}$$

Heat Kernel on S^2

Let y^a be the normal coordinates (a is the radius of the sphere)

$$-a\pi \leq y^a \leq a\pi$$

Polar normal coordinates r and φ

$$y^1 = r \cos \varphi, \quad y^2 = r \sin \varphi,$$

so that $0 \leq r \leq a\pi$ and $0 \leq \varphi \leq 2\pi$.

Orthonormal frame of 1-forms

$$e^1 = dr, \quad e^2 = a \sin\left(\frac{r}{a}\right) d\varphi,$$

Curvature (ε_{ab} is the antisymmetric Levi-Civita symbol)

$$R_{abcd} = \frac{1}{a^2} \varepsilon_{ab} \varepsilon_{cd} = \frac{1}{a^2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}),$$

Holonomy algebra is one-dimensional (Abelian)

$$D_{ab} = -\frac{1}{a^2}E_{ab} = -\frac{1}{a^2}\varepsilon_{ab}.$$

$$F^\alpha{}_{\mu\beta} = 0, \quad \beta = \frac{1}{a^2}$$

Lie derivatives

$$\mathcal{L}_1 = \cos \varphi \partial_r - \frac{\sin \varphi}{a} \cot \left(\frac{r}{a} \right) \partial_\varphi,$$

$$\mathcal{L}_2 = \sin \varphi \partial_r + \frac{\cos \varphi}{a} \cot \left(\frac{r}{a} \right) \partial_\varphi,$$

$$\mathcal{L}_3 = \frac{1}{a^2} \partial_\varphi,$$

Isometry algebra $SO(3)$

$$[\mathcal{L}_1, \mathcal{L}_2] = -\mathcal{L}_3,$$

$$[\mathcal{L}_3, \mathcal{L}_1] = -\frac{1}{a^2}\mathcal{L}_2$$

$$[\mathcal{L}_3, \mathcal{L}_2] = \frac{1}{a^2}\mathcal{L}_1.$$

Laplacian

$$L = -\partial_r^2 - \frac{1}{a} \cot\left(\frac{r}{a}\right) \partial_r - \frac{1}{a^2 \sin^2(r/a)} \partial_\varphi^2.$$

Determinant

$$\det\left(\frac{\sinh[\omega D]}{\omega D}\right)^{-1/2} = \frac{\omega/(2a^2)}{\sin[\omega/(2a^2)]},$$

Heat kernel diagonal

$$U(t; x, x) = \frac{1}{4\pi t} \exp\left(\frac{t}{4a^2}\right) \times P \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp\left(-\frac{\omega^2}{4}\right) \frac{\omega\sqrt{t}/(2a)}{\sin\left[\omega\sqrt{t}/(2a)\right]},$$

where P denotes the Cauchy principal value of the integral, which can also be written as

$$U(t; x, x) = \frac{1}{4\pi t} \exp\left(\frac{t}{4a^2}\right) \sum_{k=-\infty}^{\infty} (-1)^k \times \int_0^{2\pi a/\sqrt{t}} \frac{d\omega}{\sqrt{4\pi}} \exp\left[-\frac{1}{4}\left(\omega + \frac{2\pi a}{\sqrt{t}}k\right)^2\right] \frac{\sqrt{t}\left(\omega + \frac{2\pi a}{\sqrt{t}}k\right)}{2a \sin\left[\omega\sqrt{t}/(2a)\right]}.$$

Heat Kernel on H^2

Normal coordinates $-\infty \leq y^a \leq \infty$

Polar normal coordinates

$$y^1 = r \cos \varphi, \quad y^2 = r \sin \varphi,$$

so that $0 \leq r \leq \infty$ and $0 \leq \varphi \leq 2\pi$.

Orthonormal frame of 1-forms

$$e^1 = dr, \quad e^2 = a \sinh \left(\frac{r}{a} \right) d\varphi,$$

Curvature

$$R_{abcd} = -\frac{1}{a^2} \varepsilon_{ab} \varepsilon_{cd} = -\frac{1}{a^2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}),$$

Holonomy algebra

$$D_{ab} = \frac{1}{a^2} E_{ab} = \frac{1}{a^2} \varepsilon_{ab}.$$

$$F^\alpha{}_{\mu\beta} = 0, \quad \beta = -\frac{1}{a^2}$$

Lie derivatives

$$\mathcal{L}_1 = \cos \varphi \partial_r - \frac{\sin \varphi}{a} \coth \left(\frac{r}{a} \right) \partial_\varphi,$$

$$\mathcal{L}_2 = \sin \varphi \partial_r + \frac{\cos \varphi}{a} \coth \left(\frac{r}{a} \right) \partial_\varphi,$$

$$\mathcal{L}_3 = -\frac{1}{a^2} \partial_\varphi,$$

Isometry algebra $SO(1, 2)$

$$[\mathcal{L}_1, \mathcal{L}_2] = -\mathcal{L}_3,$$

$$[\mathcal{L}_3, \mathcal{L}_1] = \frac{1}{a^2} \mathcal{L}_2$$

$$[\mathcal{L}_3, \mathcal{L}_2] = -\frac{1}{a^2} \mathcal{L}_1.$$

Laplacian

$$L = -\partial_r^2 - \frac{1}{a} \coth\left(\frac{r}{a}\right) \partial_r - \frac{1}{a^2 \sinh^2(r/a)} \partial_\varphi^2.$$

Heat kernel diagonal

$$U(t; x, x) = \frac{1}{4\pi t} \exp\left(-\frac{t}{4a^2}\right) \times \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp\left(-\frac{\omega^2}{4}\right) \frac{\omega\sqrt{t}/(2a)}{\sinh\left[\omega\sqrt{t}/(2a)\right]}.$$

Heat kernel for S^2 is related to the heat kernel on H^2 by analytic continuation

$$a \mapsto ia, \quad r \mapsto ir \quad \text{or} \quad t \mapsto -t$$

Spectral representation (good for large t)

For H^2

$$U(t; x, x) = \frac{1}{4\pi a^2} \int_{-\infty}^{\infty} d\nu \nu \tanh(\nu) \exp \left\{ - \left(\frac{1}{4} + \nu^2 \right) \frac{t}{a^2} \right\},$$

For S^2

$$U(t; x, x) = \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} d_k \exp(-\lambda_k t),$$

where

$$\lambda_k = \frac{1}{a^2} k(k+1), \quad d_k = k + \frac{1}{2}.$$