Analytic and Geometric Methods for Heat Kernel Applications in Finance

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Lecture 5

Asymptotic Expansion of Heat Kernel

Asymptotic Ansatz

Consider a n-dim Riemannian manifold M without boundary

Elliptic second-order PDO

$$\begin{aligned} L &= -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q\,, \\ &= -g^{-1/2}(\partial_i + \mathcal{A}_i)g^{1/2}g^{ij}(\partial_j + \mathcal{A}_j) + Q\,. \end{aligned}$$

Adjoint operator

$$\begin{aligned} L^* &= -g^{ij}(\nabla_i - \mathcal{A}_i)(\nabla_j - \mathcal{A}_j) + Q \\ &= -g^{-1/2}(\partial_i - \mathcal{A}_i)g^{1/2}g^{ij}(\partial_j - \mathcal{A}_j) + Q\,. \end{aligned}$$

Heat kernel

$$(\partial_t + L_x)U(t; x, x') = 0\,,$$

Initial condition

$$U(0; x, x') = \delta(x, x')\,.$$

Covariant delta-function

$$\delta(x, x') = g^{-1/4}(x)\delta(x - x')g^{-1/4}(x')\,.$$

Adjoint equation with respect to x'

$$(\partial_t + L^*_{x'})U(t; x, x') = 0\,.$$

For $t > 0$ the heat kernel is a smooth function of the coordinates of both points x and x'.

Remarks.

Heat kernel defined with the initial condition $\tilde{U}(0; x, x') = \delta(x - x')$ is not a scalar (not invariant under diffeomorphisms). It is a two-point density.

Heat kernel defined with the initial condition $U(0; x, x') = \delta(x, x')$ is a *two-point scalar* (invariant under changes of coordinates at x and x').

Relation

$$\tilde{U}(t; x, x') = g^{1/4}(x) g^{1/4}(x') U(t; x, x')$$

If one uses the scalar heat kernel $U(t; x, x')$, then one has to use the *invariant Riemannian volume element* $\sqrt{g(x)}dx$.

Asymptotic ansatz

Heat kernel in Euclidean space $\mathbb{R}^n$,

$$U(t;x,x') = (4\pi t)^{-n/2} \exp\left(-\frac{|x-x'|^2}{4t}\right) .$$

Our *main idea* is now to exhibit an *asymptotic factor* that reproduces the initial data on the curved manifold

$$\boxed{\begin{aligned} U(t;x,x') &= (4\pi t)^{-n/2}\mathcal{P}(x,x')\Delta^{1/2}(x,x') \\ &\quad \times \exp\left(-\frac{\sigma(x,x')}{2t}\right)\Omega(t;x,x') , \end{aligned}}$$

Transport equation

Here $\sigma(x,x')$ is the world function, $\Delta(x,x')$ is the Van Vleck determinant and $\mathcal{P}(x,x')$ is the function defined by

$$\sigma^i(\nabla_i + \mathcal{A}_i)\mathcal{P} = 0\,, \qquad [\mathcal{P}] = 1$$

Transport equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{t}D + \hat{L}\right)\Omega(t;x,x') = 0\,,$$

where

$$D = \sigma^i\nabla_i\,, \qquad \hat{L} = \mathcal{P}^{-1}\Delta^{-1/2}L\Delta^{1/2}\mathcal{P}\,,$$

Initial condition

$$\Omega(0;x,x') = 1\,.$$

Properties of transport kernel $\Omega(t;x,x')$

Assume that $Q(x)$ is *bounded below* by a sufficiently large positive parameter m^2, that is, for any x

$$Q(x) \geq m^2 \,.$$

Then the operator L is *positive*.

As $t \to \infty$

$$\Omega(t;x,x') \sim f(t;x,x')e^{-\lambda t} + \cdots$$

As $t \to 0$

$$\Omega(t;x,x') \sim 1 - a_1(x,x')t + \frac{1}{2}a_2(x,x')t^2 + \cdots$$

Mellin Transform of the Heat Kernel

$$b_q(x,x') = \frac{1}{\Gamma(-q)} \int\limits_0^\infty dt\, t^{-q-1} \Omega(t;x,x')\,,$$

This integral converges for $\mathrm{Re}\, q < 0$.

Integration by parts defines an *entire function* (analytic everywhere) of q

For $\mathrm{Re}\, q < N$ (with arbitrary $N > 0$)

$$b_q(x,x') = \frac{1}{\Gamma(-q+N)} \int\limits_0^\infty dt\, t^{-q-1+N} \left(-\frac{\partial}{\partial t}\right)^N \Omega(t;x,x')\,,$$

Ansatz for transport kernel

Inverse Mellin transform

$$\boxed{\Omega(t;x,x') = \int\limits_{c-i\infty}^{c+i\infty} \frac{dq}{2\pi i}\, t^q\, \Gamma(-q) b_q(x,x')\,,}$$

where c is a negative constant.

Taylor coefficients

$$b_k(x,x') = \left(-\frac{\partial}{\partial t}\right)^k \Omega(t;x,x')\bigg|_{t=0}\,.$$

Minackshisundaram-Pleijel Expansion

Deforming the contour of integration we get

$$\Omega(t;x,x') = \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} b_k(x,x') + R_N(t;x,x')\,,$$

where the *remainder term*

$$R_N(t;x,x') = \int\limits_{c_N-i\infty}^{c_N+i\infty} \frac{dq}{2\pi i}\, t^q \Gamma(-q) b_q(x,x')\,,$$

where $N-1 < c_N < N$.

Here $R_N(t;x,x')$ is of order $O(t^N)$ as $t \to 0$ and is smaller than the last term of the sum in this limit.

Asymptotic expansion as $t \to 0$

$$\boxed{\Omega(t;x,x') \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k(x,x')\,.}$$

Asymptotic expansion converges only in the case when the remainder term $R_N(t;x,x')$ vanishes as $N \to \infty$ in a neighborhood of the point $t = 0$. In this case the transport kernel $\Omega(t;x,x')$ is *analytic* at $t = 0$.

In general, $\Omega(t;x,x')$ is *not analytic* at $t = 0$. That is, for any fixed $t > 0$ the remainder $R_N(t;x,x')$ does not vanish as $N \to \infty$ and the *asymptotic expansion diverges*.

Recurrence Relations for Heat Kernel Coefficients

For $q = 0$ we get *transport equation*

$$D b_0(x, x') = 0$$

with the initial condition

$$b_0(x, x') = 1\,.$$

Functional equation

$$\boxed{\left(1 + \frac{1}{q} D\right) b_q(x, x') = \hat{L} b_{q-1}(x, x')\,.}$$

For positive integer $q = 1, 2, \ldots$ this equation gives a *differential recursive system* for the heat kernel coefficients b_k.

Integration Along Geodesics

Consider a geodesic $x(s)$ connecting the points x' and x (so that $x(0) = x'$ and $x(\tau) = x$). Then

$$\sigma^i = \tau \frac{dx^i}{d\tau}$$

and

$$D = \sigma^i \nabla_i = \tau \frac{d}{d\tau}$$

Solution of the recursion system

$$\boxed{b_k(x, x') = k\tau^{-k} \int_0^\tau ds \; s^{k-1} \hat{L}(s) b_{k-1}(x(s), x') \, .}$$

For example,

$$b_1(x,x') = \tau^{-1}\int_0^\tau ds\; \hat{L}(s)\cdot 1\,.$$

$$b_2(x,x') = 2\tau^{-2}\int_0^\tau ds_2\int_0^{s_2} ds_1\; s_2\hat{L}(s_2)s_2^{-1}\hat{L}(s_1)\cdot 1\,.$$

To be able to use these formulas *one has to solve the equations of geodesics*

Non-recursive Solution of Recurrence Relations

Formal operator solution of the recursive system

$$b_k = \left(1 + \frac{1}{k}D\right)^{-1} \hat{L} \left(1 + \frac{1}{k-1}D\right)^{-1} \hat{L} \cdots (1+D)^{-1} \hat{L} \cdot 1 \,.$$

Covariant Taylor series

$$\begin{aligned} b_k &= \sum_{n=0}^{\infty} |n\rangle\langle n|b_k\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sigma^{i'_1} \cdots \sigma^{i'_n} \left[\nabla_{(i_1} \cdots \nabla_{i_n)} b_k\right](x') \end{aligned}$$

Inverse operator

$$\left(1+\frac{1}{k}D\right)^{-1} = \sum_{n=0}^{\infty}\left(1+\frac{n}{k}\right)^{-1}|n\rangle\langle n|$$

Solution

$$\boxed{\begin{aligned}\langle n|b_k\rangle \;\; = \;\; & \sum_{n_1,\cdots,n_{k-1}\geq 0} N(n,k;n_1,\ldots,n_k) \\ & \times\langle n|\widehat{L}|n_{k-1}\rangle\langle n_{k-1}|\widehat{L}|n_{k-2}\rangle\cdots\langle n_1|\widehat{L}|0\rangle\,,\end{aligned}}$$

where

$$N(n,k;n_1,\ldots,n_k) =$$

$$\frac{k}{(k+n)}\cdot\frac{(k-1)}{(k-1+n_{k-1})}\cdots\frac{2}{(2+n_2)}\cdot\frac{1}{(1+n_1)}\,,$$

Matrix elements of the operator $\widehat{L}$

$$\begin{aligned}\langle m|\widehat{L}|n\rangle &= \langle i_1\cdots i_m|\widehat{L}|j_1\cdots j_n\rangle \\ &= \left[\nabla_{(i_1}\cdots\nabla_{i_m)}\widehat{L}\frac{(-1)^n}{n!}\sigma^{j'_1}\cdots\sigma^{j'_n}\right]_{x=x'} \,.\end{aligned}$$

Taylor coefficients $\langle n|b_k\rangle$ are *symmetric tensors* of type $(0,n)$

$$\langle n|b_k\rangle = b_{(k)i_1\dots i_n}\,,$$

Matrix elements $\langle m|\widehat{L}|n\rangle$ are *symmeric tensors* of type (n,m) which are *symmetric in all upper indices and all lower indices separately*

$$\langle m|\widehat{L}|n\rangle = L^{j_1\cdots j_n}_{i_1\cdots i_m}\,.$$

Of course, the matrix element $\langle m|\widehat{L}|0\rangle$ is just a symmetric tensor of type $(0, m)$

$$\langle m|\widehat{L}|0\rangle = L_{i_1 \dots i_m}$$

Product of matrix elements is

$$\langle m|\widehat{L}|n\rangle\langle n|\widehat{L}|p\rangle = L^{j_1 \dots j_n}_{i_1 \dots i_m} L^{l_1 \dots l_p}_{j_1 \dots j_n} \,,$$

where the contraction over all indices $j_1, \dots j_n$ is understood.

In usual index notation

$$\begin{aligned} b_{(k) i_1 \dots i_n} &= \sum_{n_1, \cdots, n_{k-1} \geq 0} N(n, k; n_1, \dots, n_k) \\ & \quad \times L^{j_1 \dots j_{n_{k-1}}}_{i_1 \dots i_n} L^{l_1 \dots l_{n_{k-2}}}_{j_1 \dots j_{n_{k-1}}} \cdots L^{p_1 \dots p_{n_1}}_{m_1 \dots m_{n_2}} L_{p_1 \dots p_{n_1}} \,. \end{aligned}$$

Summation limits

Matrix elements $\langle m|\widehat{L}|n\rangle$ are non-zero only if $n \le m + 2$.

Therefore, the summation over $n_1, \dots, n_{k-1}$ in is limited from above, that is, $n_1 \ge 0$ and

$$n_i \le n_{i+1} + 2\,, \qquad i = 1, 2, \dots, k-1\,,$$

where $n_k = n$, or

$$0 \le n_1 \le n_2 + 2 \le \cdots \le n_{k-1} + 2(k-2) \le n + 2(k-1)\,,$$

Thus the sum for the coefficient $\langle n|b_k\rangle$ contains only a finite number of terms.

Calculation of Matrix Elements

Let ${\eta^{i'}}_j = \nabla_j \sigma^{i'}$ and ${\gamma^i}_{j'}$ be the inverse of the matrix ${\eta^{i'}}_j$.

Recall

$$\begin{aligned} \zeta &= \log \Delta^{1/2}\,, \\ \zeta_{i'} &= \mathcal{D}_{i'}\zeta\,, \\ \hat{\mathcal{A}}_{i'} &= {\gamma^j}_{i'}\mathcal{P}^{-1}(\nabla_j + \mathcal{A}_j)\mathcal{P}\,. \end{aligned}$$

The idea is to introduce *new differential operators*

$$\mathcal{D}_{i'} = {\gamma^j}_{i'}\nabla_j\,.$$

These operators have a nice property that

$$\boxed{\mathcal{D}_{i'}\sigma^{j'} = {\delta^{j'}}_{i'}\,.}$$

Operator $\widehat{L}$ in terms of $\mathcal{D}_{j'}$

$$\widehat{L} = -X^{i'j'}\mathcal{D}_{i'}\mathcal{D}_{j'} - Y^{i'}\mathcal{D}_{i'} + Z$$

where

$$X^{i'j'} = \eta^{i'}{}_k\eta^{j'k}\,, \qquad Y^{i'} = \mathcal{D}_{j'}X^{i'j'} + 2X^{i'j'}\widehat{\mathcal{A}}_{j'}$$

$$Z = X^{i'j'}(\zeta_{i'}\zeta_{j'} - \widehat{\mathcal{A}}_{i'}\widehat{\mathcal{A}}_{j'}) - \mathcal{D}_{j'}\left[X^{i'j'}\left(\zeta_{i'} + \widehat{\mathcal{A}}_{i'}\right)\right] + Q\,.$$

Matrix elements

$$\boxed{\begin{aligned}&\langle i_1\cdots i_m|\widehat{L}|j_1\cdots j_n\rangle = \frac{(-1)^n}{n!}(-1)^m \\ &\times\left[\mathcal{D}_{(i'_1}\cdots\mathcal{D}_{i'_m)}\left(-X^{i'j'}\mathcal{D}_{i'}\mathcal{D}_{j'} - Y^{i'}\mathcal{D}_{i'} + Z\right)\sigma^{j'_1}\cdots\sigma^{j'_n}\right]_{x=x'}\,.\end{aligned}}$$

Algorithm

Expand the functions $X^{i'j'}$, $Y^{i'}$ and Z in covariant Taylor series

Act with the operators $\mathcal{D}_{i'}$ on products of $\sigma^{j'}$

Take the coincidence limit $x = x'$.

Result is expressed in terms of Taylor coefficients of the functions $X^{i'j'}$, $Y^{i'}$ and Z

Finally, compute Taylor coefficients of the functions $X^{i'j'}$, $Y^{i'}$ and Z in terms of *Taylor coefficients of two-point functions ζ, $\hat{A}_i$ and $\eta^{i'}{}_j$*

Result for matrix elements

For $n > m + 2$ and $n = m + 1$,

$$\langle m|\widehat{L}|n\rangle = 0$$

For $n = m + 2$,

$$\langle i_1 \cdots i_m|\widehat{L}|j_1 \cdots j_{m+2}\rangle = -\delta^{(j_1 \cdots j_m}_{i_1 \cdots i_m} g^{j_{m+1} j_{m+2})} \, ,$$

For $n \leq m$,

$$\boxed{\begin{aligned} \langle i_1 \cdots i_m|\widehat{L}|j_1 \cdots j_n\rangle &= \binom{m}{n} \delta^{j_1 \cdots j_n}_{(i_1 \cdots i_n} Z_{i_{n+1} \cdots i_m)} \\ &+\binom{m}{n-1} \delta^{(j_1 \cdots j_{n-1}}_{(i_1 \cdots i_{n-1}} Y^{j_n)}{}_{i_n \cdots i_m)} - \binom{m}{n-2} \delta^{(j_1 \cdots j_{n-2}}_{(i_1 \cdots i_{n-2}} X^{j_{n-1} j_n)}{}_{i_{n-1} \cdots i_m)} \end{aligned}},$$

Result for Taylor coefficients of $X^{i'j'}$, $Y^{i'}$ and Z

$$X^{ij}{}_{l_1\cdots l_n} = \sum_{k=0}^{n} \binom{n}{k} \eta^{(i}{}_{m(l_1\cdots l_k}\eta^{j)m}{}_{l_{k+1}\cdots l_n)}\,,$$

$$Y^{j}{}_{l_1\cdots l_n} = -X^{ji}{}_{il_1\cdots l_n} + 2\sum_{k=0}^{n} \binom{n}{k} X^{j}{}_{m(l_1\cdots l_k}\mathcal{A}^{m}{}_{l_{k+1}\cdots l_n)}\,,$$

$$\begin{aligned} Z_{l_1\cdots l_n} &= Q_{;l_1\cdots l_n} + \sum_{k=0}^{n}\binom{n}{k}\Big\{X_{ij(l_1\cdots l_k}\left[\zeta^{ij}{}_{l_{k+1}\cdots l_n)} - \mathcal{A}^{ij}{}_{l_{k+1}\cdots l_n)}\right] \\ &\quad + X^{i}{}_{ji(l_1\cdots l_k}\left[\zeta^{j}{}_{l_{k+1}\cdots l_n)} - \mathcal{A}^{j}{}_{l_{k+1}\cdots l_n)}\right]\Big\} \\ &\quad + \sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{n!}{k!m!(n-k-m)!}X_{ij(l_1\cdots l_k} \\ &\quad \times\Big[\mathcal{A}^{i}{}_{l_{k+1}\cdots l_{k+m}}\mathcal{A}^{j}{}_{l_{k+m+1}\cdots l_n)} - \zeta^{i}{}_{l_{k+1}\cdots l_{k+m}}\zeta^{j}{}_{l_{k+m+1}\cdots l_n)}\Big]\,, \end{aligned}$$

Diagrammatic Technique

Graphic method for enumerating the different terms

Matrix elements $\langle m|L|n\rangle$ are presented by some *blocks with m lines coming in from the left and n lines going out to the right*

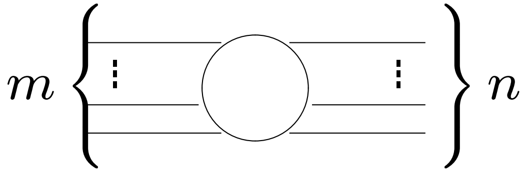


Fig. 1

Product of the matrix elements $\langle m|L|k\rangle\langle k|L|n\rangle$ is represented by *two blocks connected by k intermediate lines*

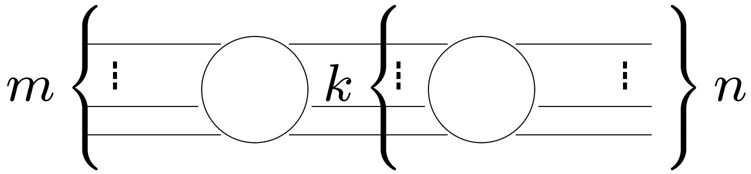


Fig. 2

Each intermediate line represents *a pair of contracted tensor indices*

To obtain the coefficient $\langle n|b_k\rangle$ one should draw, first, *all possible diagrams* which have n lines incoming from the left and which are *constructed from k blocks* connected in all possible ways by any number of intermediate lines.

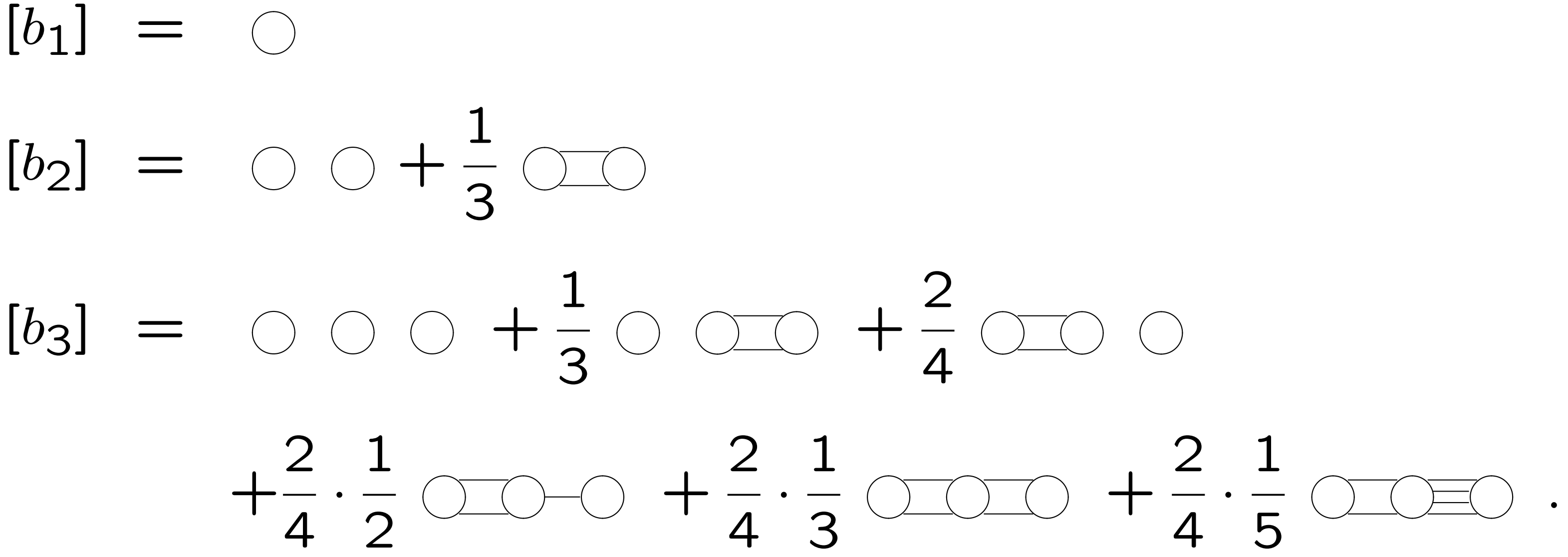
When doing this, one should keep in mind that the number of lines, going out of any block, cannot be greater than the number of lines, coming in, by more than two and by exactly one.

Then one should *sum up all diagrams* with the *weight* determined for each diagram by the number of intermediate lines from the analytic formula.

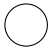
Advantage: The problem is reduced to the *computation of some standard blocks*, which can be computed *once and for all*.

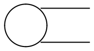
Example: Calculation of low order coefficients

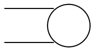
Diagrams for the coincidence limits

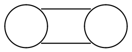


Standard blocks

 $= \langle 0|L|0\rangle = Z_{(0)} = Q - \frac{1}{6}R\,,$

 $= \langle 0|L|j_1 j_2\rangle = -g^{j_1 j_2}$

 $= \langle i_1 i_2|L|0\rangle = Z_{(2)i_1 i_2}$

 $= \langle 0|L|2\rangle\langle 2|L|0\rangle = -g^{ij} Z_{(2)ij}\,,$

Here

$$\begin{aligned} Z_{(2)ij} &= \nabla_{(i}\nabla_{j)}Q - \frac{1}{2}\mathcal{R}_{k(i}\mathcal{R}^{k}{}_{j)} + \frac{1}{2}\nabla_{(i}\nabla_{|k|}\mathcal{R}^{k}{}_{j)} - \frac{3}{20}\nabla_i\nabla_j R \\ &\quad -\frac{1}{20}\Delta R_{ij} + \frac{1}{15}R_{il}R^{l}{}_{j} - \frac{1}{30}R_{iklm}R_{j}{}^{klm} - \frac{1}{30}R_{kl}R^{k}{}_{i}{}^{l}{}_{j}\,. \end{aligned}$$

This immediately gives

$$\boxed{\begin{aligned} [b_1] &= Q - \frac{1}{6}R\,, \\ [b_2] &= \left(Q - \frac{1}{6}R\right)^2 - \frac{1}{3}\Delta Q + \frac{1}{6}\mathcal{R}_{ij}\mathcal{R}^{ij} + \frac{1}{15}\Delta R \\ &\quad -\frac{1}{90}R_{ij}R^{ij} + \frac{1}{90}R_{ijkl}R^{ijkl}\,. \end{aligned}}$$

Summary

Leading asymptotics as $t \to 0$ of the heat kernel is described by the two-point functions $\sigma(x, x')$, $\Delta(x, x')$ and $\mathcal{P}(x, x')$

There is an *asymptotic expansion* as $t \to 0$ in positive powers of t with two-point coefficients $b_k(x, x')$

The heat kernel coefficients can be computed in form of a *covariant Taylor series*

There are *explicit formulas for all Taylor coefficients* $\langle n|b_k\rangle$ of the heat kernel coefficients b_k in terms of R_{ijkl}, $\mathcal{R}_{ij}$ and Q and their covariant derivatives

One does not have to compute them every time. They are given by *explicit formulas*.

Heat Kernel Coefficients for Constant Curvature

Constant curvature manifolds (*spheres S^n* and *hyperbolic spaces H^n*)

$$R^i{}_{jkl} = -\rho^2(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk})\,,$$

Heat kernel for scalar Laplacian $L = -g^{ij}\nabla_i\nabla_j$ as well as the heat kernel coefficients *depend only on the geodesic distance*

$$r = \sqrt{2\sigma}$$

Van Vleck determinant

$$\boxed{\Delta^{1/2}(r) = \left(\frac{\sinh(\rho r)}{\rho r}\right)^{-(n-1)/2}\,.}$$

Differential operators

$$Df(r) = \sigma^i \nabla_i f(r) = r\frac{\partial}{\partial r} f(r)$$

Laplacian

$$Lf(r) = -\left[\partial_r^2 + (n-1)\rho \coth(\rho r)\partial_r\right] f(r)\,.$$

Recursion relations

$$\left(1 + \frac{1}{k} r\partial_r\right) b_k = \Delta^{-1/2} L \Delta^{1/2} b_{k-1}\,,$$

These relations can be easily integrated to get

$$\boxed{b_k(r) = k\frac{1}{r^k}\int\limits_0^r dr'\; r'^{k-1}\Delta^{-1/2}(r') L_{r'} \Delta^{1/2}(r') b_{k-1}(r')\,,}$$

Closed form for coefficient b_1

By using $b_0 = 1$ we have

$$b_1(r) = \frac{1}{r} \int_0^r dr' \, \Delta^{-1/2}(r') L_{r'} \Delta^{1/2}(r') \cdot 1 \,.$$

We remind that

$$\Delta^{-1/2} L \Delta^{1/2} = \frac{(n-1)}{4} \rho^2 \left\{ (n-3) \left[\coth^2(\rho r) - \frac{1}{\rho^2 r^2} \right] + 2 \right\} .$$

This integral can be computed exactly

$$\boxed{b_1 = \frac{(n-1)}{4} \rho^2 \left\{ n - 1 - \frac{(n-3)}{\rho^2 r^2} \left[\rho r \coth(\rho r) - 1 \right] \right\} .}$$

Coefficient b_2

$$\boxed{b_2 = -\frac{(n-1)}{2}\frac{\rho}{r^2}\int\limits_0^{\rho r} dx\; xB(x)}$$

where

$$\begin{aligned} B(x) \;&=\; \left[\partial_x + \frac{(n-1)}{2}f(x) + \frac{(n-1)}{x}\right]\left[\partial_x - \frac{(n-1)}{2}f(x)\right] \\ &\quad \times \left[n-1-\frac{(n-3)}{x}f(x)\right] . \end{aligned}$$

and

$$f(x) = \coth x - \frac{1}{x}$$

Heat kernel on hyperbolic spaces

$$\boxed{\begin{aligned} U(t;x,x') &= (4\pi t)^{-n/2}\left(\frac{\sinh(\rho r)}{\rho r}\right)^{-(n-1)/2}\exp\left(-\frac{r^2}{4t}\right) \\ &\times\left\{1-\frac{(n-1)}{4}\rho^2\left[n-1-\frac{(n-3)}{\rho^2 r^2}\left[\rho r\coth(\rho r)-1\right]\right]t+O(t^2)\right\} \end{aligned}}$$

For $n = 2$ this takes the form

$$\boxed{\begin{aligned} U(t;x,x') &= \frac{1}{4\pi t}\sqrt{\frac{\rho r}{\sinh(\rho r)}}\exp\left(-\frac{r^2}{4t}\right) \\ &\quad\times\left\{1-\frac{t}{4r^2}\left[\rho^2 r^2+\rho r\coth(\rho r)-1\right]+O(t^2)\right\}. \end{aligned}}$$

This is the *basis for Hagan formula in SABR model*

Summary

For manifolds of constant curvature the heat kernel of the Laplacian and the heat kernel coefficients *depend only on the geodesic distance*

One can compute the heat kernel *exactly* in terms of an *integral of special functions*

The heat kernel coefficients b_k can be computed exactly *in terms of definite integrals*.

To use these formulas one has to *find the expression for the geodesic distance* in terms of original coordinates

Heat Kernel Coefficients in One Dimension

Every second-order DO in one dimension has the form

$$\tilde{L} \; = \; -\mu(\partial_x + A)\mu(\partial_x + A) + Q$$

By changing the coordinate *one can always make the metric flat*. Thus, without loss of generality, $\mu(x) = 1$.

Similarity transformation

$$\tilde{L} = e^{-\omega} L e^{\omega}$$

where

$$L = -\partial^2 + Q$$

The heat kernel of the operator $\tilde{L}$ is

$$\tilde{U}(t; x, x') = \exp\{-\omega(x) + \omega(x')\} U(t; x, x')$$

Ordinary DO acting on functions of one real variable

$$\boxed{L = -\partial_x^2 + Q\,,}$$

Asymptotic expansion as $t \to 0$

$$U(t;x,x') \quad \sim \quad (4\pi t)^{-1/2} \exp\left[-\frac{1}{4t}(x-x')^2\right] \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k(x,x').$$

Recursion system

$$b_0(x,x') = 1\,, \qquad \left[1 + \frac{1}{k}(x-x')\frac{\partial}{\partial x}\right] b_k = L a_{k-1}\,.$$

Taylor series

$$\boxed{b_k(x,x') = \sum_{n=0}^{\infty} \frac{1}{n!}(x-x')^n b_k^{(n)}(x')\,.}$$

Solution

$$b_k^{(n)} = \sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{k}{k+n} \cdot \frac{k-1}{k-1+n_{k-1}} \cdots \frac{1}{1+n_1}$$
$$\times L_{nn_{k-1}} L_{n_{k-1}n_{k-2}} \cdots L_{n_1 0} \,.$$

Matrix elements of the operator L

$$L_{mn} = -\delta_{n,(m+2)} + \binom{m}{n} Q^{(m-n)} \,,$$

where $Q^{(n)} = \partial_x^n Q$

Summation limits

$$0 \leq n_1 \leq n_2 + 2 \leq \cdots \leq n_{k-1} + 2(k-1) \leq n + 2(k-1) \,.$$

Explicit results

$$\boxed{b_k^{(n)} = \sum_{d=1}^{k} \sum_{|\mathbf{m}|=n+2k-2d} c(\mathbf{m}) Q^{(m_d)} \ldots Q^{(m_2)} Q^{(m_1)}\,,}$$

where the second sum goes over <u>*multi-indices*</u> $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ with non-negative integer components such that

$$|\mathbf{m}| = m_1 + m_2 + \cdots + m_d = n + 2k - 2d\,,$$

and $c(\mathbf{m})$ is a numerical factor.

Let $|\mathbf{m}|_p$ denote the sum of the first p components of the multi-index $\mathbf{m}$, that is,

$$|\mathbf{m}|_p = m_1 + m_2 + \cdots + m_p\,.$$

Then, after some combinatorial gymnastics, one can obtain

$$c(\mathbf{m}) = \sum_{i_1,\cdots,i_{d-1}} \prod_{p=1}^{d} \binom{i_p}{i_{p-1}} \frac{\binom{|\mathbf{m}|_p - 2i_{p-1} + 2p}{m_p}}{\binom{|\mathbf{m}|_p - i_{p-1} + 2p + 1}{i_p - i_{p-1}}} \,,$$

where the summation goes now over all non-negative $i_1, \ldots, i_{d-1}$ such that

$$0 \equiv i_0 < i_1 < i_2 < \cdots < i_{d-1} < i_d \equiv k,$$

and

$$2i_p \leq |\mathbf{m}|_{p+1} + 2p \,.$$

Let us also list some low-order coefficients

$$
\begin{aligned}
[b_1] &= Q\,, \\
[b_2] &= Q^2 - \frac{1}{3}Q''\,, \\
[b_3] &= Q^3 - \frac{1}{2}\Big(QQ'' + Q''Q + Q'Q'\Big) + \frac{1}{10}Q^{(4)}\,.
\end{aligned}
$$

Summary

In one dimension *one can find all heat kernel coefficients* in terms of Taylor series