Analytic and Geometric Methods for Heat Kernel Applications in Finance

Ivan G. Avramidi

New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

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Lecture 4

Singularly Perturbed PDE

Motivation

Singularly perturbed PDE are equations with a *small parameter at highest derivatives*.

To construct approximate solutions of such equations one has to use *asymptotic methods (semi-classical approximation)*.

Semi-classical approximation is based on a deep physical principle, namely, the *duality* between *waves* and *particles*, between *quantum mechanics* and *classical mechanics*, between *electrodynamics* and *geometric optics*.

Mathematical basis of semi-classical approximation is Fourier transform.

Wave (quantum) aspects of the problem are described by a PDE and the particle (classical) aspects of the problem are described by a system of ODE's.

Semi-classical approximation can be applied to study: i) *singularities of fundamental solutions* of PDE, ii) *asymptotics of Cauchy problems* with fast oscillating initial conditions, iii) *high-frequency (short-wave) asymptotics* of diffraction problems, iv) *asymptotics of eigenvalues* and eigenfunctions of PDO (*spectral asymptotics*), v) *spectral geometry* etc.

Semi-classical Ansatz

Consider a parabolic PDE

$$(\partial_t + L)\varphi = 0$$

where L is an elliptic PDO with variable coefficients acting on functions on $\mathbb{R}^n$

$$L(x, \partial) = -\alpha^{jk}(x)\partial_j\partial_k + \beta^j(x)\partial_j + \gamma(x)\,,$$

Assume that all coefficient functions are smooth and go to some constant values at infinity. We will assume that the matrix α^{ij} is positive definite and bounded for all x.

Singularly Perturbed PDO

By *scaling* $\partial_i \mapsto \varepsilon\partial_i$, with a *small positive parameter* $\varepsilon > 0$ we obtain a *singularly perturbed PDO*

$$L(x, \varepsilon\partial) = -\varepsilon^2 \alpha^{jk}(x)\partial_j\partial_k + \varepsilon\beta^j(x)\partial_j + \gamma(x)\,.$$

More generally, we can also assume that the coefficients $\alpha^{ij}(x, \varepsilon)$, $\beta^j(x, \varepsilon)$, and $\gamma(x, \varepsilon)$ depend on ε in such a way that they have well-defined values at $\varepsilon = 0$.

Singularly perturbed PDE

$$(\varepsilon\partial_t + L(x, \varepsilon\partial))\,\varphi = 0\,.$$

For constant coefficients this equation has the following *"plane wave"* solution

$$\varphi(t,x) = \exp\left[-\frac{1}{\varepsilon}\left(\langle p,x\rangle - tH(p)\right)\right] ,$$

where $\langle p,x\rangle = p_i x^i$ and

$$H(p) = \alpha^{jk} p_j p_k + \beta^j p_j - \gamma \, ,$$

The *main idea of the semi-classical approximation* for non-constant coefficients is to replace this plane wave in the limit $\varepsilon \to 0$ by a *"distorted plane wave"*

$$\exp\left[-\frac{1}{\varepsilon} S(t,x)\right] .$$

where $S(x)$ is some function called the *action*.

Asymptotic Ansatz

More precisely, one looks for the solution of this equation as $\varepsilon \to 0$ in form of the following *asymptotic ansatz*

$$\varphi(t,x) \sim \exp\left[-\frac{1}{\varepsilon}S(t,x)\right] \sum_{k=0}^{\infty} \varepsilon^k a_k(t,x)\,,$$

with some coefficients a_k.

Then one substitutes the ansatz in the differential equation and equates to zero the coefficients at ε^k.

Algorithm of Semi-Classical Approximation

For $k = 0$ get a non-linear *first-order PDE* for the action S, called *Hamilton-Jacobi Equation*.

To solve this equation introduce the corresponding *equations of characteristics* (a system of ODE) called *Hamiltonian system*.

Solve the Hamiltonian system and find the action $S(t, x)$.

For $k \geq 1$ get a system of differential recursion relations for $a_k(x)$, called the *transport equations*

Find as many coefficients $a_k(x)$ as needed by integrating the transport equations

Hamilton-Jacobi Equation

Commutation formula

$$\exp\left(\frac{1}{\varepsilon}S\right) L(x,\varepsilon\partial) \exp\left(-\frac{1}{\varepsilon}S\right) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2$$

where T_0 is a function,

$$T_0 = -\partial_t S - \alpha^{ij} S_i S_j - \beta^j S_j + \gamma\,,$$

T_1 is a first order PDO

$$T_1 = \partial_t + \left(\beta^i + 2\alpha^{ij} S_j\right)\partial_i + \alpha^{ij} S_{ij}\,,$$

T_2 is a second-order PDO

$$T_2 = -\alpha^{ij}\partial_i\partial_j\,.$$

where

$$S_i = \partial_i S\,, \qquad S_{ij} = \partial_i\partial_j S$$

Hamilton-Jacobi equation

$$\partial_t S + H(x, \partial_x S) = 0\,,$$

where

$$H(x,p) = \alpha^{ij}(x)p_i p_j + \beta^j(x)p_j - \gamma(x)\,,$$

Transport equations

$$\begin{aligned} T_1 a_0 &= 0 \\ T_1 a_{k+1} &= -T_2 a_k\,. \end{aligned}$$

Initial conditions

$$\varphi(t,x)\Big|_{t=0} = \varphi_0(x)\exp\left[-\frac{1}{\varepsilon}S_0(x)\right]$$

The initial conditions for the action

$$S(t,x)\Big|_{t=0} = S_0(x)\,,$$

Hamiltonian System

The integration of a *first-order nonlinear PDE* can be reduced to the integration of a *system of ODE*

$$\frac{dx^k}{dt} = \frac{\partial H(x,p)}{\partial p_k} = 2\alpha^{jk} p_j + \beta^k,$$

$$\frac{dp_k}{dt} = -\frac{\partial H(x,p)}{\partial x^k} = -\partial_k \alpha^{ij}(x) p_i p_j - \partial_k \beta^j(x) p_j + \partial_k \gamma(x)\,,$$

with the initial conditions

$$x^i\Big|_{t=0} = x_0^i\,, \qquad p_i\Big|_{t=0} = \frac{\partial S_0(x)}{\partial x^i}\Big|_{x=x_0},$$

Example

For the parabolic PDE

$$(\partial_t - \delta^{ij}\partial_i\partial_j + \gamma(x))\varphi = 0$$

the Hamilton-Jacobi equation is

$$\partial_t S + \delta^{ij}(\partial_i S)(\partial_j S) - \gamma(x) = 0\,.$$

Hamiltonian system

$$\frac{dx^i}{dt} \;=\; 2p^i, \qquad \frac{dp_i}{dt} = -\frac{\partial\gamma(x)}{\partial x^i}$$

describes a particle in a potential $\gamma(x)$

Focal Points and Caustics

A solution of Hamiltonian system defines a *local diffeomorphism* $x_0 \mapsto x(\tau)$ with the *Jacobian*

$$J(\tau) = \text{det} \left(\frac{\partial x^i(\tau)}{\partial x_0^j} \right) .$$

Classical trajectories in the coordinate space do intersect, touch, collect at a single point etc., forming so-called *caustics*.

Focal points are the points along the trajectory $x(\tau)$ such that the Jacobian $J(\tau)$ vanishes.

The *caustics* are sets of focal points.

Semi-classical approximation breaks down at caustics.

Action

Let $x(\tau)$, $p(\tau)$ be the solution of Hamiltonian system with the boundary conditions

$$x(0) = x_0 \qquad x(t) = x\,.$$

The solution depends on t, x and x_0 as parameters.

Along phase trajectories the action varies according to

$$dS = \langle p, dx\rangle - H(x,p)\,.$$

In particular,

$$\frac{\partial S(t,x)}{\partial x^i} = p_i(t)$$

Therefore, the solution of Hamilton-Jacobi equation is

$$S(t,x) \;=\; S_0(x_0) + \int_0^t d\tau \left\{ \left\langle p(\tau), \frac{dx(\tau)}{d\tau} \right\rangle - H(x(\tau), p(\tau)) \right\},$$

where the integral is taken along the phase trajectories.

The *point x_0 should be considered a function of t and x* so that $x(t, x_0) = x$. That is, x_0 is a point such that the trajectory that starts at x_0 at the time $\tau = 0$ reaches the point x at the time $\tau = t$.

Transport Equations

Leading order

$$\left\{\partial_t + \left(2\alpha^{jk}S_j + \beta^k\right)\partial_k + \alpha^{jk}S_{kj}\right\} a_0 = 0\,,$$

Along the phase trajectory $x = x(\tau)$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx^k}{dt}\frac{\partial}{\partial x^k} = \partial_t + \left(2\alpha^{jk}S_j + \beta^k\right)\partial_k\,.$$

Transport equation (first-order ODE along the phase trajectory)

$$\left(\frac{d}{dt} + \alpha^{jk}S_{kj}\right) a_0 = 0\,.$$

Lioville Theorem

Let $x^i(t)$ be the solution of a system of ODE

$$\frac{dx^i}{dt} = F^i(x)\,.$$

with the initial conditions

$$x(0) = x_0\,.$$

Then the Jacobian

$$J(t) = \det \left(\frac{\partial x^i(t)}{\partial x_0^j}\right)\,.$$

satisfies the differential equation

$$\frac{d}{dt}J(t) = \frac{\partial F^i(x(t))}{\partial x^i}J(t)\,.$$

Therefore, Jacobian $J = J(t)$ of our Hamiltonian system satisfies the transport equation

$$\left\{\frac{d}{dt} - \alpha^{ij}S_{ij} - M\right\} J^{1/2} = 0\,,$$

where

$$M = S_k \partial_j \alpha^{jk} + \frac{1}{2}\partial_j \beta^j\,.$$

By using this equation the transport equation for a_0 can be written now in the form

$$\left[\frac{1}{\sqrt{J}}\frac{d}{dt}\sqrt{J} - M\right] a_0 = 0\,.$$

This equation can be now integrated along the trajectories

$$a_0(t,x) = \varphi_0(x)\left(\frac{J(0)}{J(t)}\right)^{1/2} \exp\left\{\int_0^t d\tau\ M(x(\tau))\right\} ,$$

Thus, finally, we obtain the *leading asymptotics*

$$\varphi(t,x) = \varphi_0(x)\left(\frac{J(0)}{J(t)}\right)^{1/2} \exp\left\{-\frac{1}{\varepsilon}S_0(x_0)\right\}$$

$$\times \exp\left\{-\frac{1}{\varepsilon}\int_0^t d\tau\ \left[\left\langle p(\tau), \frac{dx(\tau)}{d\tau}\right\rangle - H(x(\tau),p(\tau))\right]\right\}$$

$$\times \exp\left\{\int_0^t d\tau\ M(x(\tau))\right\} .$$

Heat Kernel Asymptotics

Singularly perturbed heat equation

$$[\varepsilon\partial_t + L(x, \varepsilon\partial_x)]\, U(t; x, x') = 0\,,$$

with the initial condition

$$U(0; x, x') = \delta(x - x')\,.$$

Asymptotic ansatz

$$U(t; x, x') \sim \exp\left[-\frac{1}{\varepsilon} S(t; x, x')\right] \sum_{k=0}^{\infty} \varepsilon^k b_k(t; x, x')\,.$$

Leading asymptotics

$$U(t; x, x') \sim \exp\left[-\frac{1}{\varepsilon} S(t; x, x')\right] b_0(t; x, x')\,.$$

Initial conditions

As $t \to 0$

$$S(t; x, x') \sim \frac{1}{4t}\Phi(x, x'),$$

$$b_0(t; x, x') \sim (4\pi t)^{-n/2}\Big(\det\left[\partial_i \partial_{j'}\Phi(x, x')\right]\Big)^{1/2},$$

Non-degenerate Hessian (at least for x close to x')

$$\det\left[\partial_i \partial_{j'}\Phi(x, x')\right] \neq 0,$$

If the operator L is *self-adjoint* then the function S and all coefficients b_k are *symmetric* in x and x'.

Hamiltonian system

Solution of the Hamiltonian system with the *boundary conditions*

$$x(0) = x', \qquad x(t) = x\,.$$

Of course, the solution, $x(\tau)$ and $p(\tau)$, depends on t, x, and x' as parameters.

This boundary value problem has a unique solution, at least when the points x and x' are close to each other.

Action

Let us define

$$S(t;x,x') = \int_0^t d\tau \ \left[p_i(\tau)\frac{dx^i(\tau)}{d\tau} - H(x(\tau),p(\tau))\right]$$

$$= \int_0^t d\tau \ \left[\frac{1}{4}\alpha_{ij}(x(\tau))\frac{dx^i(\tau)}{d\tau}\frac{dx^j(\tau)}{d\tau} - \frac{1}{2}\alpha_{ij}(x(\tau))\beta^j(x(\tau))\frac{dx^i(\tau)}{d\tau}\right.$$

$$\left.+\frac{1}{4}\alpha_{ij}(x(\tau))\beta^i(x(\tau))\beta^j(x(\tau)) + \gamma(x(\tau))\right],$$

where α_{ij} is the matrix inverse to α^{ij} and the integral is taken along the phase trajectory.

Then one can show that

$$\frac{\partial S(t;x,x')}{\partial x^i} = p_i(t)\,, \qquad \frac{\partial S(t;x,x')}{\partial x'^i} = -p_i(0)\,,$$

and that $S(t;x,x')$ satisfies Hamilton-Jacobi equation with respect to both x and x'

$$\partial_t S + H(x,\partial_x S) = 0\,, \qquad -\partial_t S + H(x',\partial_{x'} S) = 0\,.$$

Transport operator

$$T_1 = \frac{d}{dt} + \alpha^{ij} S_{ij}\,,$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx^i}{dt}\frac{\partial}{\partial x^i}\,.$$

Let us define the determinant

$$Z(t; x, x') = \text{det}\left[-\partial_i \partial_{j'} S(t; x, x')\right].$$

By using Liouville theorem we have

$$\left(\frac{d}{dt} + \alpha^{ij} S_{ij} + M\right) Z^{1/2} = 0$$

where

$$M = (\partial_i \alpha^{ij}) S_j + \frac{1}{2}(\partial_i \beta^i).$$

Transport operator

$$T_1 = Z^{1/2} \left(\frac{d}{dt} - M\right) Z^{-1/2}.$$

Leading asymptotics

By integrating the transport equation we get

$$b_0(t;x,x') = (2\pi)^{-n/2} Z^{1/2}(t;x,x') \exp\left\{\int_0^t d\tau M(\tau;t,x,x')\right\}.$$

Finally, the leading asymptotics of the singularly perturbed heat kernel as $\varepsilon \to 0$ has the form

$$U(t;x,x') \sim (2\pi)^{-n/2} Z^{1/2}(t,x,x') \times \exp\left\{-\frac{1}{\varepsilon}S(t;x,x') + \int_0^t d\tau M(\tau;t,x,x')\right\}.$$

Higher-order coefficients

$$b_k(t;x,x') = -b_0(t;x,x') \int_0^t d\tau \; b_0^{-1}(\tau;x(\tau),x') T_2 b_k(\tau;x(\tau),x')$$

Modified Singular Perturbation

Singular perturbation theory can be modified according to the dependence of the coefficients on the small parameter ε.

For example, if β^i and $\gamma(x)$ have an extra factor of ε, then the Hamiltonian is simply

$$H(x,p) = \alpha^{ij}(x) p_i p_j \,,$$

Hamilton-Jacobi equation and the Hamiltonian system read

$$\partial_t S + \alpha^{ij}(x)(\partial_i S)(\partial_j S) = 0$$

$$\frac{dx^k}{dt} = 2\alpha^{jk} p_j, \qquad \frac{dp_k}{dt} = -\partial_k \alpha^{ij}(x) p_i p_j$$

This is nothing but the *equations of geodesics* for the metric $g^{ij} = \alpha^{ij}$.

In this case the action is determined by the world function

$$S(t; x, x') = \frac{1}{2t}\sigma(x, x')$$

The determinant $Z(t; x, x')$ is then

$$Z(t; x, x') = (2t)^{-n} g^{1/2}(x) g^{1/2}(x') \Delta(x, x')$$

Leading asymptotics of the heat kernel

$$U(t; x, x') \sim (4\pi t)^{-n/2} g^{1/4}(x) g^{1/4}(x') \Delta^{1/2}(x, x') \times \exp\left\{-\frac{1}{2\varepsilon t}\sigma(x, x')\right\} .$$

Summary

The method of semi-classical approximation consists of three main ingredients:

Hamilton-Jacobi equation

Hamiltonian system

Transport equations

For this method to work *one has to solve the Hamiltonian system*.

Of course, one can easily solve them if they are *linear*, which is the case when the Hamiltonian $H(x,p)$ is *quadratic function* of x and p, that is, when α^{ij} is constant, $\beta^i(x)$ is linear and $\gamma(x)$ is quadratic.

If Hamiltonian equations are non-linear, it is usually very difficult to solve them exactly. It can be done only in rare cases of highly symmetric systems, called *integrable systems*.

There is a list of integrable systems in low dimensions. They are usually related to some *deep underlying algebraic structure* (symmetry groups).