

# **Analytic and Geometric Methods for Heat Kernel Applications in Finance**

**Ivan G. Avramidi**

*New Mexico Institute of Mining and Technology  
Socorro, NM 87801, USA*

March 15-16, 2007

**Lecture 1-2**

# Stochastic Models in Mathematical Finance

## Black-Scholes Model

Prices  $S_i(t)$  of stocks at time  $t$  satisfy *SDE*

$$dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t) dW_i(t)$$

where  $W_i(t)$  are *Wiener processes* with

*correlation matrix*  $\rho_{ij}$ ,

$\mu_i$  are *drifts* and

$\sigma_i$  are *volatilities*

Price  $V(t, S_1, \dots, S_n)$  of an option at time  $t$  satisfies *PDE*

$$\left(\frac{\partial}{\partial t} + L\right) V = 0,$$

$$L = \frac{1}{2} \sum_{i,j=1}^n C_{ij} S_i S_j \frac{\partial^2}{\partial S_i \partial S_j} + \sum_{i=1}^n (r - q_i) S_i \frac{\partial}{\partial S_i} - r$$

where  $r$  is the *risk-free interest rate*,

$q_i$  are the *dividend rates* and

$$C_{ij} = \rho_{ij} \sigma_i \sigma_j.$$

*Expiration condition at a strike time  $T$*

$$V(T, S_1, \dots, S_n) = P(S_1, \dots, S_n)$$

## Stochastic Volatility Models

Stock price  $S(t)$  and its variance  $v(t)$  satisfy SDE

$$dS(t) = \mu(t) S(t) dt + \sqrt{v(t)} S(t) dW_1(t),$$

$$dv(t) = \alpha(t, S, v) dt + \eta \beta(t, S, v) \sqrt{v(t)} dW_2(t),$$

where  $\mu(t)$  is drift,

$\eta$  is volatility of volatility,

$dW_1(t)$  and  $dW_2(t)$  are Wiener processes with correlation  $\rho$ .

Price  $V(t, S, v)$  of an option satisfies a PDE

$$\left(\frac{\partial}{\partial t} + L\right) V = 0,$$

$$\begin{aligned} L = & \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \rho\eta\beta(t, S, v)vS\frac{\partial^2}{\partial S\partial v} + \frac{1}{2}\eta^2\beta^2(t, S, v)v\frac{\partial^2}{\partial v^2} \\ & + rS\frac{\partial}{\partial S} + [\alpha(t, S, v) - \varphi(t, S, v)\beta(t, S, v)]\frac{\partial}{\partial v} - r, \end{aligned}$$

where  $\varphi = \varphi(t, S, v)$  is the *market price of volatility risk*.

## Multidimensional Stochastic Volatility Models

Assume that the volatility of a stock price  $S(t)$  is a function of  $(n - 1)$  stochastic factors  $y^i(t)$  so that

$$dS(t) = rS(t) dt + \sigma(S, y, t)S(t) dW_0(t)$$

$$dy^i(t) = \theta^i(t, y) dt + \sum_{j=1}^{n-1} \nu^{ij}(t, y) dW_j(t),$$

where  $\theta^i(t, y)$  are *drift coefficients*,

$dW_j(t)$  are Wiener processes with correlation matrix  $\rho_{ij}$ ,  
 $i, j = 0, 1, \dots, (n - 1)$ , and

$\nu^{ij}(t, y)$  is a *diffusion matrix*.

Price  $C(t, S, y, ; T, K)$  of an option satisfies PDE

$$\left( \frac{\partial}{\partial t} + L \right) C = 0,$$

$$L = \frac{1}{2} \sigma^2(S, y, t) S^2 \frac{\partial^2}{\partial S^2} + \sum_{j,k=1}^{n-1} A^k(y, t) \sigma(S, y, t) S \frac{\partial^2}{\partial S \partial y^k} \\ + \frac{1}{2} \sum_{k,l=1}^{n-1} B^{kl}(y, t) \frac{\partial^2}{\partial y^k \partial y^l} + rS \frac{\partial}{\partial S} + \sum_{i=1}^{n-1} \theta^i(t, y) \frac{\partial}{\partial y^i} - r,$$

where

$$A^k(y, t) = \sum_{j=1}^{n-1} \nu^{kj}(y, t) \rho_{j0},$$

$$B^{kl}(y, t) = \frac{1}{2} \sum_{i,j=1}^{n-1} \nu^{ki}(y, t) \rho_{ij} \nu^{jl}(y, t)$$

## Summary

- Option price satisfies *parabolic* PDE
- Operator  $L$  is *elliptic* second-order PDO
- Variables vary in a domain  $M$  of a Euclidean space  $\mathbb{R}^n$  with a boundary  $\partial M$
- There is a *terminal condition* at  $t = T$
- There must be *boundary conditions* either at the boundary or at infinity, as  $S, v, y^i \rightarrow 0, \infty$ .

# Methods for Parabolic PDE

## Parabolic PDE

### Heat (Diffusion) Equation

$$(\partial_t + L) V(t; x) = 0$$

$$L = - \sum_{i,j=1}^n \alpha^{ij}(t, x) \partial_i \partial_j + \sum_{i=1}^n \beta^i(t, x) \partial_i + \gamma(t, x),$$

Notation:

$$x = (x^1, \dots, x^n), \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

## Initial and Boundary Conditions

Initial condition

$$V(0; x) = f(x) ,$$

Dirichlet boundary conditions

$$V(t, x) \Big|_{\partial M} = 0 .$$

Neumann boundary conditions

$$\sum_{i=1}^n N^i(x) \partial_i V(t, x) \Big|_{\partial M} = 0 .$$

where  $N^i$  is the normal vector to the boundary

## Heat Kernel

$$(\partial_t + L)U(t, x|t', x') = 0,$$

Initial condition

$$U(t', x|t', x') = \delta(x - x'),$$

Delta-function

$$\delta(x - x') = \delta(x^1 - x'^1) \cdots \delta(x^n - x'^n).$$

For time-independent operators

$$U(t, x|t', x') = \tilde{U}(t - t'; x, x'),$$

## Cauchy Problem

Non-homogeneous initial value problem

$$(\partial_t + L)V(t, x) = g(t, x),$$

$$V(0, x) = f(x),$$

Solution

$$V(t, x) = \int_{\bar{M}} dx' U(t, x|0, x') f(x') \\ + \int_0^t dt' \int_{\bar{M}} dx' U(t, x|t', x') g(t', x')$$

where

$$dx = dx^1 \dots dx^n.$$

## Probabilistic Interpretation

Heat kernel  $U(t, x|t', x')$  is the *conditional probability* of reaching the point  $x$  at the time  $t$  if one starts at the point  $x'$  at the time  $t'$ .

**Similarity Transformation** Heat kernel  $U_\omega$  of the operator

$$L_\omega = e^{-\omega} L e^\omega$$

where  $\omega(x)$  is a smooth function, is given by

$$U_\omega(t, x|t', x') = e^{-\omega(x)} U(t, x|t', x') e^{\omega(x')}$$

Similarity transformation is *isospectral*

## Elliptic Operators

Dual variables (momenta)  $p = (p_1, \dots, p_n)$

Symbol

$$\sigma(x, p) = \sum_{j,k=1}^n \alpha^{jk}(x) p_j p_k + i \sum_{j=1}^n \beta^j(x) p_j + \gamma(x).$$

Leading (principal) symbol

$$\sigma_L(x, p) = \sum_{j,k=1}^n \alpha^{jk}(x) p_j p_k.$$

Operator  $L$  is *elliptic* if for any point  $x$  in  $M$  and for any real  $p \neq 0$  the leading symbol  $\sigma_L(x, p)$  is positive definite

# Spectral Theory of Operators in Hilbert Spaces

**Hilbert Space**  $L^2(M, \mu)$

*A Hilbert space is a complete infinite-dimensional vector space with an inner product.*

$L^2$  inner product

$$(f, g) = \int_M dx \mu(x) \overline{f(x)} g(x),$$

where  $\mu(x)$  be a positive *weight* function

## Operators on Hilbert Spaces

A linear *operator* on a Hilbert space  $H$  is a linear map  $A : H \rightarrow H$ .

The *adjoint* of the operator  $A$  is an operator  $A^*$  such that for any vectors  $f$  and  $g$ ,

$$(A^*f, g) = (f, Ag).$$

An operator  $A$  is called *self-adjoint* if

$$A = A^*$$

An operator  $U$  is called *unitary* if

$$UU^* = U^*U = I,$$

Every unitary operator  $U$  can be represented in the form

$$U = \exp(iA),$$

with some self-adjoint operator  $A$ .

A self-adjoint operator  $A$  is called *positive* if for all  $f \neq 0$

$$(f, Af) > 0.$$

A self-adjoint operator  $P$  is a *projection* if

$$P^2 = P.$$

For any projection operator  $P$  there is a vector subspace  $S$  it projects onto.

## Integral Operators

An operator  $G$  defined by

$$(Gf)(x) = \int_M dx' \mu(x') G(x, x') f(x'),$$

is called an *integral operator* with the *kernel*  $G(x, x')$

*Diagonal*

$$G^{\text{diag}}(x) = G(x, x).$$

*Trace*

$$\text{Tr } G = \int_M dx \mu(x) G(x, x).$$

Kernel of the adjoint operator  $G^*$

$$(G^*)(x, x') = \overline{G(x', x)}.$$

## Partial Differential Operators

Elliptic second-order PDO

$$L = - \sum_{i,j=1}^n \alpha^{ij}(x) \partial_i \partial_j + \sum_{j=1}^n \beta^j(x) \partial_j + \gamma(x).$$

Adjoint operator  $L^*$

$$L^* = - \sum_{i,j=1}^n \alpha^{ij}(x) \partial_i \partial_j - \sum_{j=1}^n \tilde{\beta}^j(x) \partial_j + \tilde{\gamma}(x),$$

where

$$\tilde{\beta}^j = -\beta^j - 2 \sum_{i=1}^n \mu^{-1} \partial_i (\mu \alpha^{ij}),$$

$$\tilde{\gamma} = \gamma - \sum_{i,j=1}^n \mu^{-1} \partial_i \partial_j (\mu \alpha^{ij}) - \sum_{i=1}^n \mu^{-1} \partial_i (\mu \beta^i).$$

Operator  $L$  is self-adjoint if

$$\beta^j = - \sum_{i=1}^n \mu^{-1} \partial_i (\mu \alpha^{ij}).$$

Then

$$L = - \sum_{i,j=1}^n \mu^{-1}(x) \partial_i \mu(x) \alpha^{ij}(x) \partial_j + \gamma(x).$$

Similar operator (with  $\omega = -\frac{1}{2} \log \mu$ )

$$L_\omega = e^{-\omega} L e^\omega = - \sum_{i,j=1}^n \partial_i \alpha^{ij}(x) \partial_j + \gamma_\omega(x),$$

where

$$\gamma_\omega = \gamma + \sum_{i,j=1}^n \left[ \alpha^{ij} (\partial_i \omega) (\partial_j \omega) - \partial_i (\alpha^{ij} \partial_j \omega) \right].$$

## Geometric form

Every elliptic second-order PDO can be written in the form

$$L = - \sum_{i,j=1}^n g^{-1/2} (\partial_i + \mathcal{A}_i) g^{1/2} g^{ij} (\partial_j + \mathcal{A}_j) + Q.$$

where

$$g^{ij} = \alpha^{ij}, \quad (g_{ij}) = (\alpha^{ij})^{-1},$$

$$g = \det g_{ij} = (\det \alpha^{ij})^{-1},$$

$$\mathcal{A}_i = - \sum_{j=1}^n \frac{1}{2} g_{ij} \beta^j - \sum_{j,k=1}^n \frac{1}{2} g_{ij} g^{-1/2} \partial_k (g^{1/2} g^{jk})$$

$$Q = \gamma + \sum_{i,j=1}^n \left[ g^{ij} \mathcal{A}_i \mathcal{A}_j + g^{-1/2} \partial_i (g^{1/2} g^{ij} \mathcal{A}_j) \right].$$

If the vector  $\mathcal{A}_i$  is non-zero, then the operator is not self-adjoint.

The tensor

$$\mathcal{R}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

measures the extent to which the operator  $L$  is non-self-adjoint.

The operator  $L$  is similar to a self-adjoint operator if  $\mathcal{R}_{ij} = 0$ .

## Resolvent

A complex number  $\lambda$  is an *eigenvalue* of an operator  $A$  if there is a non-zero vector (*eigenvector*)  $\varphi$  such that

$$A\varphi = \lambda\varphi.$$

Set of all eigenvectors corresponding to an eigenvalue forms a vector subspace (*eigenspace*)

Dimension of an eigenspace is the *multiplicity* of the eigenvalue

The operator

$$G(\lambda) = (A - \lambda I)^{-1}$$

is the *resolvent* of the operator  $A$ .

## Spectrum

The set of complex numbers  $\lambda$  for which the resolvent  $G(\lambda)$  is well defined is the *resolvent set* of the operator  $A$ .

The set of complex numbers  $\lambda$  for which the resolvent is not well defined is the *spectrum* of the operator  $A$ .

The eigenvalues form the *point spectrum*.

The remaining part of the spectrum is called the *continuous spectrum*.

## Spectra of Operators

The eigenvalues of self-adjoint operators are *real*

The eigenvalues of unitary operators are complex numbers with modulus equal to 1

The eigenvectors corresponding to distinct eigenvalues of self-adjoint operators are *mutually orthogonal*.

The eigenvalues of positive operators are positive.

The eigenvalues of a projection can only be either 1 or 0.

## Functions of Self-Adjoint Operators

Every self-adjoint operator  $A$  can be presented in the form

$$A = \sum_{n=1}^{\infty} \lambda_n P_n ,$$

where  $\lambda_n$  are the eigenvalues and  $P_n$  are the projections onto the corresponding eigenspaces

Functions of operator  $A$  are defined by

$$f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n .$$

For example, the *heat semi-group* is defined by

$$U(t) = \exp(-tA) = \sum_{n=1}^{\infty} e^{-t\lambda_n} P_n .$$

## Self-adjoint PDO on Compact Manifolds

Spectrum is *real and bounded from below*.

There is no continuous spectrum (*only point spectrum*).

Eigenvalues have *finite multiplicities*

There are only *finitely many* non-positive eigenvalues.

Eigenvalues form an *increasing sequence*  $(\lambda_k)_{k=1}^{\infty}$  which grows like  $k^2$  as  $k \rightarrow \infty$ .

The eigenfunctions  $(\varphi_k(x))_{k=1}^{\infty}$  are smooth functions that form an orthonormal basis.

## Resolvent and Heat Kernel

The resolvent and the heat kernel are given by

$$G(\lambda; x, x') = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} \varphi_k(x) \overline{\varphi_k(x')},$$

$$U(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k(x) \overline{\varphi_k(x')}.$$

## Remarks.

Eigenvalues can be computed explicitly in *rare highly symmetric cases* (constant curvature spaces, integrable potentials, separable operators)

There is an extensive list of such integrable cases (see references in lecture notes).

For non-compact manifolds the spectrum is *not discrete*

For non-self-adjoint operators the spectrum is *not real*

For singular boundary value problems the structure of the spectrum is very complicated

## Spectral Functions

Spectral functions are a powerful tool to study the spectrum, in particular, spectral asymptotics

*Heat trace*

$$\text{Tr} \exp(-tL) = \int_M dx g^{1/2}(x) U(t; x, x) = \sum_{k=1}^{\infty} d_k e^{-t\lambda_k} .$$

*Zeta-function and Functional determinant*

$$\zeta(s) = \text{Tr} L^{-s} = \sum_{k=1}^{\infty} d_k \lambda_k^{-s}, \quad \text{Det} L = \exp \left[ -\zeta'(0) \right] .$$

Functional determinant is used in the definition of Gaussian path integrals

# Operators with Constant Coefficients

## Fourier Transform

Definition

$$\hat{f}(p) = \int_{-\infty}^{\infty} dx e^{-ipx} f(x),$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \hat{f}(p).$$

Properties

$$\widehat{\partial_x f(x)}(p) = ip \hat{f}(p).$$

$$\widehat{xf(x)}(p) = i \partial_p \hat{f}(p),$$

## Step Function and Delta-Function

Fourier representation of the step function

$$\theta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} e^{ipx} \frac{1}{p - i\varepsilon} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{for } x < 0 \end{cases},$$

where  $\varepsilon > 0$  is an infinitesimal positive parameter.

Fourier representation of the delta-function

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx}.$$

Relation

$$\partial_x \theta(x) = \delta(x).$$

## Solving Heat Equation by Fourier Transform

Elliptic PDO with real *constant coefficients*

$$L = - \sum_{j,k=1}^n \alpha^{jk} \partial_j \partial_k + \sum_{j=1}^n \beta^j \partial_j + \gamma,$$

where  $\alpha^{ij}$  is a symmetric positive matrix,  $\beta^j$  be a vector and  $\gamma$  be a constant.

Heat equation

$$(\partial_t + L)U(t; x, x') = 0$$

Initial condition

$$U(0; x, x') = \delta(x - x')$$

Fourier transform

$$[\partial_t + \sigma(p)]\hat{U}(t; p) = 0$$

where

$$\sigma(p) = \sum_{j,k=1}^n \alpha^{jk} p_j p_k + i \sum_{j=1}^n \beta^j p_j + \gamma,$$

Initial condition

$$\hat{U}(0; p) = 1$$

*Heat Kernel*

$$U(t; x, x') = \int_{\mathbb{R}^n} \frac{dp}{(2\pi)^n} e^{-t\sigma(p) + i\langle p, (x-x') \rangle}$$

## Gaussian integrals

$$\int_{\mathbb{R}^n} \frac{dp}{(2\pi)^n} \exp \{ -t \langle p, Ap \rangle + i \langle x, p \rangle \}$$
$$= (4\pi t)^{-n/2} (\det A)^{-1/2} \exp \left\{ -\frac{1}{4t} \langle x, A^{-1} x \rangle \right\} .$$

## Heat Kernel

$$U(t; x, x') = (4\pi t)^{-n/2} [\det A]^{-1/2}$$
$$\times \exp \left\{ \frac{1}{2} \langle (x - x'), A^{-1} \beta \rangle - t \left[ \gamma + \frac{1}{4} \langle \beta, A^{-1} \beta \rangle \right] \right\}$$
$$\times \exp \left\{ -\frac{1}{4t} \langle (x - x'), A^{-1} (x - x') \rangle \right\} ,$$

and  $A$  is the positive definite matrix  $A = (\alpha^{ij})$ .

Initial condition

$$\lim_{t \rightarrow 0^+} (4\pi t)^{-n/2} [\det A]^{-1/2} \exp \left\{ -\frac{1}{4t} \langle x, A^{-1}x \rangle \right\} = \delta(x),$$

## Spectrum

Spectrum is continuous.

Spectrum is located in the half-plane  $\operatorname{Re} \lambda < \gamma$ .

In the self-adjoint case, when  $\beta^i = 0$ , the spectrum is the interval  $[\gamma, \infty)$  on the real line.

## Indegro-Differential Equations

*Integro-differential heat equation*

$$(\partial_t + L + K)U = 0$$

Differential operator with constant coefficients

$$L = -\alpha \partial_x^2 + \beta \partial_x + \gamma$$

Integral operator

$$(Kf)(x) = \int_{-\infty}^{\infty} dx' K(x - x') f(x'),$$

with *convolution kernel* (with positive  $\hat{K}(p)$ )

$$K(x - x') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} \hat{K}(p),$$

Fourier transform of convolution

$$\widehat{(Kf)}(p) = \widehat{K}(p)\widehat{f}(p).$$

Therefore

$$[\partial_t + \sigma(p) + \widehat{K}(p)]\widehat{U}(t; p) = 0,$$

where

$$\sigma(p) = \alpha p^2 + i\beta p + \gamma.$$

Solution

$$U(\lambda; x, x') = \int_{\mathbb{R}^n} \frac{dp}{(2\pi)^n} e^{-t[\sigma(p) + \widehat{K}(p)] + i\langle p, (x-x') \rangle}$$

## Integral Jump Operator

$$(Tf)(x) = \int_{-\infty}^{\infty} dx' \omega(x') [f(x+x') - f(x)] ,$$

where

$$\omega(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \hat{\omega}(p)$$

is some probability distribution with positive *characteristic function*  $\hat{\omega}(p)$

Integro-differential heat equation

$$(\partial_t + L + \lambda T)U = 0 ,$$

Jump operator has the form

$$T = K - I,$$

where  $I$  is the identity operator and  $K$  is a convolution operator with the kernel

$$K(x) = \omega(-x).$$

Heat Kernel

$$U(\lambda; x, x') = \int_{\mathbb{R}^n} \frac{dp}{(2\pi)^n} e^{-t[\sigma(p) + \lambda \hat{\omega}(-p) - \lambda] + i\langle p, (x - x') \rangle}$$

## Laplace Transform

Definition

$$F(s) = (\mathcal{L}f)(s) = \int_0^{\infty} dt e^{-st} f(t).$$

$$f(t) = (\mathcal{L}^{-1}F)(t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{st} F(s).$$

where  $c$  is a sufficiently large constant

Properties

$$(\mathcal{L}[\partial_t f(t)])(s) = sF(s) - f(0),$$

$$(\mathcal{L}[tf(t)])(s) = -\partial_s F(s).$$

## Solving Heat Equation by Laplace Transform

Heat kernel of an elliptic time-independent operator  $L$

$$(\partial_t + L)U(t; x, x') = 0$$

$$U(0; x, x') = \delta(x - x')$$

Resolvent

$$(L - \lambda)G(\lambda; x, x') = \delta(x - x').$$

Laplace transform

$$G(\lambda; x, x') = \int_0^{\infty} dt e^{t\lambda} U(t; x, x')$$

$$U(t; x, x') = \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{2\pi i} e^{-\lambda t} G(\lambda; x, x'),$$

# Homogeneous Differential Operators

## Mellin Transform

Definition

$$F(s) = (\mathcal{M}f)(s) = \int_0^{\infty} dt t^{s-1} f(t).$$

$F(s)$  is analytic in an infinite strip  $a < \operatorname{Re} s < b$

Inverse Mellin transform

$$f(t) = (\mathcal{M}^{-1}F)(t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} t^{-s} F(s),$$

where  $c$  is a real number such that  $a < c < b$

Relation to Fourier transform

$$(\mathcal{M}f(t))(s) = \widehat{f(e^x)}(is),$$

$$\widehat{f(x)}(p) = (\mathcal{M}f(\log t))(-ip),$$

where  $t = e^x, x = \log t$

Properties

$$[\mathcal{M}(tf(t))](s) = (\mathcal{M}f)(s + 1)$$

$$[\mathcal{M}(\partial_t f(t))](s) = -(s - 1)(\mathcal{M}f)(s - 1)$$

$$[\mathcal{M}(t\partial_t f(t))](s) = -s(\mathcal{M}f)(s)$$

## Homogeneous Differential Equations

Second-order homogeneous PDO with real coefficients

$$L = -\alpha x^2 \partial_x^2 + \beta x \partial_x + \gamma,$$

with  $\alpha > 0$  (*singular point at  $x = 0$* )

Heat equation

$$(\partial_t + L)U(t; x, x') = 0,$$

$$U(0; x, x') = \delta(x - x')$$

and the boundary condition

$$\lim_{x, x' \rightarrow 0} U(t; x, x') = \lim_{x, x' \rightarrow \infty} U(t; x, x') = 0.$$

Mellin transform

$$[\partial_t + \sigma^{\mathcal{M}}(s)]F(t, s) = 0.$$

*Mellin symbol*

$$\sigma^{\mathcal{M}}(s) = -\alpha s^2 - (\alpha + \beta)s + \gamma.$$

Initial condition

$$F(0, s) = (x')^{s-1}$$

Heat Kernel

$$U(t; x, x') = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{1}{x'} \left(\frac{x}{x'}\right)^{-s} \exp\{-t\sigma^{\mathcal{M}}(s)\}$$

# Asymptotic Expansion of Integrals

## Asymptotic Expansions

A function  $f(x)$  is *infinitesimal* with respect to a function  $g(x)$  as  $x \rightarrow a$

$$f(x) = o(g(x)) \quad \text{if} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0.$$

The function  $f(x)$  is *bounded* with respect to  $g(x)$  as  $x \rightarrow a$

$$f(x) = O(g(x)) \quad \text{if} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = C,$$

with some constant  $C$ .

We write

$$f(x) \sim g(x) \quad \text{if} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1,$$

An *asymptotic sequence* at  $x \rightarrow a$  is a sequence  $(\varphi_n)_{n=1}^{\infty}$  of real valued functions such that  $\varphi_n(x) \neq 0$  in a neighborhood of  $a$  and

$$\varphi_{n+1}(x) = o(\varphi_n(x)).$$

Example,  $\{(x - a)^n\}_{n=0}^{\infty}$

A function  $f$  is *expanded in an asymptotic series*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n(x),$$

where  $a_n$  are constants, if for all  $N \geq 0$

$$f(x) - \sum_{n=1}^N a_n \varphi_n(x) = o(\varphi_N(x)).$$

This series is called *asymptotic expansion* of the function  $f$  with respect to the asymptotic sequence  $(\varphi_n)$ .

The function

$$R_N(x) = f(x) - \sum_{n=1}^N a_n \varphi_n(x)$$

is the *remainder term* of the asymptotic series.

The condition  $R_N(x) = o(\varphi_N(x))$  means that for any fixed  $N$

$$\lim_{x \rightarrow a} R_N(x) = 0.$$

However, if for some fixed  $x$

$$\lim_{N \rightarrow \infty} R_N(x) \neq 0$$

then the asymptotic series *diverges*.

There are three possibilities:

- a) asymptotic series converges to the original function;
- b) asymptotic series converges to a different function;
- c) asymptotic series diverges.

The asymptotic expansion of a function with respect to an asymptotic sequence is *unique*.

However, two different functions can have the same asymptotic expansion.

## Gaussian Integrals

Standard one-dimensional Gaussian integral

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} = 1.$$

By *scaling* the variable  $x \rightarrow \alpha x$  with  $\alpha > 0$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\alpha x^2} = \alpha^{-1/2},$$

By *shifting* the variable  $x \rightarrow x - i\beta/(2\alpha)$  we get

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\alpha x^2 + i\beta x} = \alpha^{-1/2} \exp\left(-\frac{\beta^2}{4\alpha}\right).$$

By expanding in power series in  $\beta$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\alpha x^2} x^{2k+1} = 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\alpha x^2} x^{2k} = \frac{(2k)!}{2^{2k} k!} \alpha^{-k-1/2}.$$

## Multidimensional Gaussian integrals

Let  $A = (a_{ij})$  be an  $n \times n$  real symmetric positive matrix

Then for any vector  $p$  there holds

$$\int_{\mathbb{R}^n} \frac{dx}{\pi^{n/2}} \exp(-\langle x, Ax \rangle + i \langle p, x \rangle)$$
$$= (\det A)^{-1/2} \exp\left(-\frac{1}{4} \langle p, A^{-1}p \rangle\right).$$

Proof: by diagonalizing the matrix  $A$  and using one-dimensional Gaussian integrals.

By expanding in power series in  $p$

$$\int_{\mathbb{R}^n} \frac{dx}{\pi^{n/2}} \exp(-\langle x, Ax \rangle) x^{i_1} \dots x^{i_{2k+1}} = 0$$

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{dx}{\pi^{n/2}} \exp(-\langle x, Ax \rangle) x^{i_1} \dots x^{i_{2k}} \\ = (\det A)^{-1/2} \frac{(2k)!}{2^{2k} k!} G^{(i_1 i_2 \dots i_{2k-1} i_{2k})}. \end{aligned}$$

Here  $G = A^{-1}$  is the inverse matrix and the parenthesis denote *complete symmetrization* over all indices included.

Very important: the right-hand side does not depend on the dimension of the space  $\mathbb{R}^n$ . One can take the limit  $n \rightarrow \infty$  and define the infinite-dimensional *Gaussian path integrals*.

## Laplace Integrals in One Dimension

$$F(\lambda) = \int_a^b dx \varphi(x) e^{-\lambda S(x)},$$

where  $S$  and  $\varphi$  are smooth functions and  $\lambda$  is a *large positive parameter*.

*Non-degenerate critical point* (minimum at an interior point  $a < x_0 < b$ )

$$S'(x_0) = 0, \quad S''(x_0) > 0$$

Taylor expansion

$$S(x) = S(x_0) + \frac{1}{2} S''(x_0) (x - x_0)^2 + O((x - x_0)^3).$$

## Asymptotic Expansion

If the function  $S$  has a minimum only at a single non-degenerate interior critical point  $x_0$ , then as  $\lambda \rightarrow \infty$  there is an asymptotic expansion

$$F(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{1/2} [S''(x_0)]^{-1/2} e^{-\lambda S(x_0)} \sum_{k=0}^{\infty} a_k \lambda^{-k},$$

The coefficients  $a_k$  are *polynomial* in the higher derivatives  $S^{(k)}(x_0)$ ,  $k \geq 3$ , the derivatives  $\varphi^{(l)}(x_0)$ ,  $l \geq 0$ , and  $[S''(x_0)]^{-1}$ .

Leading asymptotics

$$F(t) \sim \left(\frac{2\pi}{\lambda}\right)^{1/2} [S''(x_0)]^{-1/2} e^{-\lambda S(x_0)} \varphi(x_0).$$

**Proof.**

Change of variable

$$x = x_0 + \lambda^{-1/2}y .$$

Taylor series

$$\begin{aligned} S(x_0 + \lambda^{-1/2}y) &= S(x_0) + \frac{1}{2}\lambda^{-1}S''(x_0)y^2 \\ &+ \sum_{n=3}^{\infty} \frac{S^{(n)}(x_0)}{n!} y^n \lambda^{-n/2} , \end{aligned}$$

$$\varphi(x_0 + \lambda^{-1/2}y) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_0)}{n!} y^n \lambda^{-n/2} .$$

Expand the exponent in a power series in inverse powers of  $\lambda$ .

Extend the integration interval to the whole real line and compute the standard Gaussian integrals.

The half-integer powers of  $\lambda^{-1}$  always come with half-integer powers of  $y$  and, therefore, vanish after integration.

The coefficients are *polynomial* in the higher derivatives  $S^{(k)}(x_0)$ ,  $k \geq 3$ , the derivatives  $\varphi^{(l)}(x_0)$ ,  $l \geq 0$ , and  $[S''(x_0)]^{-1}$ .

## Laplace Integral in Multiple Dimensions

$$F(\lambda) = \int_M dx \varphi(x) \exp[-\lambda S(x)],$$

where  $M$  is bounded connected open set in  $\mathbb{R}^n$ ,  $S$  and  $\varphi$  are some real-valued smooth functions on  $M$  and  $\lambda > 0$  is a large positive parameter.

A point  $x_0$  in  $M$  is called a *critical point* of the function  $S$  if

$$\partial_i S(x_0) = 0.$$

*Hessian matrix*

$$H = \left( \partial_i \partial_j S(x_0) \right),$$

A critical point  $x_0$  is called *non-degenerate* if the Hessian matrix is non-degenerate at  $x_0$ , that is,

$$\det H(x_0) \neq 0.$$

Non-degenerate critical points are isolated.

Suppose that the function  $S$  has a minimum only at a single interior non-degenerate critical point  $x_0$  in  $M$ . Then

$$\partial_i S(x_0) = 0, \quad H > 0$$

that is, Hessian matrix is positive

Taylor expansion

$$S(x) = S(x_0) + \frac{1}{2} \langle (x - x_0), H(x - x_0) \rangle + O((x - x_0)^3).$$

## Asymptotic Expansion

If the function  $S$  has only one non-degenerate critical point  $x_0$  in  $M$ , where it has the only minimum in  $M$ , then there is an asymptotic expansion as  $\lambda \rightarrow \infty$

$$F(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} [\det H]^{-1/2} \exp[-\lambda S(x_0)] \sum_{k=0}^{\infty} a_k \lambda^{-k}.$$

The coefficients  $a_k$  are polynomial in the higher derivatives  $[\partial_{i_1} \cdots \partial_{i_m} S(x_0)]$ ,  $m \geq 3$ , of the function  $S$ , the derivatives  $[\partial_{i_1} \cdots \partial_{i_m} \varphi(x_0)]$ ,  $m \geq 0$ , of the function  $\varphi$  and the inverse Hessian matrix  $H^{-1}$ .

Leading asymptotics

$$F(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} [\det H]^{-1/2} \exp[-\lambda S(x_0)] \varphi(x_0).$$

## Proof.

Change of the integration variables

$$x^i = x_0^i + \lambda^{-1/2} y^i .$$

Taylor series

$$S(x_0 + \lambda^{-1/2} y) = S(x_0) + \frac{1}{2} \lambda^{-1} \langle y, H y \rangle$$

$$+ \sum_{m=3}^{\infty} \sum_{i_1, \dots, i_m=1}^n \frac{\lambda^{-m/2}}{i_1! \dots i_m!} [\partial_{i_1} \dots \partial_{i_m} S(x_0)] y^{i_1} \dots y^{i_m} ,$$

$$\varphi(x_0 + \lambda^{-1/2} y) = \sum_{m=0}^{\infty} \sum_{i_1, \dots, i_m=1}^n \frac{\lambda^{-m/2}}{i_1! \dots i_m!} [\partial_{i_1} \dots \partial_{i_m} \varphi(x_0)] y^{i_1} \dots y^{i_m} .$$

Expand the exponent in a power series in inverse powers of  $\lambda$ .

Extend the integration domain to the whole  $\mathbb{R}^n$  and compute the standard Gaussian integrals.

Finally, we get a power series in inverse powers of  $\lambda$  with coefficients polynomial in the higher derivatives  $[\partial_{i_1} \cdots \partial_{i_m} S(x_0)]$ ,  $m \geq 3$ , of the function  $S$ , the derivatives  $[\partial_{i_1} \cdots \partial_{i_m} \varphi(x_0)]$ ,  $m \geq 0$ , of the function  $\varphi$  and the inverse Hessian matrix  $H^{-1}$ .

## Feynmann Diagrams

Let us represent the derivatives  $[\partial_{i_1} \cdots \partial_{i_m} S(x_0)]$ ,  $m \geq 3$ , by *vertices* with  $m$  lines attached to it

and the derivatives  $[\partial_{i_1} \cdots \partial_{i_m} \varphi(x_0)]$ ,  $m \geq 0$ , by another type of vertices with  $m$  lines attached to them.

Let us represent the inverse Hessian  $H^{-1}$ , called the *propagator*, by a line connecting two vertices.

Then each term of the asymptotic expansion can be represented by an appropriate graph known as a *Feynmann diagram* where the corresponding legs of the vertices are linked by propagators.