Analytic and Geometric Methods for Heat Kernel Applications in Finance

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Lecture 3

Introduction to Differential Geometry

Manifolds

A *manifold* M of dimension n is a topological space that can be *locally* continuously deformed everywhere to the Euclidean space $\mathbb{R}^n$.

An *atlas* is a family of overlapping *charts* which assign *local coordinates* $x = (x^1, \ldots, x^n)$ to a point p in M.

Different local coordinates are related by smooth one-to-one and onto *transition functions*

A *diffeomorphism* is a *bijective smooth map* of manifolds of the same dimension

A *submanifold* S of a manifold M is a subset of M that is itself a manifold.

A submanifold of M of dimension $(\dim M - 1)$ is called a *hypersurface*.

Every n-dimensional manifold M can be *embedded* (realized as a smooth submanifold) in $\mathbb{R}^{2n+1}$.

Properties of manifolds that do not depend on its embedding in a Euclidean space are called *intrinsic*.

Vector Fields

Tangent vector to a curve $x^i = x^i(t)$ is

$$(\dot{x}^i) = \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right) ,$$

Tangent space T_pM is the vector space of all tangent vectors to M at p.

A *vector field* on a manifold M is a smooth assignment of a tangent vector to each point p

Flow $x \mapsto x(t)$

$$\begin{aligned} \frac{dx^j}{dt} &= v^j(x(t)) , \\ x^j(0) &= x^j . \end{aligned}$$

Covector Fields

Cotangent vector (*covector*) to a function f is

$$(\partial_i f) = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)$$

The *cotangent space* $T_p^* M$ is the vector space of all covectors on M at p.

A *covector field* (*one-form*) on a manifold M is a smooth assignment of a covector to each point p in M.

Natural pairing of covectors and vectors

$$\langle \alpha, v \rangle = \sum_{i=1}^{n} \alpha_i v^i \, .$$

Riemannian Metric

Inner product

$$(v, w) = \sum_{i,j=1}^{n} g_{ij} v^i w^j \,, \qquad (\alpha, \beta) = \sum_{i,j=1}^{n} g^{ij} \alpha_i \beta_j \,.$$

Metric $\quad g_{ij}, \qquad (g^{ij}) = (g_{ij})^{-1}$

A *Riemannian metric* on a manifold M is a smooth assignment of a positive definite inner product in each tangent space T_pM

Distance

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)\, dx^i\, dx^j \,.$$

Riemannian Volume Element

Determinant of the metric

$$g = \text{det}\, g_{ij}\ .$$

<u>*Riemannian volume element*</u>

$$\sqrt{g(x)}\, dx$$

remains *invariant* under the change of coordinates

Tensor Fields

A <u>*tensor field*</u> of type (p, q) on a manifold M is a map that assigns to each local coordinate system an collection of smooth functions $T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ that transform like the product of p vectors and q covectors

Einstein Summation Convention

An index cannot appear more than twice in an equation

Indices that appear only *once* are called *free indices*.

Free indices must be in the same position in all terms of an equation and are assumed to take all possible values from 1 to n.

Indices that appear twice are called *repeated indices*.

It is assumed that there is a *summation over each repeated pair of indices* from 1 to n.

Repeated indices are *dummy indices*.

Lie Derivative

Lie derivative with respect to a vector field v measures the *rate of change* of geometric objects along the flow generated by v.

Scalars (directional derivative)

$$L_v f = v^i \partial_i f \, .$$

Vectors (*Lie bracket*)

$$(L_v w)^i = [v, w]^i = v^j \partial_j w^i - w^j \partial_j v^i \, .$$

Covectors

$$(L_v \alpha)_i = v^j \partial_j \alpha_i + \alpha_j \partial_i v^j \, .$$

Isometry

Lie derivative of the metric tensor

$$(L_v g)_{ij} = v^k \partial_k g_{ij} + g_{ik} \partial_j v^k + g_{kj} \partial_i v^k \, .$$

If the metric remains invariant under the flow generated by the vector field v, then the vector field v is called a *Killing vector field* and its flow is called an *isometry*.

Killing vector fields satisfy the equation

$$(L_v g)_{ij} = v^k \partial_k g_{ij} + g_{ik} \partial_j v^k + g_{kj} \partial_i v^k = 0$$

A map $F : M \to M$ described by

$$x^i = F^i(x')$$

is called an *isometry* if it preserves the metric structure (the distances and angles), that is,

$$g_{km}(x') = \frac{\partial F^i}{\partial x'^k}\frac{\partial F^j}{\partial x'^m} g_{ij}(x)$$

In this case the distance between two close points x and $x + dx$ does not change

$$ds^2 = g_{km}(x')\, dx'^k dx'^m = g_{ij}(x)\, dx^i dx^j$$

From the geometric point of view two isometric manifolds are considered to be the *same manifold* with two different coordinate systems.

Connection and Curvature

Problem: partial derivatives $\partial_i v^j$ of a vector field v^j do *not* form tensors.

Usual calculus with partial derivatives is *not invariant*. The meaning of PDE depends on local coordinate system

Solution: Develop an *invariant calculus* on manifolds so that the meaning of PDE is the same in all coordinate systems

Need to *generalize partial derivatives* in such a way that they become tensors (covariant derivatives)

Covariant derivatives of vectors and covectors

$$\nabla_j v^i = v^i{}_{;j} = \partial_j v^i + \Gamma^i{}_{kj} v^k \,.$$

$$\nabla_j \alpha_i = \alpha_{i;j} = \partial_j \alpha_i - \Gamma^k{}_{ij} \alpha_k \,.$$

where $\Gamma^i{}_{jk}$ is an *affine connection*

A *affine connection* is called *compatible with the metric* if

$$\nabla_j g_{ik} = \partial_j g_{ik} - \Gamma^m ij g_{mk} - \Gamma^m{}_{kj} g_{im} = 0 \,,$$

and *torsion-free* if

$$\Gamma^i{}_{jk} - \Gamma^i{}_{kj} = 0 \,.$$

Levi-Civita Connection

An affine connection that is torsion-free and compatible with the metric is called *Levi-Civita connection*.

Every Riemannian manifold has a unique Levi-Civita connection (*Christoffel symbols*)

$$\Gamma^{i}{}_{jk} = \frac{1}{2} g^{im} \left(\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk} \right) \, .$$

Laplace-Beltrami operator (on functions)

$$\Delta = g^{ij} \nabla_i \nabla_j = g^{-1/2} \partial_i g^{1/2} g^{ij} \partial_j$$

Riemann Tensor

Riemann curvature tensor

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{jk} + \Gamma^i{}_{mk}\Gamma^m{}_{jl} - \Gamma^i{}_{ml}\Gamma^m{}_{jk} \,.$$

Commutators of covariant derivatives

$$[\nabla_i, \nabla_j] v^k = R^k{}_{lij} v^l$$

$$[\nabla_i, \nabla_j] \alpha_k = -R^l{}_{kij} \alpha_l$$

Ricci tensor

$$R_{ij} = R^k{}_{ikj} \,,$$

Scalar curvature

$$R = g^{ij} R_{ij} = g^{ij} R^k{}_{ikj} \,,$$

Geometry of Two-dimensional Manifolds

Curvature tensor has just one independent component, K, called *Gauss curvature*

$$R^1{}_2{}^1{}_2 = R^2{}_1{}^2{}_1 = R^1{}_1 = R^2{}_2 = \frac{1}{2}R = K\,.$$

Orthogonal coordinate system

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2\,,$$

$$\begin{aligned} K \;=\; & -\frac{1}{2}(g_{11}g_{22})^{-1/2}\Big\{\partial_1\left[(g_{11}g_{22})^{-1/2}\partial_1 g_{22}\right] \\ & +\partial_2\left[(g_{11}g_{22})^{-1/2}\partial_2 g_{11}\right]\Big\}\,. \end{aligned}$$

Conformal Transformation

The map

$$g_{ij}(x) \mapsto \bar{g}_{ij}(x) = e^{2\omega(x)} g_{ij}(x)\,,$$

where $\omega(x)$ is a smooth function, is called *conformal transformation*

Gauss curvature of conformally transformed metric

$$\bar{K} = e^{-2\omega}(K - \Delta\omega)\,,$$

where Δ is Laplace operator and K is Gauss curvature of the metric g_{ij}

Laplacian of conformally transformed metric

$$\bar{\Delta} = e^{-2\omega}\Delta$$

We say that Laplacian is a *conformally covariant operator*

Conformal coordinates

$$ds^2 = e^{2\omega}\left[(dx^1)^2 + (dx^2)^2\right] \,,$$

$$K = -e^{-2\omega}\left(\partial_1^2 + \partial_2^2\right)\omega \,.$$

Constant curvature

$$K = \varkappa \qquad \text{for} \qquad \omega = -\log\frac{(1+\varkappa\rho^2)}{2} \,,$$

where

$$\rho = \sqrt{(x^1)^2 + (x^2)^2} \,.$$

Geodesic polar coordinates

$$ds^2 = dr^2 + a^2(r,\theta) d\theta^2\,,$$

$$K = -\frac{1}{a}\partial_r^2 a\,.$$

Constant curvature

$$K = 1 \qquad \text{for } a = \sin r\,,$$

and

$$K = -1 \qquad \text{for } a = \sinh r\,.$$

Two-point Functions

Local geometry of every manifold can be described by so-called *two-point functions*

Two-point functions are geometric objects that depend on two points, say, x and x'.

We denote tensor components at the point x' by *primed indices*

Two-point functions can have different geometric meaning at the point x' and the point x.

For example, $A^{i'}{}_{jk}$ denotes a vector at x' and a 2-tensor at x

Below, x' is a *fixed point* and we prefer to have two-point functions which are *scalars* at x and some tensors at x'

The derivatives with respect to the coordinates of x' are denoted by primed indices, $\nabla_{i'}$

Coincidence limits of two-point functions

$$[F(x,x')] = \lim_{x \to x'} F(x,x')$$

Parallel Transport and Geodesics

Smooth curve C

$$x^i(\tau)\,, \qquad x(0) = x'\,, \qquad \text{and} \qquad x(t) = x$$

A tensor T is *parallel transported along C* if

$$\dot{x}^i \nabla_i T = 0\,.$$

A curve C such that the tangent vector $\dot{x}$ to C is transported parallel along C is called a *geodesics* and such parameter τ is called an *affine parameter*.

Equations of geodesic

$$\dot{x}^j \nabla_j \dot{x}^i = \ddot{x}^i + \Gamma^i{}_{jk}(x(\tau))\dot{x}^k \dot{x}^j = 0\,.$$

Geodesic Distance

Length of the curve C

$$d_C(x,x') = \int_0^t d\tau \, \sqrt{g_{ij}(x(\tau))\dot{x}^i\dot{x}^j} \; .$$

Geodesics is the *shortest curve* between the points x and x'.

The distance $d(x,x')$ between the points x and x' along the shortest geodesics is called the *geodesic distance*.

Every two sufficiently close points can be connected by a single shortest geodesics.

Parallel Transport Operator

Differential equation

$$\dot{x}^k \nabla_k g^i{}_{j'} = 0\,,$$

with the initial condition

$$[g^i{}_{j'}] = g^i{}_{j'}(x,x')\Big|_{x=x'} = \delta^i{}_j\,.$$

Parallel transport from x' to x

$$(\mathcal{G}v)^i(x,x') = g^i{}_{j'}(x,x')v^{j'}(x')\,,$$

Parallel transport from x to x'

$$(\mathcal{G}^{-1}u)^{i'}(x,x') = g_j{}^{i'}(x,x')u^j(x)$$

World Function

$$\sigma(x,x') = \frac{1}{2} d^2(x,x')$$

Differential equations

$$\sigma = \frac{1}{2} g^{ij}\sigma_i\sigma_j = \frac{1}{2} g^{i'j'}\sigma_{i'}\sigma_{j'}\,,$$

$$\sigma_i = \nabla_i\sigma\,, \qquad \sigma_{i'} = \nabla_{i'}\sigma$$

Initial conditions

$$[\sigma] = [\sigma_i] = [\sigma_{i'}] = 0\,.$$

Tangent vectors at endpoints

$$\dot{x}^i(t) = \frac{1}{t}\sigma^i\,, \qquad \dot{x}^{j'}(0) = -\frac{1}{t}\sigma^{j'}\,,$$

$$\sigma^i = -g^i{}_{j'}\sigma^{j'}\,.$$

Caustics

Geodesics coming out of a fixed point x' can intersect. The set of intersection points forms a hypersurface that is called *caustics*

Second derivatives

$$\xi^i{}_j = \nabla_j \nabla^i \sigma \,, \qquad \eta^{j'}{}_i = \nabla_i \nabla^{j'} \sigma \,, \qquad (\gamma^i{}_{j'}) = (\eta^{j'}{}_i)^{-1}$$

Coincidence limits

$$[\xi^i{}_j] = -[\eta^{i'}{}_j] = -[\gamma^i{}_{j'}] = \delta^i{}_j \,.$$

Change of variables

$$d\sigma^{i'} = \eta^{i'}{}_j \, dx^j \,, \qquad dx^j = \gamma^j{}_{i'} d\sigma^{i'}$$

If the matrix $\gamma^i{}_{j'}$ is singular, then there is a variation of $\sigma^{i'}$ which do not change x^j. In this case the point x lies on a caustics.

Van Vleck Determinant

Van Vleck-Morette determinant

$$\Delta(x,x') = g^{-1/2}(x)\,\mathrm{det}\,\Big(-\sigma_{ij'}(x,x')\Big)\,g^{-1/2}(x')\,.$$

Equation of caustics

$$\Delta^{-1}(x,x') = 0$$

Continuity equation

$$\Delta^{-1}\nabla_i(\sigma^i\Delta) = n$$

or (transport equation)

$$t\frac{d}{dt}\log\Delta = \sigma^i\nabla_i\log\Delta = n - \sigma^{;i}{}_i$$

with the initial condition

$$[\Delta] = 1\,.$$

Deviation of geodesics

Van Vleck determinant represents the *density of geodesics*. It determines the *rate of convergence (or divergence)* of close geodesics coming out of a fixed point. The quantity $(n - \sigma^{;i}{}_{i})$ measures the *rate of change* of $\log \Delta$ along geodesics.

In flat space $\sigma^{;i}{}_{i} = n$. If the curvature is positive, then Δ increases until a caustics is formed where $\Delta \to \infty$.

If the curvature is negative then Δ decreases. There are no caustics.

(For more details, see *Bourgade & Croissant*, arXiv:cs.CE/0511024)

Auxiliary Function $\mathcal{P}$

For a vector field $\mathcal{A}_i$ we define

$$\mathcal{P}(x,x') = \exp\left(-\int\limits_0^t d\tau\ \dot{x}^i(\tau)\mathcal{A}_i(x(\tau))\right)\ ,$$

where the integral is taken along the geodesic

Differential equation

$$\sigma^i(\nabla_i + \mathcal{A}_i)\mathcal{P} = 0\,,$$

with the initial condition

$$[\mathcal{P}] = 1\,.$$

Covariant Expansions on Riemannian Manifolds

Covariant Taylor Series

$$f(x) = \sum_{n=0}^{\infty} |n\rangle\langle n|f\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sigma^{i'_1} \cdots \sigma^{i'_n} \left[\nabla_{(i_1} \cdots \nabla_{i_k)} f\right](x') \,.$$

Taylor basis

$$|0\rangle = 1\,, \qquad |n\rangle = |i'_1 \cdots i'_n\rangle = \frac{(-1)^n}{n!} \sigma^{i'_1} \cdots \sigma^{i'_n} \,.$$

Dual basis

$$\langle m|f\rangle = \langle i_1 \cdots i_m|f\rangle = \left[\nabla_{(i_1} \cdots \nabla_{i_m)} f\right] \,,$$

Duality

$$\langle m|n\rangle = \delta_{mn} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} \,, \qquad \text{where} \qquad \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} = \delta^{i_1}_{(j_1} \cdots \delta^{i_n}_{j_n)} \,.$$

Covariant Fourier Transform

$$\hat{f}(k;x') = \int\limits_{M} dx\; g^{1/2}(x)\Delta(x,x') \exp\left(ik_{j'}\sigma^{j'}(x,x')\right) f(x)\,,$$

Inverse Fourier transform

$$f(x) = \int\limits_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') \exp\left(-ik_{j'}\sigma^{j'}(x,x')\right) \hat{f}(k;x')$$

Delta-function

$$\delta(x,y) = \Delta^{1/2}(x,x')\Delta^{1/2}(y,x')$$

$$\times \int\limits_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') \exp\left\{ik_{j'}\left(\sigma^{j'}(y,x') - \sigma^{j'}(x,x')\right)\right\}\ .$$

Application of covariant Fourier transform

Differential operators

$$\mathcal{D}_{i'} = \gamma^{j}{}_{i'} \nabla_{j} \, ,$$

where $\eta^{i'}{}_{j} = \nabla_{j} \sigma^{i'}$ and $\gamma^{i}{}_{j'}$ be the inverse of the matrix $\eta^{i'}{}_{j}$

These operators *act like in flat space*

$$\mathcal{D}_{i'} \sigma^{j'} = \delta^{j'}{}_{i'}$$

$$\mathcal{D}_{i'} f(x) = \int\limits_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') \exp\Big(-i k_{j'} \sigma^{j'}(x, x')\Big) (-i k_{i'}) \hat{f}(k; x')$$

Elliptic Laplace type operator

$$\begin{aligned} L &= -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q \\ &= -X^{i'j'}\mathcal{D}_{i'}\mathcal{D}_{j'} - Y^{i'}\mathcal{D}_{i'} + Z \end{aligned}$$

where

$$X^{i'j'} = g^{i'j'}(x') + \dots$$

Heat kernel in leading order

$$\begin{aligned} U(t;x,x') &= \text{exp}(-tL)\delta(x,x') = \mathcal{P}(x,x')\Delta^{1/2}(x,x') \\ &\times \int\limits_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x')\,\text{exp}\left(-tg^{i'j'}(x')k_{i'}k_{j'} - ik_{j'}\sigma^{j'}\right) + \dots \\ &= (4\pi t)^{-n/2}\mathcal{P}(x,x')\Delta^{1/2}(x,x')\,\text{exp}\left(-\frac{1}{2t}\sigma(x,x')\right) + \dots \end{aligned}$$

Road Map

We need to compute covariant Taylor series of the following quantities:

Second derivatives of world function

$$\eta^{i'}{}_{j} = \nabla_j \nabla^{i'} \sigma, \qquad \xi^{i}{}_{j} = \nabla^i \nabla_j \sigma, \qquad (\gamma^{i}{}_{j'}) = (\eta^{i'}{}_{j})^{-1},$$

Van Vleck determinant Δ and the function $\zeta = \log \Delta^{1/2}$.

Derivatives of operator of parallel transport and function $\mathcal{P}$

$$\mathcal{G}^{k'}{}_{j'i'} = g_l{}^{k'} \mathcal{D}_{i'} g^l{}_{j'} = \gamma^m{}_{i'} g_l{}^{k'} \nabla_m g^l{}_{j'}$$

$$\hat{\mathcal{A}}_{i'} = \gamma^m{}_{i'} \mathcal{A}_m + \mathcal{P}^{-1} \mathcal{D}_{i'} \mathcal{P} = \gamma^m{}_{i'} \mathcal{P}^{-1} (\nabla_m + \mathcal{A}_m) \mathcal{P}$$

Covariant Taylor Series of Two-Point Functions

Matrices of second derivatives of world function

$$\xi = (\xi^i{}_j) = (\sigma^i{}_j)\,, \qquad \eta = (\eta^{i'}{}_j) = (\sigma^{i'}{}_j)\,, \qquad \gamma = (\gamma^i{}_{j'}) = \eta^{-1}$$

Symmetric derivatives of curvature

$$K^i{}_{jl_1\cdots l_n} = \nabla_{(l_1}\cdots\nabla_{l_{n-2}}R^i{}_{l_{n-1}|j|l_n)}\,.$$

Matrices

$$K_{(n)} = \left(K_{(n)}{}^{i'}{}_{j'}\right) \qquad \text{where} \qquad K_{(n)}{}^{i'}{}_{j'} = K^{i'}{}_{j'l'_1\cdots l'_n}\sigma^{l'_1}\cdots\sigma^{l'_n}\,,$$

Matrix γ

$$\boxed{{\gamma^i}_{j'} = -{g^i}_{j'} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} {g^i}_{k'} {\gamma_{(n)}}^{k'}{}_{j'} \,.}$$

$$\gamma_{(n)} \;=\; \left({\gamma_{(n)}}^{i'}{}_{j'}\right) = \sum_{k=1}^{[\frac{n}{2}]} (-1)^{k+1} \sum_{\substack{n_1,\cdots,n_k \geq 2 \\ n_1+\cdots+n_k=n}} N^{-1}(n_1,\cdots,n_k)$$

$$\times K_{(n_k)} K_{(n_{k-1})} \cdots K_{(n_2)} K_{(n_1)} \,,$$

where $[x]$ denotes the integer part of x, and

$$N(n_1,\cdots,n_k) = \frac{(n+1)}{(n-1)!}(n_1-2)!\cdots(n_k-2)!$$

$$\times n_1(n_1+1)(n_1+n_2)(n_1+n_2+1)\cdots$$

$$\times(n_1+\cdots+n_{k-1})(n_1+\cdots+n_{k-1}+1)\,.$$

First coefficients

$$\begin{aligned}
\gamma^{i}{}_{jk_1k_2} &= \frac{1}{3}R^{i}{}_{(k_1|j|k_2)}\,, \\
\gamma^{i}{}_{jk_1k_2k_3} &= \frac{1}{2}\nabla_{(k_1}R^{i}{}_{k_2|j|k_3)}\,, \\
\gamma^{i}{}_{jk_1k_2k_3k_4} &= \frac{3}{5}\nabla_{(k_1k_2}R^{i}{}_{k_3|j|k_4)} - \frac{1}{5}R^{i}{}_{(k_1|m|k_2}R^{m}{}_{k_3|j|k_4)}\,.
\end{aligned}$$

Matrix η

$$\boxed{{\eta^{i'}}_j = -{g_j}^{i'} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} {g_j}^{k'} {\eta_{(n)}}^{i'}{}_{k'}}$$

where

$$\eta_{(n)} = \left({\eta_{(n)}}^{i'}{}_{j'}\right) = -\sum_{k=1}^{[\frac{n}{2}]} \sum_{\substack{n_1,\cdots,n_k \geq 2 \\ n_1+\cdots+n_k=n}} \frac{n!}{n_1! \cdots n_k!} \gamma_{n_k} \cdots \gamma_{n_1} \,.$$

First coefficients

$$\begin{aligned} {\eta^i}_{jk_1k_2} &= -\frac{1}{3} R^i{}_{(k_1|j|k_2)} \,, \\ {\eta^i}_{jk_1k_2k_3} &= -\frac{1}{2} \nabla_{(k_1} R^i{}_{k_2|j|k_3)} \,, \\ {\eta^i}_{jk_1k_2k_3k_4} &= -\frac{3}{5} \nabla_{(k_1} \nabla_{k_2} R^i{}_{k_3|j|k_4)} - \frac{7}{15} R^i{}_{(k_1|m|k_2} R^m{}_{k_3|j|k_4)} \,. \end{aligned}$$

Van Vleck-Morette determinant

$$\Delta = \exp(2\zeta)\,, \qquad \text{where} \qquad \zeta = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}\zeta_{(n)}$$

with the coefficients

$$\zeta_{(n)} = \sum_{k=1}^{[\frac{n}{2}]} \frac{1}{2k} \sum_{\substack{n_1,\cdots,n_k\ge 2\\ n_1+\cdots+n_k=n}} \frac{n!}{n_1!\cdots n_k!}\,\mathrm{tr}\,\left(\gamma_{(n_1)}\cdots\gamma_{(n_k)}\right)\,.$$

First coefficients

$$\begin{aligned}
\zeta_{k_1k_2} &= \frac{1}{6}R_{k_1k_2},\\
\zeta_{k_1k_2k_3} &= \frac{1}{4}\nabla_{(k_1}R_{k_2k_3)},\\
\zeta_{k_1k_2k_3k_4} &= \frac{3}{10}\nabla_{(k_1}\nabla_{k_2}R_{k_3k_4)} + \frac{1}{15}R^i{}_{(k_1|m|k_2}R^m{}_{k_3|i|k_4)}\,.
\end{aligned}$$

$$\Delta^{1/2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \Delta^{1/2}_{(n)}$$

First coefficients

$$\begin{aligned}
\Delta^{1/2}_{k_1 k_2} &= \frac{1}{6} R_{k_1 k_2}, \\
\Delta^{1/2}_{k_1 k_2 k_3} &= \frac{1}{4} \nabla_{(k_1} R_{k_2 k_3)}, \\
\Delta^{1/2}_{k_1 k_2 k_3 k_4} &= \frac{3}{10} \nabla_{(k_1} \nabla_{k_2} R_{k_3 k_4)} + \frac{1}{15} R^{i}{}_{(k_1|m|k_2} R^{m}{}_{k_3|i|k_4)} \\
&\quad + \frac{1}{12} R_{(k_1 k_2} R_{k_3 k_4)} \,.
\end{aligned}$$

Operator of parallel transport

Derivative of operator of parallel transport

$$\mathcal{G}^{k'}{}_{j'i'} = \gamma^{m}{}_{i'} g_{l}{}^{k'} \nabla_m g^{l}{}_{j'}$$

Taylor series

$$\boxed{\mathcal{G}^{i'}{}_{j'm'} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathcal{G}^{i'}{}_{j'm'k'_1\cdots k'_n} \sigma^{k'_1} \cdots \sigma^{k'_n}\,,}$$

where

$$\mathcal{G}^{i}{}_{jmk_1\cdots k_n} = \frac{n}{n+1} \left\{ \mathcal{B}^{i}{}_{jmk_1\cdots k_n} - \sum_{p=1}^{n-2} \binom{n-1}{p-1} \mathcal{B}^{i}{}_{jl(k_1\cdots k_p} \gamma^{l}{}_{|m|k_{p+1}\cdots k_n)} \right\}$$

$$\mathcal{B}^{i}{}_{jmk_1\cdots k_n} = \nabla_{(k_1} \cdots \nabla_{k_{n-1}} R^{i}{}_{|jm|k_n)}\,.$$

First coefficients

$$
\begin{aligned}
\mathcal{G}^i{}_{jmk_1} &= \frac{1}{2} R^i{}_{jmk_1}\,, \\
\mathcal{G}^i{}_{jmk_1k_2} &= \frac{2}{3} \nabla_{(k_1} R^i{}_{j|m|k_2)}\,, \\
\mathcal{G}^i{}_{jmk_1k_2k_3} &= \frac{3}{4} \nabla_{(k_1} \nabla_{k_2} R^i{}_{j|m|k_3)} - \frac{1}{4} R^i{}_{jl(k_1} R^l{}_{k_2|m|k_3)}\,.
\end{aligned}
$$

Auxiliary Function $\mathcal{P}$

Derivative

$$\hat{\mathcal{A}}_{i'} = \gamma^{m}{}_{i'}\mathcal{P}^{-1}(\nabla_m + \mathcal{A}_m)\mathcal{P}$$

Taylor series

$$\boxed{\hat{\mathcal{A}}_{j'} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\mathcal{A}_{j'i'_1\cdots i'_n}\sigma^{i'_1}\cdots\sigma^{i'_n}\,,}$$

where

$$\mathcal{A}_{ji_1\cdots i_n} = \frac{n}{n+1}\left\{\mathcal{R}_{ji_1\cdots i_n} - \sum_{k=1}^{n-2}\binom{n-1}{k-1}\mathcal{R}_{l(i_1\cdots i_k}\gamma^{l}{}_{|j|i_{k+1}\cdots i_n)}\right\}\,.$$

$$\mathcal{R}^{k}{}_{i_1\cdots i_n} = \nabla_{(i_1}\cdots\nabla_{i_{n-1}}\mathcal{R}^{k}{}_{i_n)}\,,$$

First coefficients

$$\begin{aligned}
\mathcal{A}_{mk_1} &= \frac{1}{2}\mathcal{R}^{i}{}_{mk_1}\,, \\
\mathcal{A}_{mk_1k_2} &= \frac{2}{3}\nabla_{(k_1}\mathcal{R}_{|m|k_2)}\,, \\
\mathcal{A}_{mk_1k_2k_3} &= \frac{3}{4}\nabla_{(k_1}\nabla_{k_2}\mathcal{R}_{|m|k_3)} - \frac{1}{4}\mathcal{R}_{l(k_1}R^{l}{}_{k_2|m|k_3)}\,.
\end{aligned}$$

Two-point Functions in Symmetric Spaces

Symmetric spaces

$$\nabla_m R^i{}_{jkl} = 0\,.$$

Closed formulas (here $\bar\eta^{i'}j' = g^k{}_{j'}\eta^{i'}{}_k$, $\bar\gamma^{i'}j' = g_k{}^{i'}\gamma^k{}_{j'}$)

$$\bar\gamma = -\frac{\sin\sqrt{\bar K}}{\sqrt{\bar K}}\,, \qquad \bar\eta = -\frac{\sqrt{\bar K}}{\sin\sqrt{\bar K}}\,,$$

$$\Delta = \det\frac{\sqrt{\bar K}}{\sin\sqrt{\bar K}}\,.$$

where the matrix $\bar K = (\bar K^{i'}{}_{j'})$ is

$$\bar K^{i'}{}_{j'} = R^{i'}{}_{k'j'l'}\sigma^{k'}\sigma^{l'}\,.$$

Similarly,

$$\mathcal{G}^{i'}{}_{j'm'} = -R^{i'}{}_{j'l'p'}\sigma^{p'} \left(\frac{1 - \cos\sqrt{\bar{K}}}{\bar{K}} \right)^{l'}{}_{m'} .$$

$$\hat{\mathcal{A}}_{m'} = -\mathcal{R}_{l'p'}\sigma^{p'} \left(\frac{1 - \cos\sqrt{\bar{K}}}{\bar{K}} \right)^{l'}{}_{m'} .$$

Maximally Symmetric Spaces of Constant Curvature

Curvature

$$R^i{}_{jkl} = \Lambda(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk})\,,$$

If $\Lambda > 0$ then this symmetric space is a *n-sphere* S^n (compact),
if $\Lambda < 0$, it is the *hyperbolic space* H^n (non-compact).

In this case

$$\bar{K}^{i'}{}_{j'} = 2\Lambda\sigma\Pi_\perp{}^{i'}{}_{j'}\,,$$

where $\Pi_\perp$ is an orthogonal projection

$$\Pi_\perp{}^{i'}{}_{j'} = \delta^{i'}_{j'} - \frac{\sigma^{i'}\sigma_{j'}}{2\sigma}\,.$$

Projection properties

$$\Pi_\perp{}^2 = \Pi_\perp, \qquad \Pi_\perp{}^{j'}{}_{i'}\sigma^{i'} = \sigma_{j'}\Pi_\perp{}^{j'}{}_{i'} = 0\,.$$

Function of the matrix $\bar{K}$

$$f(\bar{K}) = f(0)(I - \Pi_\perp) + f(2\Lambda\sigma)\Pi_\perp\,.$$

Closed explicit form

$$\eta^{i'}{}_l \;=\; \nabla_l\nabla^{i'}\sigma = -g^{i'}{}_l\Phi^{-1} - \frac{\sigma^{i'}\sigma_l}{2\sigma}(\Phi^{-1} - 1)\,,$$

where

$$\Phi = \frac{\sinh\sqrt{-2\Lambda\sigma}}{\sqrt{-2\Lambda\sigma}}\,,$$

Van Vleck determinant

$$\Delta = \Phi^{-n+1}\,.$$

and

$$\mathcal{G}^{i'}{}_{j'm'} = -\Psi R^{i'}{}_{j'm'k'}\sigma^{k'}\,, \qquad \hat{\mathcal{A}}_{m'} = -\Psi \mathcal{R}_{m'k'}\sigma^{k'}\,,$$

where

$$\Psi = \frac{1 - \cosh\sqrt{-2\Lambda\sigma}}{2\Lambda\sigma}\,.$$

By using

$$\xi = I + (D\gamma)\eta$$

we compute the matrix ξ

$$\xi^i{}_k \;=\; \nabla_k \nabla^i \sigma = \delta^i{}_j (1 + F) - \frac{\sigma^i \sigma_k}{2\sigma} F \,,$$

where

$$F = \Phi^{-1} D \Phi$$

When acting on scalar functions that depend only on σ

$$D f(\sigma) = 2\sigma \frac{\partial}{\partial \sigma} f(\sigma)$$

Therefore

$$F = \sqrt{-2\Lambda\sigma}\, \coth\left(\sqrt{-2\Lambda\sigma}\right) - 1 \,.$$

Laplacian of the world function

$$\sigma^{;i}{}_{i} = \nabla_i \nabla^i \sigma = n + (n-1)F$$

When acting on a scalar functions that depend only on σ

$$\nabla_i \nabla^i f(\sigma) = \left(2\sigma \frac{\partial^2}{\partial \sigma^2} + \sigma^{;i}{}_{i} \frac{\partial}{\partial \sigma} \right) f(\sigma)$$

Laplacian of $\Delta^{1/2}$

$$\Delta^{-\frac{1}{2}} \nabla^i \nabla_i \Delta^{\frac{1}{2}} = \frac{(n-1)}{2} \Lambda \left[\frac{(n-3)}{2} \left(\coth^2(\sqrt{-2\Lambda\sigma}) + \frac{1}{2\Lambda\sigma} \right) + 1 \right] .$$

For $n = 3$, $\Delta^{1/2}$ is an eigenfunction of the Laplacian with the eigenvalue $[(n-1)/2]\Lambda$.

For $n = 2$ this will be *directly applied to the SABR model* to derive the *Hagan formula*

Geometric Interpretation of Elliptic PDO

Elliptic PDO

$$L = -\alpha^{ij}(x)\partial_i\partial_j + \beta^j(x)\partial_j + \gamma(x)\,.$$

Since the matrix α^{ij} is positive it naturally defines a Riemannian metric. Let

$$\begin{aligned} g^{ij} &= \alpha^{ij} \\ \mathcal{A}_i &= -\frac{1}{2}g_{ij}\beta^j - \frac{1}{2}g_{ij}g^{-1/2}\partial_k(g^{1/2}g^{jk}) \\ Q &= \gamma + g^{ij}\mathcal{A}_i\mathcal{A}_j + g^{-1/2}\partial_i(g^{1/2}g^{ij}\mathcal{A}_j)\,. \end{aligned}$$

Geometric form of the operator

$$\begin{aligned} L &= -g^{ij}(\nabla_i + \mathcal{A}_i)(\nabla_j + \mathcal{A}_j) + Q \\ &= -g^{-1/2}(\partial_i + \mathcal{A}_i) g^{1/2} g^{ij}(\partial_j + \mathcal{A}_j) + Q \,. \end{aligned}$$

If the vector $\mathcal{A}_i$ is non-zero, then the operator L is not self-adjoint.

If $\qquad \mathcal{R}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i = 0$

then $\qquad \mathcal{A}_i = \partial_i \omega \qquad$ for some ω

and the operator L is similar to a self-adjoint operator

$$L = e^{-\omega} \left(-g^{ij} \nabla_i \nabla_j + Q \right) e^{\omega}$$

Road Map

A PDO defines a metric g_{ij} and a vector $\mathcal{A}_i$

The metric defines geodesics, world function σ, Van Vleck determinant Δ, the parallel transport operator $g^i{}_{j'}$, and their derivatives, $\sigma^{i'}$, $\eta^{i'}{}_j$, and $\gamma^i{}_{j'}$

The vector $\mathcal{A}_i$ defines the function $\mathcal{P}$ and its derivative $\hat{A}_{j'}$

We can compute all these two-point functions in form of a covariant Taylor series

The *heat kernel coefficients are expressed directly in terms of the Taylor coefficients of the two-point functions*